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Generalizations of growth constants, 1

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Abstract. For entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, M. N. Seremeta defined the order ϱ as follows

$$(*)\qquad \varrho = \limsup_{r\to\infty} a[\log M(r)]/\beta[\log r],$$

where $M(r) = \max|f(z)|$ on $|z| = r$ and a, β are the real valued monotonically indefinitely increasing functions defined on $[a, \infty)$ with a real, such that $a(x) > 0, a[x(1+\delta(x))]/a(x)\to 1$ as $x\to\infty$, for x in $[a, \infty)$ and $\delta(x) > 0$ as $x\to\infty$ and $\beta(cx)/\beta(x)\to 1$ as $x\to\infty$ for $0 < c < \infty$. Seremeta established coefficient equivalents for ϱ defined by $(*)$. In the present paper, we complimented the above result by defining the lower order $\lambda = \liminf_{r\to\infty} a[\log M(r)]/\beta[(\log r)]$ and establishing the coefficients equivalents for λ. Our sample result is

$$\lambda = \max_{n_k}\{\liminf_{k\to\infty} a(m_{n_{k-1}})/\beta((1/m_{n_k})\log|a_{n_k}|^{-1})\},$$

where $\{m_{n_k}\}$ is a subsequence of $\{m_n\}$ in $f(z) = \sum_{n=0}^{\infty} a_n z^{m_n}$. Our results include the results obtained recently by the authors in Trans. Amer. Math. Soc. 203 (1975), p. 275–297.

1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$, $a_k \neq 0$, be an entire function. Following Seremeta [5], define the following:

$$(1.1)\qquad \varrho = \limsup_{r\to\infty} \frac{a[\log M(r)]}{\beta[\log r]},$$

where

$$M(r) = \max_{|z|=r} |f(z)|$$

and $a \in \Lambda$ and $\beta \in L^0$. By L^0 we denote the class of functions h defined on $[a, \infty)$ such that h is differentiable, monotonically strictly increasing, approaches ∞ as x approaches ∞ and

$$(1.2)\qquad \lim_{x\to\infty} \frac{h[(1+\delta(x))x]}{h(x)} = 1, \quad h(x) > 0,\ x \in [a, \infty),$$

for every $\delta(x)$ such that $\delta(x) \to 0$ as $x \to \infty$. Further, if h belongs to L^0 and satisfies

(1.3) $$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1, \quad 0 < c < \infty,$$

with h belongs to the class Λ, provided that convergence in (1.3) is uniform then respect to c, $0 < c_1 \leqslant c \leqslant c_2 < \infty$.

Seremeta proved the following:

THEOREM [Seremeta]. *Let $a \in \Lambda$, $\beta \in L^0$. We set $F(x;\ c) = \beta^{-1}[ca(x)]$. If $dF(x;\ c)/d\log x = O(1)$ as $x \to \infty$ for all c, $0 < c < \infty$, then*

$$\varrho = \limsup_{k\to\infty} \frac{a(k)}{\beta\left(\frac{1}{k}\log\left|\frac{1}{a_k}\right|\right)}$$

for $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and f entire.

Remark. Note that Seremeta's theorem for $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$, $a_k \neq 0$, takes the form

(1.4) $$\varrho = \limsup_{k\to\infty} \frac{a(\lambda_k)}{\beta\left[\frac{1}{\lambda_k}\log\left|\frac{1}{a_k}\right|\right]}.$$

We prove the following theorem which is an analogue of Shah [7] and the first author [1].

THEOREM 1. *Let $a \in \Lambda$, $\beta \in L^0$ and let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ be entire, $a_k \neq 0$. Set*

(1.5) $$F(x;\ c) = \beta^{-1}[c(a(x))].$$

If $dF(x;\ c)/d\log x = O(1)$ as $x \to \infty$ for all c, $0 < c < \infty$ and $\psi(k) = \left\{\log\left|\frac{a_k}{a_{k+1}}\right|\right\}/(\lambda_{k+1} - \lambda_k)$ is non-decreasing, then

(1.6) $$\varrho = \limsup_{k\to\infty} \frac{a(\lambda_k)}{\beta\left[\left(\frac{1}{\lambda_k - \lambda_{k-1}}\right)\log\left|\frac{a_{k-1}}{a_k}\right|\right]}.$$

Proof. Let

(1.7) $$q = \limsup_{k\to\infty} \frac{a(\lambda_k)}{\beta\left[\left(\frac{1}{\lambda_k - \lambda_{k-1}}\right)\log\left|\frac{a_{k-1}}{a_k}\right|\right]}.$$

Since $\psi(k)$ is non-decreasing, we can always find, for all $k \geqslant k_0$, $\psi(k) > 0$. Thus, without loss of generality we may assume $\psi(k) > 0$ for all k in (1.7). Thus, for given $\varepsilon > 0$, there exists k_0 such that, for all $k \geqslant k_0(\varepsilon) \equiv k_0$,

(1.8) $$a(\lambda_k) < (q+\varepsilon)\beta\left[\left(\frac{1}{\lambda_k - \lambda_{k-1}}\right)\log\left|\frac{a_{k-1}}{a_k}\right|\right].$$

From (1.8), we have

$$(1.9)\qquad (\lambda_k-\lambda_{k-1})\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]<\log\left|\frac{a_{k-1}}{a_k}\right|.$$

Adding inequalities (1.9) for $k = k_0, k_0+1, \ldots, k$, we infer

$$(1.10)\qquad \sum_{m=k_0}^{k}(\lambda_m-\lambda_{m-1})\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_m)\right]<\log\left|\frac{a_{k_0-1}}{a_k}\right|.$$

From (1.10) we have

$$(1.11)\qquad \beta\left[\frac{1}{\lambda_k}\sum_{m=k_0}^{k}(\lambda_m-\lambda_{m-1})\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_m)\right]\right]<\beta\left[\frac{1}{\lambda_k}\log\left|\frac{a_{k_0-1}}{a_k}\right|\right].$$

This gives

$$(1.12)\qquad \varrho\leqslant\limsup_{k\to\infty}\frac{\alpha(\lambda_k)}{\beta\big(D_1(\lambda_k)\big)},$$

where

$$D_1(\lambda_k)=\left[\frac{1}{\lambda_k}\sum_{m=k_0}^{k}(\lambda_m-\lambda_{m-1})\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_m)\right]\right].$$

Now we compute $D_1(\lambda_k)$. If $n(t)=\lambda_{m-1}$ if $\lambda_{m-1}\leqslant t<\lambda_m$, then we have

$$\begin{aligned}
D_1(\lambda_k) &= \frac{1}{\lambda_k}\sum_{m=k_0}^{k}(\lambda_m-\lambda_{m-1})\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_m)\right]\\
&= \frac{1}{\lambda_k}\Bigg[\lambda_k\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]-\sum_{m=k_0+1}^{k}\lambda_{m-1}\Bigg\{\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_m)\right]-\\
&\qquad -\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_{m-1})\right]\Bigg\}-\lambda_{k_0-1}\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_{k_0})\right]\Bigg]\\
&= \beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]-\frac{1}{\lambda_k}\int_{\lambda_{k_0+1}}^{\lambda_k}n(t)\,d\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(t)\right]-\\
&\qquad -\left(\frac{\lambda_{k_0-1}}{\lambda_k}\right)\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_{k_0})\right]\\
&= \beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]-O(1)\frac{1}{\lambda_k}\int_{\lambda_{k_0+1}}^{\lambda_k}\frac{n(t)}{t}\,dt-\\
&\qquad -\left(\frac{\lambda_{k_0-1}}{\lambda_k}\right)\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_{k_0})\right]
\end{aligned}$$

(by using hypothesis $df(x, c) = O(1)d\log x$ for λ_{k_0} sufficiently large)

$$\geqslant \beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right] - \frac{1}{\lambda_k}\int_{\lambda_{k_0+1}}^{\lambda_k} dt - \frac{\lambda_{k_0-1}}{\lambda_k}\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_{k_0})\right]$$

$$= \left(\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]\right)\left(1+\delta(\lambda_k)\right);$$

$$\beta(D_1(\lambda_k)) = \beta\left[\left\{\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]\right\}\left(1+\delta(\lambda_k)\right)\right] \tag{1.13}$$

$$\simeq \beta\left[\beta^{-1}\left[\left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k)\right]\right]$$

$$= \left(\frac{1}{q+\varepsilon}\right)\alpha(\lambda_k),$$

since $\beta^{-1}\left[\left(1/(q+\varepsilon)\right)\alpha(\lambda_k)\right]$ is well defined.

Thus, from (1.12) and (1.13) it follows immediately that $\varrho \leqslant q$. Now we need to prove $q \leqslant \varrho$. This is very simple and can be seen as follows: Since $\psi(k)$ is non-decreasing, we have

$$\log\left|\frac{a_{k_0}}{a_{k+1}}\right| = \sum_{m=k_0}^{k}\psi(m)(\lambda_{m+1}-\lambda_m) < \psi(k)(\lambda_{k+1}-\lambda_{k_0}). \tag{1.14}$$

From (1.14) we have

$$\left(\frac{1}{\lambda_{k+1}-\lambda_{k_0}}\right)\log\left|\frac{a_{k_0}}{a_{k+1}}\right| < \left(\frac{1}{\lambda_{k+1}-\lambda_k}\right)\log\left|\frac{a_k}{a_{k+1}}\right|. \tag{1.15}$$

Therefore, we have $q \leqslant \varrho$. This completes the proof of the theorem.

2. Lower order analogues. In this section we introduce the lower order analogue by defining

$$\lambda = \liminf_{r\to\infty}\frac{\alpha[\log(M(r))]}{\beta[\log r]}, \tag{2.1}$$

where $\alpha \in L^0$ and $\beta \in \Lambda$. The growth ϱ defined by (1.1) for $\alpha = \log$, $\beta = I$ is due to Whittakar [9]. Coefficient equivalents when $\alpha = \log$, $\beta = 1$ are due to Shah [6], Roux [4] and for $\alpha = \log^p = \log\log\ldots$ p-times and $\beta = \log^q = \log\log\ldots$, q-times are due to Bajpai, Juneja, and Kapoor ([1]–[3]). In this context, we shall obtain the coefficient equivalent of (2.1). These coefficient equivalents include earlier results mentioned above under suitable selections of α and β. In fact, we have the following:

THEOREM 2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$, $a_k \neq 0$, *be an entire function of growth* λ. *If* $a \in \Lambda$ *and* $\beta \in L^0$ *and*

$$\lim_{x\to\infty} \frac{a[x\psi(x)]}{a[x]} = 1 \quad \textit{if} \quad \lim_{x\to\infty} \frac{\psi(x)}{X} = 0$$

for ψ *defined on* $[a, \infty)$ *increases indefinitely, then we have*

$$\lambda = \max_{\{n_k\}} \left\{ \liminf_{k\to\infty} \frac{a[\lambda_{n_{k-1}}]}{\beta\left[\frac{1}{\lambda_{n_k}} \log|a_{n_k}|^{-1}\right]} \right\} \tag{2.2}$$

$$= \max_{\{n_k\}} \left\{ \liminf_{k\to\infty} \frac{a[\lambda_{n_{k-1}}]}{\beta\left[\left(\frac{1}{\lambda_{n_k}-\lambda_{n_{k-1}}}\right)\log\left|\frac{a_{n_{k-1}}}{a_{n_k}}\right|\right]} \right\},$$

where $\{\lambda_{n_k}\}$ *is subsequence of integers* $\{\lambda_k\}$ *and* $a_{n_k} \in \{a_n\}$.

LEMMA 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, $a_n \neq 0$ *be an entire function of growth* ϱ, λ *defined by* (1.1) *and* (2.1). *Further, if*

$$m(r) = \max_{n\geq 0} \{|a_n| r^{\lambda_n}\} \tag{2.3}$$

and

$$\nu(r) = \max\{n \mid m(r) = |a_n| r^{\lambda_n}\}, \tag{2.4}$$

then we have

$$\varrho = \limsup_{r\to\infty} \frac{a[\log m(r)]}{\beta[\log r]} = \limsup_{r\to\infty} \frac{a[\nu(r)]}{\beta[\log r]} \tag{2.5}$$

for $a \in \Lambda$ *and* $\beta \in L^0$ *such that*

$$\lim_{x\to\infty} \frac{a(x\psi(x))}{a(x)} = 1 \quad \textit{if} \quad \lim_{x\to\infty} \frac{\psi(x)}{x} = 0. \tag{2.6}$$

Proof. From the well-known results (see e.g. [8])

$$\log m(r) = \log m(r_0) + \int_{r_0}^{r} \frac{\nu(x)}{x}\,dx \tag{2.7}$$

and

$$m(r) \leqslant M(r) < m(r)\left\{1 + 2\nu\left(r + \frac{r}{\nu(r)}\right)\right\} \tag{2.8}$$

we have

$$\nu(r)\log 2 \leqslant \int_{r}^{2r} \frac{\nu(x)}{x}\,dx < \log m(2r) \leqslant \log M(2r) \tag{2.9}$$

and

$$(2.10)\qquad M(r) < m(r)\left\{1+2\nu\left(r+\frac{r}{\nu(r)}\right)\right\} < m(r)\{1+2\nu(2r)\}.$$

Hence, for functions of growth ϱ, it follows

$$(2.11)\qquad \limsup_{r\to\infty}\frac{\alpha[\log m(r)]}{\beta[\log r]} = \limsup_{r\to\infty}\frac{\alpha(\nu(2r))}{\beta[\log 2r]} = \varrho.$$

LEMMA 2. *Under the hypotheses of Lemma* 1, *we have*

$$(2.12)\qquad \lambda = \liminf_{r\to\infty}\frac{\alpha[\log m(r)]}{\beta[\log r]} = \liminf_{r\to\infty}\frac{\alpha[\nu(2r)]}{\beta[\log 2r]}.$$

Proof. Let

$$(2.13)\qquad \delta = \liminf_{r\to\infty}\frac{\alpha[\nu(r)]}{\beta[\log r]} \quad\text{and}\quad \delta > 0.$$

Then, we have for a sequence $r = r_1, r_2, \ldots \to\infty$,

$$(2.14)\qquad \alpha[\nu(r)] < b_1\beta[\log r],\quad b_1 = bd,\quad \delta < b_1,\quad \frac{\delta}{b} < d < 1.$$

Then, if $r_n^d \leqslant r \leqslant r_n$,

$$(2.15)\qquad \alpha[\nu(r)] \leqslant \alpha[\nu(r_n)] < bd\beta[\log r_n] \leqslant bd\beta\left[\frac{1}{d}\log r\right].$$

Also, since $\beta \in L^0$, we have, for all $0 < 1/d < \infty$,

$$(2.16)\qquad bd\beta\left[\frac{1}{d}\log r\right] < (1+\varepsilon)bd\beta[\log r].$$

Then Cauchy's inequality and the relation

$$(2.17)\qquad \nu(r)\log k \leqslant \int_r^{kr}\frac{\nu(x)}{x}\,dx < \log m(kr) \leqslant \log M(kr)$$

implies, by making $k\to e$,

$$(2.18)\qquad \delta = \liminf_{r\to\infty}\frac{\alpha[\log\nu(r)]}{\beta[\log r]} \leqslant \liminf_{r\to\infty}\frac{\alpha[\log m(kr)]}{\beta[\log r]} \leqslant \liminf_{r\to\infty}\frac{\alpha[\log M(kr)]}{\beta[\log r]} = \lambda.$$

Thus, we need to prove the reverse inequality in case $0 \leqslant \delta < \infty$. Choosing the constants c, e such that $d < c < 1$, $d/c < e < 1$ and writing $S_n = R_n^c$ so that

$$R_n^d < S_n^e < S_n < \tfrac{1}{2}R_n \quad (n \geqslant n_1),$$

by relation (2.7) we have, for $R_n = r_n$,

$$\log m(S_n) = \log m(S_n^\varepsilon) + \int_{S_n^\varepsilon}^{S_n} \frac{\nu(x)}{x}\,dx$$

and

$$\log m(S_n^\varepsilon) < \varepsilon \nu(S_n^\varepsilon) \log S_n.$$

Then

$$\begin{aligned}\log m(S_n) &\geqslant \log m(S_n^\varepsilon) + \nu(S_n^\varepsilon) \int_{S_n^\varepsilon}^{S_n} \frac{dx}{x} \\ &\geqslant \log m(S_n^\varepsilon) + (1-\varepsilon)\nu(S_n^\varepsilon)\log S_n \\ &> \left[1 + \frac{1-\varepsilon}{\varepsilon}\right] \log m(S_n^\varepsilon) = \frac{1}{\varepsilon} \log m(S_n^\varepsilon).\end{aligned}$$

Now, by using (2.15), we get

$$\log m(S_n) < \varepsilon \log m(S_n) + \int_{S_n^\varepsilon}^{S_n} \frac{\nu(x)}{x}\,dx$$

or

$$\begin{aligned}(1-\varepsilon)\log m(S_n) &< \int_{S_n^\varepsilon}^{S_n} \frac{\nu(x)}{x}\,dx \\ &< \int_{S_n^\varepsilon}^{S_n} \frac{a^{-1}\left[bd\beta\left[\frac{1}{d}\log x\right]\right]}{x}\,dx \\ &\leqslant a^{-1}\left[bd\beta\left[\frac{1}{d}\log S_n\right]\right](1-\varepsilon)\log S_n.\end{aligned}$$

This gives

$$a[\log m(S_n)] < a\left[a^{-1}\left[bd\beta\left[\frac{1}{d}\log S_n\right]\right]\log S_n\right]. \tag{2.19}$$

Also from (2.10) and (2.15), we have

$$M(S_n) < m(S_n)\left[2\nu\left(S_n + \frac{S_n}{\nu(S_n)}\right) + 1\right]$$

and

$$\nu\left(S_n + \frac{S_n}{\nu(S_n)}\right) < \nu(2S_n) < a^{-1}\left[bd\beta\left[\frac{1}{d}\log 2S_n\right]\right],$$

whence

$$M(S_n) < m(S_n)\left[1+2a^{-1}\left[bd\beta\left[\frac{1}{d}\log 2S_n\right]\right]\right]. \tag{2.20}$$

Now, if the growth of a is such that $a[x\psi(x)] \sim a(x)$ if $x/\psi(x)\to\infty$, then we find that

$$\liminf_{S_n\to\infty}\frac{a(\log M(S_n))}{\beta(\log S_n)} \leqslant bd \leqslant \delta. \tag{2.21}$$

Also, from (2.20), we have

$$\log M(S_n) < \log m(S_n) + o(1) + \log\left[a^{-1}\left[bd\beta[\log S_n]\right]\right] \tag{2.22}$$

and so that

$$1 < \frac{\log m(S_n)}{\log M(S_n)} + o(1) + \frac{\log\left[a^{-1}\left[bd\beta[\log S_n]\right]\right]}{\log M(S_n)}. \tag{2.23}$$

The last term in (2.23) approaches zero if $\lambda > 0$. But if $\lambda = 0$, then trivially we have $\delta = 0$ by (2.18). Thus

$$1 = \liminf_{S_n\to\infty}\frac{\log m(S_n)}{\log M(S_n)} \quad \text{if } 0 < \lambda < \infty. \tag{2.24}$$

This implies for $0<\lambda<\infty$, by (2.21), that

$$\liminf_{S_n\to\infty}\frac{a[\log M(S_n)]}{\beta[\log S_n]} = \liminf_{S_n\to\infty}\frac{a[\log m(S_n)]}{\beta[\log S_n]} \leqslant \delta. \tag{2.25}$$

Hence, we have the result of the lemma.

LEMMA 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, $a_n \neq 0$, *be an entire function of growth* λ *defined by* (2.1), *where* a *satisfies in addition* (2.6); *then we have*

$$\eta = \liminf_{n\to\infty}\frac{a(\lambda_{n-1})}{\beta\left[\dfrac{1}{\lambda_n}\log|a_n|^{-1}\right]} \leqslant \lambda \tag{2.26}$$

and

$$\zeta = \liminf_{n\to\infty}\frac{a(\lambda_{n-1})}{\beta\left[\left(\dfrac{1}{\lambda_n-\lambda_{n-1}}\right)\log\left|\dfrac{a_{n-1}}{a_n}\right|\right]} \leqslant \lambda. \tag{2.27}$$

For (2.27), *we define* β *negatively in the complement of* $[a, \infty)$.

Proof. Assume first $0 < \eta < \infty$. Then, for $0 < \varepsilon < \eta$ and all $n \geqslant n_0(\varepsilon)$, we have

$$\exp\left\{-\lambda_n\beta^{-1}\left[\left(\frac{1}{\eta-\varepsilon}\right)a(\lambda_{n-1})\right]\right\} < |a_n|. \tag{2.28}$$

Define

$$r_k = k\exp\left\{\beta^{-1}\left[\left(\frac{1}{\eta-\varepsilon}\right)\alpha(\lambda_{n-1})\right]\right\} \quad \text{for } k = 2, 3, \ldots \tag{2.29}$$

and assume (2.28) hold for all n, since addition of a polynomial does not affect the growth λ. If $r_n \leqslant r \leqslant r_{n+1}$, then from Cauchy's inequality and (2.29), we have

$$\begin{aligned} \log M(r) &\geqslant \log|a_n| + \lambda_n \log r \\ &\geqslant \log|a_n| + \lambda_n \log r_n \geqslant \lambda_n \log k. \end{aligned} \tag{2.30}$$

But (2.30) implies

$$\begin{aligned} \alpha(\log M(r)) &\geqslant \alpha(\lambda_n \log k) \simeq \alpha(\lambda_n) \\ &\geqslant (\eta-\varepsilon)\beta\left[\log\frac{1}{k} r_{n+1}\right] \geqslant (\eta-\varepsilon)\beta[\log r - \log k] \\ &= (\eta-\varepsilon)\beta\left[(\log r)\left(1-\frac{\log k}{\log r}\right)\right] \\ &\simeq (\eta-\varepsilon)\beta[\log r] \quad \text{as } \beta \in L^0. \end{aligned} \tag{2.31}$$

This proves (2.26). Now, we establish (2.27). By hypothesis, β is either positive or negative. In case it is negative, we find (2.27) true trivially. Without loss of generality we assume, for all $n \geqslant n_0(\varepsilon)$, that β is positive. Thus, for given $\varepsilon > 0$ and $\zeta > \varepsilon > 0$, we have

$$(\zeta-\varepsilon)\beta\left[\left(\frac{1}{\lambda_n-\lambda_{n-1}}\right)\log\left|\frac{a_{n-1}}{a_n}\right|\right] < \alpha(\lambda_{n-1}). \tag{2.32}$$

Then by rearrangement and by summing the resulting inequalities for $n_0, n_0+1, \ldots, n$, we have

$$\log\left|\frac{a_{n_0-1}}{a_n}\right| < \sum_{k=n_0}^{n}(\lambda_k-\lambda_{k-1})\beta^{-1}\left[\left(\frac{1}{\zeta-\varepsilon}\right)\alpha(\lambda_{n-1})\right]. \tag{2.33}$$

Let r_n be defined by

$$r_n = 2\exp\left\{\beta^{-1}\left[\left(\frac{1}{\zeta-\varepsilon}\right)\alpha(\lambda_{n-1})\right]\right\}; \quad n = 2, 3, \ldots \tag{2.34}$$

Let $r_n \leqslant r \leqslant r_{n+1}$. Then

$$\begin{aligned} \log M(r) &\geqslant \log|a_n| + \lambda_n \log r_n \\ &> \lambda_n \log r_n - \sum_{k=n_0}^{n}(\lambda_k-\lambda_{k-1})\beta^{-1}\left[\left(\frac{1}{\zeta-\varepsilon}\right)\alpha(\lambda_{k-1})\right] + \log|a_{n_0-1}| \\ &> \lambda_n \log 2 + O(1) \sim \lambda_n \log 2 \end{aligned} \tag{2.35}$$

Thus, we have the asymptotic inequality

$$\begin{aligned} a(\log M(r)) > a(\lambda_n \log 2) &\simeq a(\lambda_n) \\ &= (\zeta-\varepsilon)\beta\left[\left(\log\left(\frac{r_{n+1}}{2}\right)\right)\right] \\ &\geqslant (\zeta-\varepsilon)\beta\left[\log\left(\frac{r}{2}\right)\right] \\ &\simeq (\zeta-\varepsilon)\beta[\log r]. \end{aligned}$$

This completes the proof of (2.27). For $\zeta = 0$ or $\eta = 0$, the lemma is trivial.

LEMMA 4. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$, $a_k \neq 0$, be such an entire function of growth λ that $\left\{\left|\dfrac{a_k}{a_{k+1}}\right|^{1/(\lambda_{k+1}-\lambda_k)}\right\}$ forms a non-decreasing function for $k \geqslant k_0$; then we have*

$$\lambda \leqslant \liminf_{n\to\infty} \frac{a(\lambda_{n-1})}{\beta\left[\dfrac{1}{\lambda_n}\log|a_n|^{-1}\right]} = \eta$$

and

$$\lambda \leqslant \liminf_{n\to\infty} \frac{a(\lambda_{n-1})}{\beta\left[\left(\dfrac{1}{\lambda_n-\lambda_{n-1}}\right)\log\left|\dfrac{a_{n-1}}{a_n}\right|\right]} = \zeta.$$

Proof. Since $\psi(k)$ forms a non-decreasing function of k for $k > k_0$, we have $\psi(k) > \psi(k-1)$ for infinity of k; if otherwise, $\psi(k) = \psi(k+1) = \ldots$ ad infinitum for $k > k_0$ say, so the radius of convergence of the series $\sum_{k=0}^{\infty} a_k z^{\lambda_k}$ would be finite, $\psi(k)$ tends to infinity with k. When $\psi(k) > \psi(k-1)$, the term $a_k z^{\lambda_k}$ becomes maximum term and we have

$$m(r) = |a_k| r^{\lambda_k}, \quad \nu(r) = \lambda_k \quad \text{for } \psi(k-1) \leqslant r < \psi(k).$$

Then, from (2.12), we have

$$\nu(r) > a^{-1}[(\lambda-\varepsilon)\beta[\log r]] \tag{2.36}$$

for $\varepsilon > 0$, and $r \geqslant r_0(\varepsilon)$.

Let $|z| = r > r_0$ and let $a_{k_1} z^{\lambda_{k_1}}$ and $a_{k_2} z^{\lambda_{k_2}}$ ($k_1 > k_0$, $\psi(k_1-1) > r_0$) be two consecutive maximum terms so that $k_1 \leqslant k_2 - 1$. Let $k_1 < k \leqslant k_2$. Since $a_{k_1} z^{\lambda_{k_1}}$ is maximum term, we have

$$\nu(r) = \lambda_{k_1} \quad \text{for } \psi(k_1-1) \leqslant r < \psi(k_1).$$

Hence for all r in this interval

$$\lambda_{k_1} = \nu(r) > a^{-1}[(\lambda-\varepsilon)\beta[\log r]]. \tag{2.37}$$

Further, since

$$\psi(k_1) = \psi(k_1+1) = \psi(k_1+2) = \ldots = \psi(k-1),$$

we have

$$\lambda_{k-1} \geqslant \lambda_{k_1} > a^{-1}\left[(\lambda-\varepsilon)\beta[\log r]\right] \geqslant a^{-1}\left[(\lambda-\varepsilon)\beta[\log\psi(k-1)-c]\right], \tag{2.38}$$

where

$$c = \min\left[1, \frac{\psi(k_1)-\psi(k_1-1)}{2}\right].$$

Thus

$$\log\left|\frac{a_{k_0}}{a_{k_0+1}}\right| + \ldots + \log\left|\frac{a_{k-1}}{a_k}\right| = \log\left|\frac{a_{k_0}}{a_k}\right| \leqslant (\lambda_k-\lambda_{k_0})\log\psi(k-1), \tag{2.39}$$

and hence, for large k,

$$\lambda_{k-1} \geqslant a^{-1}\left[(\lambda-\varepsilon)\beta\left[\frac{1}{\lambda_k-\lambda_{k_0}}\log\left|\frac{a_{k_0}}{a_k}\right| - c\right]\right] \tag{2.40}$$

or

$$\frac{a(\lambda_{k-1})}{|\beta\left[\frac{1}{\lambda_k}\log|a_k|^{-1}(1+o(1))\right]} \geqslant (\lambda-\varepsilon). \tag{2.41}$$

Since $\beta \in L^0$ and ε is arbitrary, first part of the lemma follows. For proving of the second part, we notice that (2.40) gives

$$\frac{a(\lambda_{k-1})}{\beta\left[(\log\psi(k-1))(1-o(1))\right]} \geqslant \lambda-\varepsilon. \tag{2.42}$$

Again, since $\beta \in L^0$ and ε is arbitrary, the second part of the lemma also follows by taking limit inferior in (2.38).

Proof of the theorem. It is very much obvious from the construction of Newton's polygon that growth constants ϱ and λ defined by equations (1.1) and (2.1) are the same for the function $f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}$ given in Theorem 2, and its auxiliary function $g(z) = \sum_{k=0}^{\infty} a_{n_k} z^{\lambda_{n_k}}$ constructed such that g and f have the same principal indicies, i.e. they have the same maximum terms. From Lemma 3, it is clear that λ given by f gives the inequalities

$$\lambda \geqslant \liminf_{n_k\to\infty} \frac{a(\lambda_{n_{k-1}})}{\beta\left[\frac{1}{\lambda_{n_k}}\log\left|\frac{1}{a_{n_k}}\right|\right]} \tag{2.43}$$

and

$$\lambda \geqslant \liminf_{n_k\to\infty} \frac{a(\lambda_{n_{k-1}})}{\beta\left[\left(\frac{1}{\lambda_{n_k}-\lambda_{n_{k-1}}}\right)\log\left|\frac{a_{n_{k-1}}}{a_{n_k}}\right|\right]} \tag{2.44}$$

for every subsequence $\{\lambda_{n_k}\}$; $\{\lambda_k\}$ and a_{n_k} are corresponding coefficients. But as $g(z)$ satisfies all the conditions of Lemma 4, it follows from (2.43) and (2.44) that

$$\lambda = \max_{\{n_k\}}\left\{\liminf_{k\to\infty} \frac{\alpha[\lambda_{n_{k-1}}]}{\beta\left[\dfrac{1}{\lambda_{n_k}}\log|a_{n_k}|^{-1}\right]}\right\}$$

$$= \max_{\{n_k\}}\left\{\liminf_{k\to\infty} \frac{\alpha[\lambda_{n_{k-1}}]}{\beta\left[\left(\dfrac{1}{\lambda_{n_k}-\lambda_{n_{k-1}}}\right)\log\left|\dfrac{a_{n_{k-1}}}{a_{n_k}}\right|\right]}\right\}.$$

This completes the proof of the theorem.

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