ANNALES
POLONICI MATHEMATICI
XLVIII (1988)

The logarithmic capacity in C^n

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Abstract. The logarithmic capacity defined by means of the Siciak extremal function is proved to be a Choquet capacity in C^n. A sharper version of B. A. Taylor's inequality concerning the quantitative relation between the logarithmic capacity and some other capacity in C^n is given.

1. Introduction. If L denotes the class of all plurisubharmonic functions $u(z)$ on C^n which satisfy

$$u(z) \leqslant N + \log^+ |z|, \quad z \in C^n,$$

then for E a bounded subset of C^n the real function

$$u_E(z) := \{\sup u(z)\colon u \in L \text{ and } u \leqslant 0 \text{ on } E\}$$

is called the *L-extremal* (*extremal*) *function corresponding to E*. Let

$$u_E^*(z_0) = \limsup_{z \to z_0} u_E(z)$$

be the upper semicontinuous regularization of u_E. This function and the corresponding capacities have been studied by Siciak ([4], [5]) and others (e.g. [2], [3], [6]). The number

$$c(E) := \exp\left[-\limsup_{|z| \to \infty} (u_E^*(z) - \log^+ |z|)\right]$$

is called the *L-capacity* of E. For E a compact subset of the complex plane $c(E)$ is its logarithmic capacity. Therefore c is also called the *logarithmic capacity*. In this paper we show that c is a *Choquet capacity*, i.e., it satisfies the following axioms:

$$c(E) \leqslant c(F) \quad \text{if } E \subset F, \tag{1}$$

$$c(K_j) \downarrow c(K) \quad \text{if } K_j \downarrow K \text{ as } j \to \infty, \tag{2}$$

$$c(E_j) \uparrow c(E) \quad \text{if } E_j \uparrow E \text{ as } j \to \infty, \tag{3}$$

where K, K_j are compact and E, F, E_j are arbitrary subsets of C^n. It is well known (see e.g. [5]) that c satisfies (1) and (3). We prove that (2) is also satisfied, thus giving a positive answer to the question set in [5]. Next, we will slightly improve the estimate due to Taylor [6], concerning the relation between c and the capacity α introduced in [5]. Given a subset E of C^n, α is defined by the formula

$$\alpha(E) = (\exp \|u_E^*\|_{\bar{B}_1})^{-1}, \quad \text{where } B_1 = \{z\colon |z| < 1\}.$$

As has been pointed out by Siciak, Taylor's result can be stated as follows:

THEOREM 1.1. *There exists a constant M such that for every subset K of the unit ball in C^n*

$$\alpha(K) \leqslant c(K) \leqslant M\alpha(K)^{1/n}.$$

We show that $1/n$ cannot be replaced by any number greater than $\frac{1}{2}$ if $n \geqslant 2$. Thus in the case $n = 2$ we obtain the sharp value of this exponent.

This paper was written as a Master Thesis at the Jagiellonian University during the academic year 1984/85 under the direction of Professor J. Siciak. I would like to thank him for his advice and help.

2. Preliminaries. The proof of the main theorem of this paper strongly depends on the results obtained by Bedford and Taylor (B–T) in [1], [2] and Taylor [6]. For the convenience of the reader the relevant theorems are included here.

THEOREM 2.1 (B–T, [1]). *Let Ω be an open subset of C^n. If $u_1, \ldots, u_k \in L^\infty(\Omega) \cap \mathrm{PSH}(\Omega)$, $k = 1, 2, \ldots, n$, then one can define inductively the positive current $dd^c u_1 \wedge \ldots \wedge dd^c u_k$ setting for a test form φ of bidegree $(n-k, n-k)$*

$$\int_\Omega dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge \varphi = \int_\Omega u_1\, dd^c u_2 \wedge \ldots \wedge dd^c u_k \wedge dd^c \varphi,$$

where $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$.

THEOREM 2.2 (Convergence Theorem B–T, [2]). *For $0 \leqslant i \leqslant k \leqslant n$, let $\{u_j^i\} \subset \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be a sequence converging almost everywhere on Ω to a function $u^i \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. If all but one of the sequences $\{u_j^0\}, \ldots, \{u_j^k\}$ are monotone, either increasing or decreasing, then*

$$\lim_{j\to\infty} \int_\Omega \varphi\, dd^c u_j^1 \wedge \ldots \wedge dd^c u_j^k = \int_\Omega \varphi\, dd^c u^1 \wedge \ldots \wedge dd^c u^k,$$

$$\lim_{j\to\infty} \int_\Omega \varphi u_j^0\, dd^c u_j^1 \wedge \ldots \wedge dd^c u_j^k = \int_\Omega \varphi u^0\, dd^c u^1 \wedge \ldots \wedge dd^c u^k,$$

$$\lim_{j\to\infty} \int_\Omega \varphi\, du_j^0 \wedge d^c u_j^1 \wedge dd^c u_j^2 \wedge \ldots \wedge dd^c u_j^k = \int_\Omega \varphi\, du^0 \wedge d^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^k$$

for arbitrary test form φ of bidegree $(n-k,\ n-k)$.

THEOREM 2.3 (Comparison Theorem B–T, [2]). *Let Ω be a bounded open subset of C^n. If $u,\ v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ satisfy*

$$\liminf_{z \to \partial\Omega} (u(z) - v(z)) \geqslant 0,$$

then

$$\int_{\{u<v\}} (dd^c v)^n \leqslant \int_{\{u<v\}} (dd^c u)^n.$$

LEMMA 2.4 (B–T, [2], see also [3]). *If E is a compact subset of C^n and u_E is its extremal function, then*

$$\int_E u_E^* (dd^c u_E^*)^n = 0.$$

THEOREM 2.5 (B–T, [2]). *If u_E is the extremal function of a compact set E, then*

$$(dd^c u_E^*)^n = 0 \quad \text{on } C^n \setminus E.$$

DEFINITION (see [4]). $L_+ = \{u \in L\colon u(z) \geqslant M + \log^+|z|$ for some real number M, and every $z \in C^n\}$.

LEMMA 2.6 (Taylor [6]). *Let $u,\ v \in \mathrm{PSH}(C^n) \cap L^\infty_{\mathrm{loc}}(C^n)$ and $u(z),\ v(z) \to +\infty$ as $|z| \to \infty$. If $u(z) = v(z) + o(v(z))$, then*

$$\int_{C^n} (dd^c u)^k \wedge (dd^c v)^{n-k} = \int_{C^n} (dd^c v)^n$$

for $0 \leqslant k \leqslant n$. If, moreover, $u,\ v \in L_+$, then the number

$$\int_{C^n} (dd^c u)^k \wedge (dd^c v)^{n-k} = \int_{C^n} (dd^c \log^+|z|)^n =: c_n$$

depends only on the dimension of the space.

THEOREM 2.7 (Taylor [6]). *If $u,\ v \in \mathrm{PSH}(C^n) \cap L^\infty_{\mathrm{loc}}(C^n)$ satisfy the following conditions:*

$$v(z) \leqslant u(z), \quad z \in C^n, \tag{1}$$

$$\limsup_{|z| \to \infty} (u(z) - v(z)) = \gamma < +\infty, \quad \liminf_{|z| \to \infty} v(z) = +\infty, \tag{2}$$

$$\mathrm{supp}(dd^c u)^n \quad \text{is compact}, \tag{3}$$

then

$$\int_{C^n} u(z)(dd^c v)^n \leqslant \int_{C^n} v(z)(dd^c u)^n + \gamma n c_n.$$

We also recall some well-known facts.

LEMMA 2.8. *If $u, v \in C^1$ (C^n) and Φ is a smooth current of bidegree $(n-1, n-1)$, then*

$$du \wedge d^c v \wedge \Phi = dv \wedge d^c u \wedge \Phi.$$

Set $\omega(x) = a_n \exp(-1/(1-|x|^2))$ for $|x| \leqslant 1$ and $\omega(x) = 0$ for $|x| \geqslant 1$, where a_n is chosen so that

$$\int_{C^n} \omega(x)\,dx = 1.$$

For $\varepsilon > 0$ put

$$\omega_\varepsilon(x) = \varepsilon^{-2n}\omega(x/\varepsilon).$$

LEMMA 2.9 (see [4]). *If $u \in L$ and $\varepsilon > 0$, then the function $u_\varepsilon = u * \omega_\varepsilon$ given by*

$$u * \omega_\varepsilon(x) = \int_{C^n} u(x+y)\,\omega_\varepsilon(y)\,dy, \qquad x \in C^n,$$

belongs to $C^\infty \cap L$ and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.

For an arbitrary sequence $\varepsilon_j \downarrow 0$ the decreasing sequence of functions $u_j = u_{\varepsilon_j}$ is called a *standard regularization* of u.

LEMMA 2.10. *For E a compact subset of C^n let*

$$E^{(\varepsilon)} = \{x \in C^n \colon \inf_{y \in E} |x-y| \leqslant \varepsilon\}.$$

Then the set $E^{(\varepsilon)}$ is L-regular, i.e., the extremal function $u_{E^{(\varepsilon)}}$ is continuous.

LEMMA 2.11. *Let E be an L-regular compact subset of C^n and $u = u_E$ its extremal function. If $\{u_j\}$ is a standard regularization of u, then $u_j \to u$ uniformly on C^n as $j \to \infty$.*

Proof. The Dini Theorem implies that $u_j \to 0$ uniformly on E. This means that for arbitrary $\delta > 0$ and j large enough we have $u_j - \delta \leqslant u$ on E. The definition of the extremal function extends this inequality to C^n.

The following theorem will be used in the proof of the second result of this paper.

THEOREM 2.12 (M. Klimek [3]). *Let $f = (f_1, \ldots, f_n)\colon C^n \to C^n$ be a polynomial map and let $\hat{f}_j$ be the principal part of the polynomial f_j for $j = 1, 2, \ldots, n$. If $\hat{f}^{-1}(0) = \{0\}$, then the polynomial polyhedron*

$$E = \{z \in C^n \colon |f_j(z)| \leqslant 1,\ j = 1, 2, \ldots, n\}$$

is compact and its extremal function is given by

$$u_E(z) = \max\{0, (1/\deg f_1)\log|f_1(z)|, \ldots, (1/\deg f_n)\log|f_n(z)|\}.$$

3. c is a Choquet capacity. For E a subset of C^n we put

$$c(E) = \exp(-\gamma(E)),$$

where

$$\gamma(E) = \limsup_{|z| \to \infty} (u_E^*(z) - \log^+ |z|)$$

and u_E is the extremal function of E. The number $c(E)$ is called the *L-capacity* or *logarithmic capacity* of E and $\gamma(E)$ the *Robin constant* of E. We wish to show that c is a Choquet capacity. By the remarks in the introduction, it suffices to prove

THEOREM 3.1. *Let E_k, $k = 1, 2, \ldots$, be a decreasing* (with respect to inclusion) *sequence of compact subsets of C^n. Put $E = \bigcap_{k=1}^{\infty} E_k$. Then*

$$\lim_{k \to \infty} c(E_k) = c(E).$$

Proof. Without loss of generality we may assume that every E_k is a subset of the unit ball and

$$E_k = E^{(1/k)} = \{x \in C\colon \inf_{y \in E} |x-y| \leqslant 1/k\}, \quad k = 1, 2, \ldots$$

Then the extremal functions u_{E_k} are continuous (see 2.10). Furthermore, we may consider only the case $c(E) > 0$, because otherwise our conclusion is a direct consequence of Theorem 1.1 and the Choquet properties of α.

First we prove two lemmas.

LEMMA 3.2. *Let $u \in L_+ \cap C^\infty$ and $\varphi \in C_0^\infty$. Put*

$$u^s = \max\{0, u-s\}, \quad 0 < s \leqslant r.$$

If the set $\Omega = \{u < r\}$ has smooth boundary, then

$$\int_{\partial\Omega} \varphi d^c u \wedge (dd^c u)^{n-1} = \int_{\Omega} \varphi (dd^c u^r)^n.$$

Proof. Let $0 < s < r$. Applying the Stokes Theorem and Lemma 2.8 we get

$$\begin{aligned}(1) \quad \int_{\partial\Omega} \varphi d^c u \wedge (dd^c u)^{n-1} &= \int_{\partial\Omega} \varphi d^c u^s \wedge (dd^c u^s)^{n-1} \\ &= \int_{\Omega} \varphi (dd^c u^s)^n + \int_{\Omega} du^s \wedge d^c \varphi \wedge (dd^c u^s)^{n-1} \\ &= \int_{\Omega} \varphi (dd^c u^s)^n + \int_{\partial\Omega} u^s d^c \varphi \wedge (dd^c u^s)^{n-1} - \\ &\quad - \int_{\Omega} u^s dd^c \varphi \wedge (dd^c u^s)^{n-1}.\end{aligned}$$

(In fact u^s is smooth only in some neighbourhood of $\partial\Omega$ but all equalities above hold true because of Theorem 2.2.) Letting $s\uparrow r$ we see that the last two integrals on the right-hand side vanish. For the first one we have

$$(2)\qquad \int_{\Omega}\varphi(dd^c u^s)^n = \int_{\bar\Omega}\varphi(dd^c u^s)^n = \int_{C^n}\varphi(dd^c u^s)^n - \int_{C^n\setminus\bar\Omega}\varphi(dd^c u^s)^n$$
$$\to \int_{C^n}\varphi(dd^c u^r)^n - \int_{C^n\setminus\bar\Omega}\varphi(dd^c u^r)^n = \int_{\bar\Omega}\varphi(dd^c u^r)^n.$$

(Here once again we use Theorem 2.2.) Comparison of (1) and (2) completes the proof.

LEMMA 3.3. *Let* $u\in L$, $\gamma := \limsup_{|z|\to\infty}(u(z)-\log|z|)$ *and*

$$F := \{z\colon\ u(z)-\log^+|z| \geqslant \gamma-\varepsilon\},\qquad \textit{where } \varepsilon>0.$$

If λ *is the Lebesgue measure in* C^n *and* σ *the unitarily invariant surface measure on a sphere, then*:

(1) *There exists a positive number* p *such that for every* $R>2$

$$\lambda(F\cap B_R) > p\lambda(B_R),$$

where B_R *is the ball with center at* 0 *and radius* R.

(2) *There exists a positive number* q *such that for every real* R_0 *there exists* $R>R_0$ *such that*

$$\sigma(F\cap S_R) > q\sigma(S_R),\qquad \textit{where } S_R = \partial B_R.$$

Proof. Let $R>0$ and let t, ϱ, r be such that

$$0<t<\min\{1, \exp\tfrac{1}{2}\varepsilon-1\},\qquad \varrho = R/(1+t),\qquad r = t\varrho.$$

If m is defined by

$$m(x) = \max_{|z|=x} u(z),\qquad x>0,$$

then

$$u(z)-m(x)\leqslant 0\qquad \text{for } z\in B_x.$$

Since $\log^+(|z|/x)$ is the extremal function of B_x, we have

$$u(z)-m(x)\leqslant \log(y/x)\qquad \text{on } B_y.$$

Hence

$$(3)\qquad m(y)-\log y\leqslant m(x)-\log x,\qquad 0<x<y.$$

If we set $x=\varrho$ ($\varrho>1$), then letting $y\to+\infty$ we see from (3) that there exists

$z_0 \in S_\varrho$ such that

$$u(z_0) = \max_{|z|=\varrho} u(z) \geqslant \gamma + \log \varrho .$$

Let B denote the ball with center at z_0 and radius r. From (3) we derive

$$u(z) - \log(\varrho + r) \leqslant u(z_0) - \log \varrho \quad \text{if} \quad z \in B.$$

Hence

$$(4) \qquad \begin{aligned} u(z) &\leqslant u(z_0) + \log(\varrho + r) - \log \varrho = u(z_0) + \log(1+t) \\ &< u(z_0) + \tfrac{1}{2}\varepsilon \quad \text{for } z \in B. \end{aligned}$$

If $z \in B \setminus F$, we have

$$(5) \qquad \begin{aligned} u(z) &\leqslant \log|z| + \gamma - \varepsilon \leqslant \log(\varrho + r) + \gamma - \varepsilon \\ &\leqslant \log(\varrho + r) + \big(u(z_0) - \log \varrho\big) - \varepsilon < \tfrac{1}{2}\varepsilon - \varepsilon + u(z_0) \\ &= u(z_0) - \tfrac{1}{2}\varepsilon. \end{aligned}$$

Since $u \in \mathrm{PSH}(C^n)$ its value at z_0 does not exceed the mean value of u on the ball B. So (4) and (5) imply

$$u(z_0)\lambda(B) < \lambda(B \setminus F)\big(u(z_0) - \tfrac{1}{2}\varepsilon\big) + \lambda(B \cap F)\big(u(z_0) + \tfrac{1}{2}\varepsilon\big).$$

Therefore

$$\lambda(B \setminus F) < \lambda(B \cap F).$$

However, $B \subset B_R$ and $r = \big(t/(1+t)\big)R$, so

$$\lambda(B_R \cap F) \geqslant \lambda(B \cap F) > \tfrac{1}{2}\lambda(B) = \tfrac{1}{2}\big(t^{2n}/(1+t)^{2n}\big)\lambda(B_R) = p\lambda(B_R)$$

and the first part of the lemma follows.

Now, put $q = \frac{1}{2}p$ and suppose that for every $r > R_0$

$$(\tfrac{1}{2}p)\,\sigma(S_r) \geqslant \sigma(S_r \cap F).$$

Choose $R > 2$ so large that

$$(\tfrac{1}{2}p)\,\lambda(B_R) > \lambda(B_{R_0}),$$

Fubini's Theorem implies

$$\lambda\big((B_R \setminus B_{R_0}) \cap F\big) \leqslant (\tfrac{1}{2}p)\,\lambda(B_R \setminus B_{R_0}).$$

This contradicts the first conclusion of our lemma, for we have

$$\begin{aligned} \lambda(B_R \cap F) &\leqslant \lambda\big((B_R \setminus B_{R_0}) \cap F\big) + \lambda(B_{R_0}) \\ &\leqslant (\tfrac{1}{2}p)\,\lambda(B_R \setminus B_{R_0}) + (\tfrac{1}{2}p)\,\lambda(B_R) < p\lambda(B_R). \end{aligned}$$

This completes the proof of the lemma.

Put $u = u_E$ and $u_k = u_{E_k}$, where u_E, u_{E_k} are the extremal functions of E and E_k respectively. Let $\{u_k^j\}$, $j = 1, 2, \ldots$, be a standard regularization of u_k.

LEMMA 3.4. $\forall \delta > 0 \exists k_0 \forall k > k_0 \exists j_k \forall j > j_k$: $\int_{C^n} u(dd^c u_k^j)^n < \delta$.

Proof. From Lemma 2.11 we have $u_k^j \to u_k$ uniformly on C^n as $j \to \infty$. So for every $\zeta > 0$ there exists j_0 such that

$$u_k + \eta \geq u_k^j \geq u_k \quad \text{for } j > j_0.$$

Theorem 2.7 now implies

$$\int_{C^n} u_k (dd^c u_k^j)^n \leq \int_{C^n} (u_k + \zeta)(dd^c u_k^j)^n \leq \int_{C^n} u_k^j (dd^c u_k)^n + \zeta n c_n, \tag{6}$$

where

$$c_n = \int_{C^n} (dd^c u_k^j)^n = \int_{C^n} (dd^c u_k)^n$$

is a constant depending only on the dimension of the space. From Theorem 2.5 we know that

$$\operatorname{supp}(dd^c u_k)^n \subset E_k.$$

So if we apply Theorem 2.2, then

$$\lim_{j \to \infty} \int_{C^n} u_k^j (dd^c u_k)^n = \int_{C^n} u_k (dd^c u_k)^n = 0. \tag{7}$$

Combining (6) and (7) gives

$$\lim_{j \to \infty} \int_{C^n} u_k (dd^c u_k^j)^n = 0. \tag{8}$$

Now given $\varphi \in C_0^\infty$ such that $\varphi \geq 0$ and $\varphi = 1$ on $\bar{B}_1$, we may apply Theorems 2.2 and 2.5 to get

$$\begin{aligned} \limsup_{j \to \infty} \int_{B_1} (u - u_k)(dd^c u_k^j)^n &\leq \lim_{j \to \infty} \int_{C^n} \varphi(u - u_k)(dd^c u_k^j)^n \\ &= \int_{C^n} \varphi(u - u_k)(dd^c u_k)^n = \int_{C^n} u(dd^c u_k)^n. \end{aligned} \tag{9}$$

If $\psi \in C_0^\infty$, $0 \leq \psi \leq 1$, $\psi = 1$ on E_k and $\operatorname{supp}\psi \subset B_1$, then the Convergence Theorem 2.2 with Lemma 2.6 implies

$$\begin{aligned} c_n = \int_{B_1} (dd^c u_k)^n = \int_{B_1} \psi(dd^c u_k)^n = \lim_{j \to \infty} \int_{B_1} \psi(dd^c u_k^j)^n \\ \leq \lim_{j \to \infty} \int_{B_1} (dd^c u_k^j)^n \leq c_n. \end{aligned}$$

Hence

$$\lim_{j\to\infty}\int_{B_1}(dd^c u_k^j)^n = c_n, \tag{10}$$

$$\lim_{j\to\infty}\int_{C^n\setminus B_1}(dd^c u_k^j)^n = 0. \tag{11}$$

Let φ be chosen as in (9). Then Theorem 2.2 implies

$$\lim_{k\to\infty}\int_{B_1} u(dd^c u_k)^n \leqslant \lim_{k\to\infty}\int_{C^n}\varphi u(dd^c u_k)^n = \int_{C^n}\varphi u(dd^c u)^n = \int_{C^n} u(dd^c u)^n = 0, \tag{12}$$

where the last equality follows from Lemma 2.4. So for arbitrary $\delta > 0$ one can find $k_0 \in N$ such that

$$\int_{B_1} u(dd^c u_k)^n < \tfrac{1}{4}\delta \quad \text{as } k > k_0. \tag{13}$$

Combining (8), (9) and (11) we may choose for every $k > k_0$ a number j_k such that

$$\begin{gathered}\int_{C^n} u_k(dd^c u_k^j)^n < \tfrac{1}{4}\delta,\\ \int_{B_1}(u-u_k)(dd^c u_k^j)^n < \int_{B_1} u(dd^c u_k)^n + \tfrac{1}{4}\delta < \tfrac{1}{2}\delta,\\ \int_{C^n\setminus B_1}\gamma_0(dd^c u_k^j)^n < \tfrac{1}{4}\delta,\end{gathered} \tag{14}$$

where $\gamma_0 := \{\sup(u-u_k)(z)\colon z\in C^n,\ k\in N\} < +\infty$ and $j > j_k$. Now, from (13), (14) we get the desired result:

$$\begin{aligned}\int_{C^n} u(dd^c u_k^j)^n &= \int_{C^n} u_k(dd^c u_k^j)^n + \int_{C^n}(u-u_k)(dd^c u_k^j)^n\\ &= \int_{C^n} u_k(dd^c u_k^j)^n + \int_{B_1}(u-u_k)(dd^c u_k^j)^n + \int_{C^n\setminus B_1}(u-u_k)(dd^c u_k^j)^n\\ &< \tfrac{3}{4}\delta + \int_{C^n\setminus B_1}\gamma_0(dd^c u_k^j)^n < \delta,\end{aligned}$$

when $k > k_0$ and $j > j_k$.

To prove Theorem 3.1 it is enough to show that the hypothesis that

$$\limsup_{|z|\to\infty}\big(u_k(z)-\log^+|z|\big) < \gamma_1 < \gamma = \limsup_{|z|\to\infty}\big(u(z)-\log^+|z|\big) \tag{15}$$

for $k = 1, 2, \ldots$ leads to a contradiction.

Fix $\varepsilon > 0$ and $\alpha > 0$ such that

$$4\varepsilon < \gamma-\gamma_1, \quad \alpha(\gamma_1+\varepsilon) < \varepsilon \quad \text{and} \quad (1-\alpha)(\gamma-\gamma_1-2\varepsilon) > 2\varepsilon. \tag{16}$$

Choose a sequence of numbers $R(k) > 1$, $k = 1, 2, \ldots$ so that

(i) $u_k(z) < \log^+|z| + \gamma_1$ as $|z| \geqslant R(k)$,

(ii) the inequality from the second part of Lemma 3.3 holds for $R = R(k)$ and every k with a constant q depending only on u and ε.

Further, we need the following

LEMMA 3.5. *For every $k > 1$ there exists $\hat{u}_k \in C^\infty \cap L_+$ such that*

(i') $\hat{u}_k \leqslant u_k$ *in* C^n,

(ii') $\int_{B_1} (dd^c \hat{u}_k)^n \geqslant c_n - \min\{\alpha^n q c_n/2^{n+1}, [k \log(R(k)+\gamma_1+\varepsilon)]^{-1}\} =: P$,

(iii') $\lim_{k\to\infty} \int_{C^n} u(dd^c \hat{u}_k)^n = 0$.

Proof. Let $k \geqslant 2$ be a positive integer. Since

$$E_k \subset \operatorname{int} E_{k-1}$$

for j large enough we have

$$u_{k-1}^j = 0 \quad \text{on } E_k,$$

where $\{u_{k-1}^j\}_{j\geqslant 1}$ is a standard regularization of u_{k-1}. The definition of the extremal function u_k implies

$$u_{k-1}^j \leqslant u_k \quad \text{for } j \geqslant j_k^0.$$

From (10) we know that

$$\int_{B_1} (dd^c u_{k-1}^j)^n \geqslant P \quad \text{if} \quad j \geqslant j_k^1.$$

Let $\{\delta_s\}$, $s = 1, 2, \ldots$, be a sequence of positive numbers decreasing to 0. Lemma 3.4 implies that for every s there exists k_s such that

(17) $$\forall k \geqslant k_s \exists j_k^s \forall j \geqslant j_k^s: \int_{C^n} u(dd^c u_{k-1}^j)^n < \delta_s.$$

We can assume that the sequence k_s is increasing to infinity. Put $j_k = \max\{j_k^0, j_k^1\}$ for $2 \leqslant k < k_1$. For every $k \geqslant k_1$ find s such that $k \in [k_s, k_{s+1})$ and put

$$j_k = \max\{j_k^0, j_k^1, j_k^s\}.$$

If now δ is an arbitrary positive number and $\delta_s < \delta$, then for every $k > k_s$, (17) gives

$$\int_{C^n} u(dd^c u_{k-1}^{j_k})^n \leqslant \delta_{s+r} \leqslant \delta_s < \delta, \quad \text{where } k \in [k_{s+r}, k_{s+r+1}).$$

So the choice of $\hat{u}_k$, $k = 1, 2, \ldots$, is accomplished.

For $k \geqslant 2$ we choose positive numbers $r(k)$ such that

(18) $$\log R(k) + \gamma_1 < r(k) < \log R(k) + \gamma_1 + \varepsilon$$

and the boundary of the set $\{z\colon \hat{u}_k(z) < r(k)\}$ is smooth (here we use Sard's Theorem).

We are now ready to prove the following

LEMMA 3.6. *Let* $\Omega_k = \{\hat{u}_k < r(k)\}$. *Then*

$$\lim_{k\to\infty} \int_{\partial\Omega_k} (u-\hat{u}_k)\, d^c\hat{u}_k \wedge (dd^c\hat{u}_k)^{n-1} = 0.$$

Proof. It is clear that $d^c\hat{u}_k \wedge (dd^c\hat{u}_k)^{n-1} \geqslant 0$ on $\partial\Omega_k$ (see e.g. Lemma 3.2 and Theorem 2.1) Combining (ii′) and (18) we get

$$\int_{\Omega_k} (dd^c\hat{u}_k)^n \geqslant c_n - 1/kr(k).$$

Hence Lemma 2.6 gives

$$r(k)\Big(\int_{\Omega_k} dd^c u \wedge (dd^c\hat{u}_k)^{n-1} - \int_{\Omega_k} (dd^c\hat{u}_k)^n\Big) \leqslant 1/k.$$

Repeating the argument from the proof of Lemma 3.2 we have

$$\begin{aligned}
(19)\qquad & \int_{\partial\Omega_k} (u-\hat{u}_k)\, d^c\hat{u}_k \wedge (dd^c\hat{u}_k)^{n-1} \\
&= \int_{\Omega_k} (u-\hat{u}_k)(dd^c\hat{u}_k)^n + \int_{\Omega_k} d\hat{u}_k \wedge d^c(u-\hat{u}_k) \wedge (dd^c\hat{u}_k)^{n-1} \\
&= \int_{\Omega_k} (u-\hat{u}_k)(dd^c\hat{u}_k)^n + \int_{\partial\Omega_k} \hat{u}_k\, d^c(u-\hat{u}_k) \wedge (dd^c\hat{u}_k)^{n-1} - \\
&\qquad - \int_{\Omega_k} \hat{u}_k\, dd^c(u-\hat{u}_k) \wedge (dd^c\hat{u}_k)^{n-1} \\
&= \int_{\Omega_k} u(dd^c\hat{u}_k)^n - \int_{\Omega_k} \hat{u}_k\, dd^c u \wedge (dd^c\hat{u}_k)^{n-1} + \\
&\qquad + r(k)\Big(\int_{\Omega_k} dd^c u \wedge (dd^c\hat{u}_k)^{n-1} - \int_{\Omega_k} (dd^c\hat{u}_k)^n\Big) \\
&\leqslant \int_{\Omega_k} u(dd^c\hat{u}_k)^n + 1/k.
\end{aligned}$$

From (iii′) we know that the right-hand side of (19) tends to 0 as $k\to\infty$, and the lemma follows.

We proceed to the last step of the proof of Theorem 3.1. Let

$$v_k(z) = \log^+ |z|/R(k), \qquad \tilde{u}_k(z) = \max\{0,\ \hat{u}_k(z)-r(k)\},$$
$$u^k(z) = \max\{0,\ u(z)-r(k)\}.$$

Apply the Comparison Theorem 2.3 to the functions

$$h_k = 2\tilde{u}_k + 2\varepsilon \quad\text{ and }\quad g_k = (1-\alpha)u^k + \alpha v_k$$

and to a ball with radius so large that $h_k \geq g_k$ on its boundary (observe that $\lim_{|z|\to\infty} h_k(z)/g_k(z) = 2$). Theorem 2.3 gives

$$(20) \qquad \int_{\{h_k < g_k\}} (dd^c g_k)^n \leq \int_{\{h_k < g_k\}} (dd^c h_k)^n.$$

Fix $z \in \partial B_{R(k)} \cap \{x: u(x) - \log^+ |x| > \gamma - \varepsilon\}$. From (18) we get

$$u(z) \geq \log R(k) + \gamma - \varepsilon > r(k) + \gamma - \gamma_1 - 2\varepsilon, \qquad u^k(z) > \gamma - \gamma_1 - 2\varepsilon.$$

Now, apply (16) to obtain

$$(21) \qquad (1-\alpha) u^k(z) > (1-\alpha)(\gamma - \gamma_1 - 2\varepsilon) > 2\varepsilon.$$

Since $|z| = R(k)$, from (i), (i′) and (18) we conclude that

$$\hat{u}_k(z) \leq u_k(z) \leq \log^+ |z| + \gamma_1 = \log R(k) + \gamma_1 < r(k).$$

Hence $z \in \Omega_k = \{\hat{u}_k < r(k)\}$ and (21) leads to

$$h_k(z) = 2\varepsilon < (1-\alpha) u^k(z) = g_k(z).$$

Consequently we have proved that

$$(22) \qquad F_k = \partial B_{R(k)} \cap \{z: u(z) - \log^+ |z| > \gamma - \varepsilon\} \subset \{h_k < g_k\}.$$

Theorem 2.1 gives

$$(23) \qquad (dd^c g_k)^n = \sum_{j=0}^{n} \binom{n}{j} \alpha^j (dd^c v_k)^j \wedge (1-\alpha)^{n-j} (dd^c u^k)^{n-j} \geq \alpha^n (dd^c v_k)^n.$$

Combining (20), (22) and (23) we get

$$(24) \qquad \alpha^n \int_{F_k} (dd^c v_k)^n \leq \int_{F_k} (dd^c g_k)^n \leq \int_{\{h_k < g_k\}} (dd^c g_k)^n \leq \int_{\{h_k < g_k\}} (dd^c h_k)^n.$$

But $(dd^c v_k)^n$ is the unitarily invariant surface measure on the sphere $S_{R(k)}$ and

$$\int_{S_{R(k)}} (dd^c v_k)^n = c_n.$$

Because of (ii) we may apply Lemma 3.3 to the left-hand side of (24) to obtain

$$(25) \qquad \alpha^n q c_n < \int_{\{h_k < g_k\}} (dd^c h_k)^n = 2^n \int_{\{h_k < g_k\}} (dd^c \tilde{u}_k)^n.$$

From (ii′) and Lemma 2.6 we derive

$$(26) \qquad \int_{C^n \setminus \Omega_k} (dd^c \tilde{u}_k)^n \leq \int_{C^n \setminus B_1} (dd^c \hat{u}_k)^n \leq c_n - (c_n - \alpha^n q c_n / 2^{n+1}) = \alpha^n q c_n / 2^{n+1}.$$

(We have proved that $B_{R(k)} \subset \Omega_k$, so $B_1 \subset \Omega_k$.) Since $\tilde{u}_k = 0$ on Ω_k we get

$$\int_{\partial\Omega_k \cap \{h_k < g_k\}} (dd^c \tilde{u}_k)^n = \int_{\{h_k < g_k\}} (dd^c \tilde{u}_k)^n - \int_{(C^n \setminus \Omega_k) \cap \{h_k < g_k\}} (dd^c \tilde{u}_k)^n,$$

and applying (25), (26) to the right-hand side yields

$$\int_{\partial\Omega_k \cap \{h_k < g_k\}} (dd^c \tilde{u}_k)^n > \alpha^n q c_n/2^{n+1}. \tag{27}$$

Now, we claim that

$$\alpha v_k(z) < \varepsilon \quad \text{for } z \in \Omega_k. \tag{28}$$

Indeed, take $z \in \Omega_k$ such that $v_k(z) \neq 0$. Then

$$\log|z| \leqslant \hat{u}_k(z) < r(k)$$

and (16), (18) implies

$$\alpha v_k(z) = \alpha\big(\log|z| - \log R(k)\big) \leqslant \alpha\big(r(k) - \log R(k)\big) \leqslant \alpha(\gamma_1 + \varepsilon) < \varepsilon.$$

From (28) we conclude that for $z \in \Omega_k \cap \{h_k < g_k\}$

$$2\varepsilon = h_k(z) < (1-\alpha)u^k(z) + \alpha v_k(z) < (1-\alpha)u^k(z) + \varepsilon,$$

and

$$u^k(z) > (1-\alpha)u^k(z) > \varepsilon \quad \text{for } z \in \partial\Omega_k \cap \{h_k < g_k\}.$$

So, we have proved

$$\partial\Omega_k \cap \{h_k < g_k\} \subset \partial\Omega_k \cap \{u - \hat{u}_k > \varepsilon\}.$$

Combining this inclusion with (27) gives

$$\varepsilon\alpha^n q c_n/2^{n+1} < \varepsilon \int_{\partial\Omega_k \cap \{h_k < g_k\}} (dd^c \tilde{u}_k)^n \leqslant \int_{\partial\Omega_k \cap \{h_k < g_k\}} (u - \hat{u}_k)(dd^c \tilde{u}_k)^n.$$

To the right-hand side we may apply Lemma 3.2 to obtain

$$\varepsilon\alpha^n q c_n/2^{n+1} < \int_{\partial\Omega_k} (u - \hat{u}_k) d^c \hat{u}_k \wedge (dd^c \hat{u}_k)^{n-1}.$$

This contradicts the conclusion of Lemma 3.6, so the hypothesis (15) proved false and the theorem follows.

4. Comparison of the capacities c and α. It has recently been shown by Taylor [6] that for every compact subset K of the unit ball in C^n

$$\alpha(K) \leqslant c(K) \leqslant M\alpha(K)^\delta \tag{29}$$

for some positive constants M and δ independent of K with α defined in the introduction. Combining Theorem 6.1 from [5] with Theorem 2 from [6] (Theorem 2.7 in this paper) J. Siciak has observed that (29) holds true for every $\delta \leqslant 1/n$. The following theorem yields an upper limit for δ.

THEOREM 4.1. *Given $b > \frac{1}{2}$ assume A, B, r are positive real numbers and m is a positive integer such that*

$$\big((2m-1)/m\big)b > 1, \qquad Ar^m - 3 > 0, \qquad B = (Ar^m - 3)A^{(m-1)/m}.$$

Put

$$\begin{aligned} p(x, y) &= Ax^m + Bx^{m-1} + Ay^m \\ q(x, y) &= A^{1/m} x \end{aligned} \qquad \textit{for } (x, y) \in C^2$$

and

$$E = \{(x, y) \in C^2 := |p(x, y)| \leqslant 1, \ |q(x, y)| \leqslant 1\}.$$

Then

$$E \subset \{|x| \leqslant r, \ |y| \leqslant r\} \quad \textit{and} \quad \lim_{A \to \infty} c(E)/\alpha(E)^b = +\infty.$$

Proof. For $(x, y) \in E$ we have

$$|x| \leqslant A^{-1/m}.$$

Hence

$$1 \geqslant |p(x, y)| \geqslant A|y|^m - 1 - (Ar^m - 3) = A(|y|^m - r^m) + 2.$$

So we conclude that $(x, y) \in \{|x| \leqslant r, \ |y| \leqslant r\}$.

It follows from Theorem 2.12 that

$$u_E = \max\{(1/m)\log|p|, \log|q|\}.$$

Therefore

$$\gamma(E) = \limsup_{|z| \to \infty} (u_E(z) - \log^+|z|) \leqslant (1/m)\log 2A,$$

and the L-capacity of E is not less than $(2A)^{-1/m}$. We also have

$$\alpha(E)^{-1} \geqslant |p(1, 0)|^{1/m} \geqslant B^{1/m}.$$

It is clear that

$$\begin{aligned} \left(c(E)/\alpha(E)^b\right)^m &\geqslant B^b (2A)^{-1} = \left((Ar^m - 3) A^{(m-1)/m}\right)^b (2A)^{-1} \\ &= \tfrac{1}{2}(r^m - 3/A)^b A^{((2m-1)/m)b - 1}. \end{aligned}$$

Hence

$$\lim_{A \to \infty} c(E)/\alpha(E)^b = +\infty,$$

which gives the desired result.

COROLLARY 4.2. *For* $\delta > \frac{1}{2}$, (29) *is not true. Thus in the case* $n = 2$ *we obtain the sharp value of the exponent* δ (*equal* to $\frac{1}{2}$) *in this estimate.*

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Reçu par la Rédaction le 1985.11.20