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# Function classes pertaining to differential inequalities of parabolic type in unbounded regions

by P. Besala (Gdańsk)

**1. Introduction.** We treat the problem whether some theorems on differential inequalities of parabolic type hold true in function classes in which the uniqueness of solutions of the Cauchy problem for suitable parabolic equations has been proved. The theorems on differential inequalities are more general and imply the uniqueness, the maximum principle, a continuous dependence of solutions on the initial data etc. (see [10]). For the sake of simplicity, in this introduction we confine ourselves to the heat conduction equation with one space variable

$$u_t = u_{xx} \tag{1.1}$$

in the strip $S = (0, T\rangle \times (-\infty, +\infty)$. Tihonov [11] proved that the solution of the Cauchy problem for equation (1.1) is unique in the class of such functions $u(t, x)$ that for each of them there are positive constants $M$, $K$ such that

$$|u(t, x)| \leqslant M\exp(K|x|^2) \quad \text{in } \bar{S}. \tag{1.2}$$

Denote this class by $E_2$. (This result has been extended to more general equations and systems of parabolic type.) Next it was shown that the following theorem on differential inequalities holds: *if $u(t, x)$, $v(t, x) \in E_2$ in $\bar{S}$, then the differential inequalities*

$$u_t \leqslant u_{xx} \tag{1.3}$$

$$v_t \geqslant v_{xx} \qquad \text{for } (t, x) \in S \tag{1.4}$$

*and the initial inequality*

$$u(0, x) \leqslant v(0, x) \quad \text{for } x \in (-\infty, +\infty)$$

*imply the inequality*

$$u(t, x) \leqslant v(t, x) \quad \text{for } (t, x) \in \bar{S}.$$

(Theorems of this type for more general non-linear systems of parabolic inequalities covering (1.3), (1.4) can be found in references [10], [3], [5], [6].)

As concerns the uniqueness the class $E_2$ has been extended to a wider class, say $I_2$, of functions $u(t, x)$ which satisfy the integral growth condition

$$\iint_S |u(t, x)| \exp(-K|x|^2)\, dx\, dt < \infty, \tag{1.5}$$

where the constant $K > 0$ may depend on $u$. (In [2], [4], [7], [8] one can find corresponding theorems relating to more general equations and systems of parabolic type.) Now, in a natural way the question arises whether the differential inequalities theorem holds true for functions of class $I_2$. In section 3 (Theorems 1, 2) we prove that the answer is positive. Our result is concerned with a system of semilinear inequalities of the form

$$u_t^i \leqslant \sum_{j,k=1}^{n} \big(a_{jk}^i(t, x) u^i\big)_{x_j x_k} - \sum_{j=1}^{n} \big(b_j^i(t, x) u^i\big)_{x_j} + f^i(t, x, u^1, \ldots, u^m) \tag{1.6}$$

$(i = 1, \ldots, m)$. We assume only weak parabolicity and the coefficients are allowed to grow to infinity in various ways. The proof is patterned on that of papers [2] and [4]. The maximum principle formulated in section 4 is an immediate consequence of the above result. Theorem 4 of section 4 is a further consequence of the differential inequalities theorem and reads that any solution of a system of parabolic equations, which belongs to class $I_2$ and satisfies (1.2) at the initial moment $t = 0$ is of class $E_2$ in $\bar{S}$. (Now $x$ in (1.2) and (1.5) should be understood as a vector and $S$ a corresponding zone in the $(t, x)$-space.)

Another uniqueness class for the Cauchy problem for the heat equation and also more general linear second order parabolic equations is the class of non-negative functions in $S$ (cf. e.g. [1]). However, as we show in section 5 by constructing a counter-example, in this class the theorem on differential inequalities does not hold.

**2. Preliminary considerations and assumptions.** Denote by $x = (x_1, \ldots, x_n)$ points of the Euclidean $n$-dimensional space $E_n$ $(n \geqslant 1)$ and by $t$ points of the interval $\langle 0, T\rangle$, $T > 0$. Let $S = (0, T\rangle \times E_n$, $\bar{S} = \langle 0, T\rangle \times E_n$.

LEMMA 1. *Assume $u(t, x)$ is continuous and has finite derivatives $u_{x_j}$, $u_t$, $u_{x_j x_k}$ at each point of $S$. Let $w = u^+ = \max(0, u)$ for $(t, x) \in S$. Then the function*

$$z = (w^p + \varepsilon^p)^{1/p}, \quad \textit{where } \varepsilon > 0, p > 2,$$

*is also continuous and has finite derivatives* $z_{x_j}, z_t, z_{x_jx_k}$ *at each point of* $S$. *Moreover, the derivatives can be represented by the formulae*

$$z_{x_j} = z^{1-p}w^{p-1}u_{x_j}, \quad z_t = z^{1-p}w^{p-1}u_t, \tag{2.1}$$

$$z_{x_jx_k} = \varepsilon^p(p-1)z^{1-2p}w^{p-2}u_{x_j}u_{x_k} + z^{1-p}w^{p-1}u_{x_jx_k} \tag{2.2}$$

*at each point of* $S$.

Proof. Evidently $z$ is continuous in $S$. Define

$$D = \{(t, x) \in S\colon\ u(t, x) > 0\}.$$

At points of $D$ the function $z$ has the derivatives and they can be calculated in usual way, whence we obtain (2.1), (2.2). It is obvious that at points of the complement of $\bar{D}$ in $S$ ($\bar{D}$ being the closure of $D$) we have $z_{x_j} = z_t = z_{x_jx_k} = 0$. Thus at these points (2.1), (2.2) hold either as their right-hand sides are equal to zero since $w = 0$. Now let $(t, x)$ be a point of $S$, situated on the boundary of $D$. In order to complete the proof it is enough to show that at this point derivatives $z_{x_j}, z_t, z_{x_jx_k}$ exist and are equal to zero. To this end let

$$x^{jh} = (x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n).$$

Since

$$(w^p + \varepsilon^p)^{1/p} = \varepsilon + [(\theta w)^p + \varepsilon^p]^{(1-p)/p}(\theta w)^{p-1}w, \quad 0 < \theta < 1,$$

we have

$$\begin{aligned}
&|h^{-1}[z(t, x^{jh}) - z(t, x)]| \\
&\quad \leqslant |h^{-1}\{[\theta w(t, x^{jh})]^p + \varepsilon^p\}^{(1-p)/p}w^p(t, x^{jh})| \\
&\quad \leqslant \{[\theta w(t, x^{jh})]^p + \varepsilon^p\}^{(1-p)/p}w^{p-1}(t, x^{jh})\,|h^{-1}u(t, x^{jh})| \\
&\quad \to \varepsilon^{1-p}\cdot 0 \cdot |u_{x_j}(t, x)| = 0.
\end{aligned}$$

Thus $z_{x_j}(t, x) = 0$. Similarly one can show that $z_t(t, x) = 0$ and thereby (2.1) is proved. Taking advantage of (2.1) we get

$$\begin{aligned}
|h^{-1}[z_{x_j}(t, x^{kh}) - z_{x_j}(t, x)]| &= z^{1-p}(t, x^{kh})w^{p-1}(t, x^{kh})\,|h^{-1}u_{x_j}(t, x^{kh})| \\
&\leqslant z^{1-p}(t, x^{kh})w^{p-1}(t, x^{kh})\,|h^{-1}[u_{x_j}(t, x^{kh}) - u_{x_j}(t, x)]| + \\
&\quad + z^{1-p}(t, x^{kh})w^{p-2}(t, x^{kh})\,|u_{x_j}(t, x)|\,|h^{-1}u(t, x^{kh})| \\
&\to \varepsilon^{1-p}\cdot 0 \cdot |u_{x_jx_k}(t, x)| + \varepsilon^{1-p}\cdot 0 \cdot |u_{x_j}(t, x)|\,|u_{x_k}(t, x)| = 0.
\end{aligned}$$

Therefore $z_{x_jx_k}(t, x) = 0$ which was to be proved.

CONDITION C. A function $u(t, x)$ defined in $\bar{S}$ will be said to *satisfy condition* C in $S$ if $u$ is continuous in $\bar{S}$, has the derivatives $u_{x_j}, u_t, u_{x_jx_k}$ at each point of $S$ which are measurable functions in $S$ and locally bounded.

LEMMA 2. *If $u$ satisfies condition C in $S$, so does the function $z$ defined in Lemma* 1.

This lemma follows immediatelly from formulae (2.1), (2.2), our assumptions and suitable theorems on measurability of a product of measurable functions.

Throughout the paper we make the following assumptions concerning system (1.6):

($A_1$) the coefficients $a^i_{jk}$, $b^i_j$ and their derivatives $(a^i_{jk})_{x_j}$, $(a^i_{jk})_{x_jx_k}$, $(b^i_j)_{x_j}$ are measurable and bounded in any finite cylinder $\langle 0, T\rangle \times (|x| \leqslant R)$ and, for simplicity, $a^i_{jk} = a^i_{kj}$,

($A_2$) $\sum_{j,k=1}^{n} a^i_{jk}(t, x)\xi_j\xi_k \geqslant 0$ $(i = 1, \ldots, m)$ for any real vector $(\xi_1, \ldots, \xi_n)$ and for $(t, x) \in \bar{S}$,

($A_3$) functions $f^i(t, x, u^1, \ldots, u^m)$ $(i = 1, \ldots, m)$, being defined for $(t, x) \in \bar{S}$ and $u^1, \ldots, u^m$ arbitrary, satisfy the following monotonicity condition: for any fixed $i$ $(1 \leqslant i \leqslant m)$ the relations $u^j \leqslant v^j$ $(j = 1, \ldots, m)$, $u^i = v^i$, imply the inequality

$$f^i(t, x, u^1, \ldots, u^m) \leqslant f^i(t, x, v^1, \ldots, v^m)$$

for almost all $(t, x) \in \bar{S}$,

($A_4$) there exist functions $c^i_s(t, x)$ $(i, s = 1, \ldots, m)$ continuous in $\bar{S}$, $c^i_s \geqslant 0$ for $s \neq i$, such that the inequalities

$$[f^i(t, x, u^1, \ldots, u^m) - f^i(t, x, v^1, \ldots, v^m)]\operatorname{sgn}(u^i - v^i) \leqslant \sum_{s=1}^{m} c^i_s(t, x)|u^s - v^s|$$

$(i = 1, \ldots, m)$ hold almost everywhere in $\bar{S}$.

**3. Differential inequalities.** Define the operators

$$L^i u = \sum_{j,k=1}^{n} \big(a^i_{jk}(t, x)u\big)_{x_jx_k} - \sum_{j=1}^{n} \big(b^i_j(t, x)u\big)_{x_j},$$

$$\tilde{L}^i \Phi = \sum_{j,k=1}^{n} a^i_{jk}(t, x)\Phi_{x_jx_k} + \sum_{j=1}^{n} b^i_j(t, x)\Phi_{x_j}$$

$(i = 1, \ldots, m)$, $(t, x) \in S$. Now we prove a general theorem on differential inequalities.

THEOREM 1. *Let functions $u^i(t, x)$, $v^i(t, x)$ $(i = 1, \ldots, m)$ satisfy condition C in $S$ and the initial inequalities*

$$u^i(0, x) \leqslant v^i(0, x) \quad (i = 1, \ldots, m) \text{ for } x \in E_n. \tag{3.1}$$

*Define the sets*

$$D^l = \{(t, x) \in S\colon u^l(t, x) > v^l(t, x)\} \quad (l = 1, \ldots, m) \tag{3.2}$$

*and suppose that for every fixed $i$ the differential inequalities*

$$u_t^i(t,x) \leqslant L^i[u^i(t,x)]+f^i\big(t,x,u^1(t,x),\ldots,u^m(t,x)\big), \tag{3.3}$$

$$v_t^i(t,x) \geqslant L^i[v^i(t,x)]+f^i\big(t,x,v^1(t,x),\ldots,v^m(t,x)\big) \tag{3.4}$$

*are satisfied whenever $(t,x)\in D^i$. Let assumptions $(A_1)$-$(A_4)$ related to the coefficients of $L^i$ and functions $f^i$ be fulfilled. We assume there exist functions $\Phi^i(t,x)\in C^2(\bar{S})$ $(i=1,\ldots,m)$ such that $\Phi^i(t,x)>0$ in every compact subset of $\bar{S}$,*

$$\tilde{L}^i\Phi^i+\sum_{s=1}^{m}c_i^s\Phi^s+\Phi_t^i\leqslant 0 \qquad (i=1,\ldots,m) \tag{3.5}$$

*almost everywhere in $\bar{S}$ and, moreover,*

$$\iint_S (u^i-v^i)^+\Big[\max_j\sum_k|a_{jk}^i\Phi_{x_k}^i|+\Phi^i(\max_{j,k}|a_{jk}^i|+\max_j|b_j^i|)\Big]\,dx\,dt<\infty \tag{3.6}$$

*$(i=1,\ldots,m)$. Under these assumptions we have*

$$u^i(t,x)\leqslant v^i(t,x) \qquad (i=1,\ldots,m) \tag{3.7}$$

*everywhere in $\bar{S}$.*

Proof. We shall make use of the identity

$$\sum_i(z^i\varphi^i)_t\equiv\sum_i z^i(\tilde{L}^i\varphi^i+\varphi_t^i)-\sum_i\varphi^i(L^iz^i-z_t^i)+ \\ +\sum_i\sum_j\Big[\varphi^i\sum_k(a_{jk}^iz^i)_{x_k}-z^i\sum_k a_{jk}^i\varphi_{x_k}^i-b_j^iz^i\varphi^i\Big]_{x_j}, \tag{3.8}$$

where functions $\varphi^i(t,x)$ $(i=1,\ldots,m)$ will be determined later to be smooth and non-negative in $\bar{S}$, with compact support as functions of $x$ in $E_n$. For $z^i$ we substitute in (3.8)

$$z^i=[(w^i)^p+\varepsilon^p]^{1/p} \quad (i=1,\ldots,m),\ \varepsilon>0,\ p>2,$$

where $w^i=(u^i-v^i)^+=\max(0,u^i-v^i)$ for $(t,x)\in\bar{S}$.

By Lemma 2 and assumption $(A_1)$ we can integrate (3.8) over the strip $(0,t_0)\times E_n$, $t_0\in(0,T\rangle$, to get

$$\int_{E_n}\sum_i z^i\varphi^i\Big|_{t=t_0}dx \\ =\int_{E_n}\sum_i z^i\varphi^i\Big|_{t=0}dx+\int_0^{t_0}\int_{E_n}\sum_i[z^i(\tilde{L}^i\varphi^i+\varphi_t^i)-\varphi^i(L^iz^i-z_t^i)]\,dx\,dt. \tag{3.9}$$

If for a certain $i$ the set $D^i$ defined by (3.2) were not empty, then at points of this set we would have, by (2.1), (2.2),

$$L^iz^i-z_t^i=(z^i)^{1-p}(w^i)^{p-1}(L^iw^i-w_t^i)+\varepsilon^p(p-1)\,(z^i)^{1-2p}(w^i)^{p-2}\sum_{j,k}a_{jk}^iw_{x_j}^iw_{x_k}^i+ \\ +\varepsilon^pH^i(z^i)^{1-p}, \qquad \text{where } H^i=\sum_{j,k}(a_{jk}^i)_{x_jx_k}-\sum_j(b_j^i)_{x_j}.$$

Hence, taking advantage of inequalities (3.3), (3.4) and their parabolicity we obtain

$$(3.10)\quad L^i z^i - z^i_t \geqslant -(z^i)^{1-p}(w^i)^{p-1}[f^i(t,x,u^1,\ldots,u^m)-f^i(t,x,v^1,\ldots,v^m)]+\varepsilon^p H^i(z^i)^{1-p}.$$

Note that for $(t,x)\in D^i$, $u^l-v^l\leqslant w^l$ $(l=1,\ldots,m)$ and $u^i-v^i=w^i$. Applying successively assumptions $(A_3)$, $(A_4)$ yields

$$f^i(t,x,u^1,\ldots,u^m)-f^i(t,x,v^1,\ldots,v^m) \leqslant f^i(t,x,v^1+w^1,\ldots,v^m+w^m)-f^i(t,x,v^1,\ldots,v^m)\leqslant \sum_s c^i_s w^s$$

which together with (3.10) gives

$$(3.11)\qquad L^i z^i - z^i_t \geqslant -(z^i)^{1-p}(w^i)^{p-1}\sum_s c^i_s w^s+\varepsilon^p H^i(z^i)^{1-p}.$$

Further, since $c^i_s\geqslant 0$ for $s\neq i$ and $(z^i)^{1-p}(w^i)^{p-1}\leqslant 1$ we obtain from (3.11)

$$(3.12)\qquad L^i z^i - z^i_t \geqslant -\sum_s c^i_s z^s+\varepsilon^p(H^i+c^i_i)(z^i)^{1-p}.$$

Inequality (3.12) is derived for points $(t,x)\in D^i$. Notice that this inequality holds true also in the complement of $D^i$ in $S$ and, in particular, in the intersection of the boundary of $D^i$ with $S$ for there the left-hand side of (3.12) is equal to $\varepsilon H^i$ and the right-hand side is less than or equal to $\varepsilon H^i$. Thus (3.12) is valid for all $(t,x)\in S$. By (3.12) and by relations $\varphi^i(t,x)\geqslant 0$, $w^i(0,x)=0$ we find from (3.9), after letting $\varepsilon\to 0$,

$$(3.13)\qquad \int_{E_n}\sum_i w^i\varphi^i\Big|_{t=t_0}dx\leqslant\int_0^{t_0}\int_{E_n}\sum_i w^i[\tilde{L}^i\varphi^i+\sum_s c^i_s\varphi^s+\varphi^i_t]\,dx\,dt.$$

We set in (3.13) $\varphi^i(t,x)=\gamma^R(x)\Phi^i(t,x)$, where $\Phi^i$ are the functions appearing in assumptions of the theorem and $\gamma^R(x)$, $R>1$, is a function of class $C^2(E_n)$ such that $\gamma^R(x)=1$ for $|x|\leqslant R-1$, $\gamma^R(x)=0$ for $|x|\geqslant R$, $0\leqslant\gamma^R(x)\leqslant 1$ in $E_n$ and the first and second derivatives of $\gamma^R(x)$ are bounded in $E_n$ by a constant independent of $R$. Now

$$(3.14)\quad \tilde{L}^i\varphi^i+\sum_s c^s_i\varphi^s+\varphi^i_t = \gamma^R\Big\{\tilde{L}^i\Phi^i+\sum_s c^s_i\Phi^s+\Phi^i_t\Big\}+2\sum_{j,k}a^i_{jk}\gamma^R_{x_j}\Phi^i_{x_k}+\Phi^i\Big\{\sum_{j,k}a^i_{jk}\gamma^R_{x_jx_k}+\sum_j b^i_j\gamma^R_{x_j}\Big\}.$$

By (3.5), (3.14) it follows from (3.13) that

$$(3.15)\quad \int_{E_n}\sum_i w^i\varphi^i\Big|_{t=t_0}dx \leqslant\int_0^{t_0}\int_{E_n}\sum_i w^i\Big|2\sum_{j,k}a^i_{jk}\gamma^R_{x_j}\Phi^i_{x_k}+\Phi^i\Big(\sum_{j,k}a^i_{jk}\gamma^R_{x_jx_k}+\sum_j b^i_j\gamma^R_{x_j}\Big)\Big|\,dx\,dt.$$

In view of (3.6) and the properties of $\gamma^R$, the right-hand side of (3.15) tends to zero as $R \to \infty$. Thus for any $\varrho > 0$ we have

$$\max_{\langle 0,T\rangle} \int_{|x|\leqslant\varrho} \sum_i w^i \Phi^i dx \leqslant 0 .$$

Since $\varrho$ is arbitrary, $w^i \geqslant 0$ and $\Phi^i > 0$ in $\langle 0, T\rangle \times (|x| \leqslant \varrho)$, therefore $w^i(t, x) \equiv 0$ in $\bar{S}$ which means that $u^i(t, x) \leqslant v^i(t, x)$ in $\bar{S}$ and the proof is completed.

The following theorem is a particular case of Theorem 1:

THEOREM 2. *We assume that $u^i(t, x)$, $v^i(t, x)$ satisfy condition* C *in $S$ and*

$$u^i(0, x) \leqslant v^i(0, x) \quad (i = 1, \ldots, m),\ x \in E_n. \tag{3.16}$$

*Let*

$$D^l = \{(t, x) \in S\colon\ u^l(t, x) > v^l(t, x)\} \quad (l = 1, \ldots, m). \tag{3.17}$$

*Suppose that for any fixed $i$ we have*

$$u_t^i(t, x) \leqslant L^i[u^i(t, x)] + f^i\big(t, x, u^1(t, x), \ldots, u^m(t, x)\big), \tag{3.18}$$

$$v_t^i(t, x) \geqslant L^i[v^i(t, x)] + f^i\big(t, x, v^1(t, x), \ldots, v^m(t, x)\big) \tag{3.19}$$

*whenever $(t, x) \in D^i$. We retain assumptions* $(A_1)$-$(A_4)$. *Assume that*

$$\iint_S (u^i - v^i)^+ \exp\{-K(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\} dx\, dt < \infty \qquad (i = 1, \ldots, m) \tag{3.20}$$

*for some constant $K \geqslant 0$. The constant $\lambda$ is supposed to be non-negative whereas $\mu$ may be any real number if $\lambda > 0$, and $\mu \geqslant 1$ if $\lambda = 0$. Moreover, we assume that the coefficients $a_{jk}^i$, $b_j^i$ of operators $L^i$, and functions $c_s^i$ satisfy, almost everywhere in $S$, the following growth conditions: there are constants $A$, $B$, $C \geqslant 0$ such that*

$$\begin{gathered} |a_{jk}^i| \leqslant A(|x|^2+1)^{(2-\lambda)/2}[\ln(|x|^2+1)+1]^{-\mu}, \qquad |b_j^i| \leqslant B(|x|^2+1)^{1/2}, \\ \sum_s c_i^s \leqslant C(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu \end{gathered} \tag{3.21}$$

*if $\lambda > 0$ ($\mu$-arbitrary), and*

$$\begin{gathered} |a_{jk}^i| \leqslant A(|x|^2+1)[\ln(|x|^2+1)+1]^{2-\mu}, \\ |b_j^i| \leqslant B(|x|^2+1)^{1/2}[\ln(|x|^2+1)+1], \\ \sum_s c_i^s \leqslant C[\ln(|x|^2+1)+1]^\mu \end{gathered} \tag{3.22}$$

*if $\lambda = 0$ ($\mu \geqslant 1$).*

*The above assumptions imply the inequalities*

$$u^i(t, x) \leqslant v^i(t, x) \quad (i = 1, \ldots, m) \text{ in } \bar{S}. \tag{3.23}$$

Proof. To derive this theorem from Theorem 1 we set in Theorem 1

$$\Phi^i = \Phi = \exp\left\{-\frac{K+2}{1-\nu t}(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\right\}, \tag{3.24}$$

where, for both cases (3.21), (3.22), one can take

$$\begin{aligned}\nu = nA(K+2)(\lambda+2|\mu|)^2+[nA(|\lambda-2|+1)+\sqrt{n}B](\lambda+2|\mu|)+\\ +2nA|\mu|(2|\mu-1|+\lambda)+C/(K+2).\end{aligned} \tag{3.25}$$

At first the proof is caried out for the strip $S^\nu = (0, 1/2\nu\rangle \times E_n$ and then extended in a usual way. Using (3.21), (3.22) one can show, as in [4], that $\Phi^i$ defined by (3.24) satisfy inequalities (3.5) in $S^\nu$ and that (3.20) implies (3.6) (with $S$ replaced by $S^\nu$). We omit the computational details.

**4. Maximum principle and estimates.** The next two theorems are deduced from Theorem 2.

THEOREM 3 (Maximum principle). *Suppose $u^i(t, x)$ $(i = 1, \ldots, m)$ satisfy in $S$ condition C and the system of equations*

$$u^i_t = L^i u^i + f^i(t, x, u^1, \ldots, u^m) \quad (i = 1, \ldots, m), \tag{4.1}$$

*where operators $L^i$ are defined as in section 3, whose coefficients satisfy hypotheses* $(A_1)$, $(A_2)$, *and functions $f^i$ satisfy assumptions* $(A_3)$, $(A_4)$. *Le*

$$u^i(0, x) \leqslant M^i \quad (i = 1, \ldots, m),\ x \in E_n,$$

*$M^i$ being non-negative constants, and*

$$\iint\limits_S (u^i)^+ \exp\{-K(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\}\,dx\,dt < \infty, \qquad K \geqslant 0,$$

$(i = 1, \ldots, m)$, *where $\lambda$, $\mu$ are defined as in Theorem 2. Assume that the growth conditions* (3.21), (3.22) *respectively, are satisfied almost everywhere in $S$. If, moreover,*

$$\Big[\sum_{j,k}(a^i_{jk})_{x_j x_k} - \sum_j (b^i_j)_{x_j}\Big] M^i + f^i(t, x, M^1, \ldots, M^m) \leqslant 0 \quad (i = 1, \ldots, m),$$

*then*

$$u^i(t, x) \leqslant M^i \quad (i = 1, \ldots, M) \text{ for } (t, x) \in \bar{S}.$$

Proof. We verify immediately that functions $u^i$ and $v^i = M^i$ satisfy all the assumptions of Theorem 2.

Remark. Theorems 1, 2 imply suitable theorems on the uniqueness of the Cauchy problem for system (4.1) in the strip $S$. A proof of the uniqueness not based on differential inequalities is given in [4].

THEOREM 4. *Let functions $u^i(t, x)$ $(i = 1, \ldots, m)$ fulfil condition C in $S$ and constitute, in $S$, a solution of system* (4.1). *We retain assumptions* $(A_1)$-$(A_4)$. *Suppose there exist constants $M$, $K$, $K_0$, $\lambda \geqslant 0$ and a constant $\mu$, which is arbitrary if $\lambda > 0$ and $\mu \geqslant 1$ if $\lambda = 0$, such that*

$$(4.2)\qquad |u^i(0, x)| \leqslant M\exp\{K(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\}, \qquad x \in E_n,$$

$$(4.3)\qquad |f^i(t, x, 0, \ldots, 0)| \leqslant M\exp\{K(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\}, \qquad (t, x) \in S,$$

*and*

$$(4.4)\qquad \iint_S |u^i(t, x)|\exp\{-K_0(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\}\,dx\,dt < \infty$$

$(i = 1, \ldots, m)$. *Further let the growth conditions* (3.21) *in case $\lambda > 0$, $\mu$ arbitrary, and* (3.22) *in case $\lambda = 0$, $\mu \geqslant 1$, hold true almost everywhere in $S$. Assume, moreover, that $c_s^i = c_i^s$ and that the inequalities*

$$(4.5)\qquad \begin{aligned} &\Big|\sum_k (a_{jk}^i)_{x_k}\Big| \leqslant B(|x|^2+1)^{1/2}, \\ &\sum_{j,k} (a_{jk}^i)_{x_j x_k} - \sum_j (b_j^i)_{x_j} \leqslant C(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu \end{aligned}$$

*if $\lambda > 0$, $\mu$-arbitrary, and*

$$(4.6)\qquad \begin{aligned} &\Big|\sum_k (a_{jk}^i)_{x_k}\Big| \leqslant B(|x|^2+1)^{1/2}[\ln(|x|^2+1)+1], \\ &\sum_{j,k} (a_{jk}^i)_{x_j x_k} - \sum_j (b_j^i)_{x_j} \leqslant C[\ln(|x|^2+1)+1]^\mu \end{aligned}$$

*if $\lambda = 0$, $\mu \geqslant 1$, are satisfied almost everywhere in $S$.*

*Then the estimates*

$$(4.7)\qquad |u^i(t, x)| \leqslant M\exp\{2K(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^\mu\} \qquad (i = 1, \ldots, m)$$

*hold for $(t, x) \in \bar{S}^\delta = \langle 0, \delta\rangle \times E_n$, where $\delta = \min\left(\frac{1}{2\nu}, T\right)$ and*

$$(4.8)\qquad \nu = nAK(\lambda+2|\mu|)^2+[nA(|\lambda-2|+1)+3\sqrt{n}B](\lambda+2|\mu|)+ +2nA|\mu|(2|\mu-1|+\lambda)+(2C+1)/K.$$

Proof. Denote

$$v = M\exp\left\{\frac{K}{1-\nu t}(|x|^2+1)^{\lambda/2}[\ln(|x|^2+1)+1]^{\mu}\right\}. \tag{4.9}$$

To deduce this theorem from Theorem 2 we first show that functions $v^i = v$ $(i = 1, \ldots, m)$ satisfy inequalities (3.19) for $i = 1, \ldots, m$, $(t, x) \in \bar{S}^\delta$. By assumptions $(A_4)$, (4.3) we get

$$L^i v^i - v^i_t + f^i(t, x, v^1, \ldots, v^m) \leqslant L^i v - v_t + \Big(\sum_s c^i_s + 1\Big) v. \tag{4.10}$$

One can show by direct computation that for $v$ defined by (4.9) the right-hand side of (4.10) is non-positive in $S^\delta$. We check immediately that functions $u^i$ and $v^i = v$ satisfy all the remaining assumptions of Theorem 2, whence we obtain the inequalities $u^i \leqslant v$ in $S^\delta$. Similarly it can be shown that inequalities (3.18) are satisfied with $u^i$ substituted by $-v$. Thus setting, in Theorem 2, $u^i = -v$ and $v^i = u^i$ we observe that all the assumptions of the theorem are fulfilled. Hence we get inequalities $u^i \geqslant -v$ which together with $u^i \leqslant v$ imply the assertion (4.7).

**5. Remark on positive solutions.** Now we show that in general differential inequalities theorems are not valid in the class of non-negative functions (in which the uniqueness of the Cauchy problem for a class of parabolic equations holds true). To this end we make use of a solution given by Tihonov [11]. In order to prove non-uniqueness of the Cauchy problem for the heat equation

$$w_t(t, x) = w_{xx}(t, x) \tag{5.1}$$

in the class of functions satisfying condition

$$|w(t, x)| \leqslant M\exp\{K|x|^{2+\varepsilon}\}, \qquad \varepsilon > 0 \text{ arbitrary}, \tag{5.2}$$

Tihonov constructed a (regular) solution $w(t, x)$ of (5.1) (satisfying (5.2)), such that $w(0, x) = 0$ for $x \in (-\infty, \infty)$ and $w(t, x) \not\equiv 0$ in a strip $S = (0, T\rangle \times (-\infty, \infty)$.

Now, define $u(t, x) = w^2(t, x) + C$ and $v(t, x) = C$, $C = \text{const} \geqslant 0$. It can easily be checked that functions $u$, $v$ satisfy, in $S$, inequalities $u_t \leqslant u_{xx}$, $v_t \geqslant v_{xx}$ respectively, both are non-negative and $u(0, x) \leqslant v(0, x)$. However, inequality $u(t, x) \leqslant v(t, x)$ is not satisfied in $S$.

From the above remark it also follows that it is not true that every non-negative solution of the inequality $u_t \leqslant u_{xx}$ belongs to the class $I_2$ (see introduction; although, as is known, any non-negative solution of the equation (5.1) does).

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