ANNALES
POLONICI MATHEMATICI
XLVII (1986)

On the behaviour of solutions of the differential equations $(r(t)\,y'')' + q(t)(y')^{\beta} + p(t)\,y^{\alpha} = f(t)$

by N. Parhi and S. Parhi (Berhampur, Orissa, India)

Abstract. In this paper, we consider the nonlinear non-homogeneous third order differential equation

$$(r(t)\,y'')' + q(t)(y')^{\beta} + p(t)\,y^{\alpha} = f(t),$$

where p, q, r and f are real-valued continuous functions on $[0,\infty]$ such that $r(t) > 0$, $f(t) \geqslant 0$ and both $\alpha > 0$ and $\beta > 0$ are ratios of odd integers. Sufficient conditions are obtained for non-oscillation of solutions of the equation in two cases, viz, (i) $p(t) \geqslant 0$ and $q(t) \leqslant 0$ and (ii) $p(t) \leqslant 0$ and any $q(t)$. Some results concerning the asymptotic behaviour of solutions of the equation are also given.

1. Finding sufficient conditions for non-oscillation of solutions is a problem of general interest in the theory of ordinary and delay-differential equations. In this work, we consider

$$(r(t)\,y'')' + q(t)(y')^{\beta} + p(t)\,y^{\alpha} = f(t), \tag{1}$$

where p, q, r and f are real-valued continuous functions on $[0, \infty)$ such that $r(t) > 0$ and $f(t) \geqslant 0$ and both $\alpha > 0$ and $\beta > 0$ are ratios of odd integers. In [7] N. Parhi gave sufficient conditions for non-oscillation of solutions of (1) with $p(t) \leqslant 0$ and $q(t) \leqslant 0$. In section 2 of this paper we give sufficient conditions for non-oscillation of solutions of (1) with $p(t) \geqslant 0$ and $q(t) \leqslant 0$. These results are more general than those obtained by the present authors in [8]. We also study the asymptotic behaviour of non-oscillatory solutions of (1) in this section. Sufficient conditions are given for non-oscillation of bounded solutions of (1) with $p(t) \leqslant 0$ in Section 3. We note that there is no sign restriction on q for most of the results in this section. Also we obtain results concerning the asymptotic behaviour of solutions and the existence of a positive increasing solution of (1). These results strengthen some of the results of Hanan [3].

Equation (1) with $r(t) \equiv 1$, $f(t) \equiv 0$, $\beta = 1$ has been considered by Heidel [4], Nelson [6], Waltman [10], and for the case $\alpha = \beta = 1$, $r(t) \equiv 1$ and $f(t) \equiv 0$ we should mention the papers of Barrett [1], Hanan [3], Lazer

[5] and the book of Swanson [9]. Erbe [2] has considered equation (1) with $r(t)$ once continuously differentiable, $f(t) \equiv 0$ and $\beta = 1$. The above-mentioned authors have given sufficient conditions for the existence of oscillatory and non-oscillatory solutions and have studied their asymptotic behaviour. It seems that the research on non-homogeneous third order differential equations started with the work of N. Parhi [7].

We restrict our considerations to those real solutions of (1) which exist on the half-line $[T, \infty)$, where $T \geqslant 0$ depends on the particular solution, and are non-trivial in any neighbourhood of infinity. A solution $y(t)$ of (1) on $[T, \infty)$ is said to be *non-oscillatory* if there exists a $t_1 \geqslant T$ such that $y(t) \neq 0$ for $t \geqslant t_1$; it is said to be *oscillatory* if for any $t_1 \geqslant T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$, and the solution is said to be a *Z-type solution* if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

2. In this section we obtain sufficient conditions for the non-oscillation of solutions of (1) with $p(t) \geqslant 0$ and $q(t) \leqslant 0$. Also results concerning the asymptotic behaviour of these non-oscillatory solutions are obtained.

THEOREM 1. *If $q(t)+t^\alpha p(t) \leqslant 0$ for large t, then all solutions of (1) with $\alpha = \beta$ are non-oscillatory.*

Proof. Let $q(t)+t^\alpha p(t) \leqslant 0$ for $t \geqslant t_0 > 0$. Let $y(t)$ be a solution of (1). If possible, let $y(t)$ be of non-negative Z-type with consecutive double zeros at a and b $(t_0 \leqslant a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Consequently, there exists a $d \in (a, c)$ such that $y''(d) = 0$ and $y''(t) > 0$ for $t \in (a, d)$. Clearly, for large t_0, $y'(t) \geqslant y(t)/t$ for $t \in [a, d]$. Now, integrating

$$[r(t)y'(t)y''(t)]' = r(t)(y''(t))^2 - q(t)(y'(t))^{\alpha+1} - p(t)y^\alpha(t)y'(t) + f(t)y'(t) \tag{2}$$

from a to d, we obtain

$$0 > -\int_a^d [q(t)(y'(t))^\alpha + p(t)y^\alpha(t)]\, y'(t)\, dt$$

$$> -\int_a^d [q(t)+t^\alpha p(t)] \frac{y^\alpha(t)\, y'(t)}{t^\alpha}\, dt > 0,$$

a contradiction. Let $y(t)$ be of non-positive Z-type with consecutive double zeros at a and b $(t_0 \leqslant a < b)$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. Integration of (2) from c to b yields

$$0 = \int_c^b [r(t)(y''(t))^2 - q(t)(y'(t))^{\alpha+1} - p(t)y^\alpha(t)y'(t) + f(t)y'(t)]\, dt > 0,$$

a contradiction.

If possible, let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($t_0 \leqslant a < b < a'$) such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. If possible, let $y''(b) \leqslant 0$. Integrating (2) from c to b, we get

$$\begin{aligned} 0 &\geqslant r(b)y'(b)y''(b) \\ &\geqslant \int_c^b \left[r(t)(y''(t))^2 - q(t)(y'(t))^{\alpha+1} - p(t)y^{\alpha}(t)y'(t) + f(t)y'(t)\right]dt > 0, \end{aligned}$$

a contradiction. Hence $y''(b) > 0$. Since $y''(t)$ is continuous, $y''(t) > 0$ for $t \in [b, b+\delta_1)$, $0 < \delta_1 < c'-b$. So $y'(t)$ is increasing on $[b, b+\delta_1)$. Again $y'(c') = 0$ and $y'(t) > 0$ for $t \in (b, c')$ imply that $y'(t)$ is decreasing on $[c'-\delta_2, c']$, where $0 < \delta_2 < c'-b$. This in turn implies that $y''(t) < 0$ for $t \in [c'-\delta_2, c']$. Hence $y''(d) = 0$ for some $d \in (b, c')$ and $y''(t) > 0$ for $t \in [b, d)$. Clearly, $y'(t) \geqslant y(t)/t$ for $t \in [b, d]$. Now, integrating (2) from b to d, we obtain

$$\begin{aligned} 0 &\geqslant -r(b)y'(b)y''(b) \\ &> -\int_b^d \left[q(t)(y'(t))^{\alpha} + p(t)y^{\alpha}(t)\right]y'(t)\,dt \\ &> -\int_b^d \left[q(t) + t^{\alpha}p(t)\right]\frac{y^{\alpha}(t)y'(t)}{t^{\alpha}}\,dt > 0, \end{aligned}$$

a contradiction. Hence the theorem.

EXAMPLE. All solutions of

$$(t^2y'')' - 2t^4(y')^3 + ty^3 = 1/t^2, \qquad t \geqslant 1,$$

are non-oscillatory. In particular, $y(t) = 1/t$ is a non-oscillatory solution of the equation.

THEOREM 2. *Let $q(t)$ be once continuously differentiable and such that $q'(t) \geqslant 0$ and $\lim_{t\to\infty}[q'(t)/p(t)] = \infty$. Then all bounded solutions of (1) with $\alpha \geqslant 1$ and $\beta = 1$ are non-oscillatory.*

Proof. Let $y(t)$ be a bounded solution of (1) such that $|y(t)| \leqslant K$ for $t \geqslant T$. So there exists a $t_0 \geqslant T$ such that $q'(t) \geqslant K^{\alpha-1}p(t)$ for $t \geqslant t_0$. If possible, let $y(t)$ be of non-negative Z-type with consecutive double zeros at a and b ($t_0 \leqslant a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Clearly, $y''(a) \geqslant 0$ and $y''(c) \leqslant 0$. Integrating (1) from a to c, we get

$$0 > -\int_a^c q(t)y'(t)\,dt - \int_a^c p(t)y^{\alpha}(t)\,dt$$

$$> -q(c)y(c)+\int_a^c [q'(t)-p(t)y^{\alpha-1}(t)]\,y(t)\,dt$$

$$> \int_a^c [q'(t)-K^{\alpha-1}p(t)]\,y(t)\,dt > 0,$$

a contradiction. Let $y(t)$ be of non-positive Z-type with consecutive double zeros at a and b ($t_0 \leqslant a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. Now, multiplying (1) through by $y'(t)$ and integrating the resulting identity from c to b, we get

$$0 = \int_c^b \left[r(t)\big(y''(t)\big)^2 - q(t)\big(y'(t)\big)^2 - p(t)y^{\alpha}(t)y'(t) + f(t)y'(t)\right] dt > 0,$$

a contradiction.

Let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($t_0 \leqslant a < b < a'$) such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. We consider two cases, viz., $y''(b) \leqslant 0$ and $y''(b) > 0$. If $y''(b) \leqslant 0$, then we multiply (1) through by $y'(t)$ and integrate the resulting identity from c to b to get $0 \geqslant r(b)y'(b)y''(b) > 0$, a contradiction. If $y''(b) > 0$, then there exists a $d \in (b, c')$ such that $y''(d) = 0$. Now, integrating (1) from b to d, we obtain

$$0 > -r(b)y''(b) > -q(d)y(d) + \int_b^d [q'(t)-K^{\alpha-1}p(t)]\,y(t)\,dt > 0,$$

a contradiction. This completes the proof of the theorem.

The following example illustrates the above theorem.

EXAMPLE.

$$\left(\frac{1}{10t^2}y''\right)' - \frac{1}{t}y' + \frac{1}{t^3}y^3 = \frac{1}{t^3}, \qquad t > 1.$$

All bounded solutions of this equation are non-oscillatory. In particular, $y(t) = 1/t$ is a bounded non-oscillatory solution of the equation.

Remark. It is interesting to note that Theorem 1 cannot be applied to the example illustrating Theorem 2 and Theorem 2 cannot be applied to the example illustrating Theorem 1.

In the following we study the asymptotic behaviour of solutions of (1).

THEOREM 3. *If* $q(t)+tp(t) \leqslant 0$ *for large* t, $\displaystyle\int_0^{\infty} \frac{dt}{r(t)} = \infty$, $\displaystyle\int_0^{\infty} p(t)\,dt = \infty$,

$\int_0^\infty f(t)\,dt < \infty$ *and* $q(t)$ *is bounded, then all bounded solutions of* (1) *with* $\alpha = \beta = 1$ *tend to zero as* $t \to \infty$.

Proof. Let $|q(t)| \leqslant M$ and $q(t)+tp(t) \leqslant 0$ for $t \geqslant t_0 > 0$. Let $y(t)$ be a bounded solution of (1) such that $|y(t)| \leqslant K$. From Theorem 1 it follows that $y(t)$ is non-oscillatory. So it is ultimately positive or ultimately negative.

Let $y(t) < 0$ for $t \geqslant t_1 \geqslant t_0$. If possible, let $y'(t)$ be oscillatory (or non-negative Z-type) with consecutive zeros (or double zeros) at a and b ($t_1 \leqslant a < b$) such that $y'(t) > 0$ for $t \in (a, b)$. Integrating

$$[r(t)y'(t)y''(t)]' = r(t)\big(y''(t)\big)^2 - q(t)\big(y'(t)\big)^2 - p(t)y(t)y'(t) + f(t)y'(t)$$

from a to b, we get

$$0 = \int_a^b \big[r(t)\big(y''(t)\big)^2 - q(t)\big(y'(t)\big)^2 - p(t)y(t)y'(t) + f(t)y'(t)\big]\,dt > 0,$$

a contradiction. Let $y'(t) \leqslant 0$ for $t \geqslant t_2 \geqslant t_1$. Integrating (1) from t_2 to t, we get

$$\begin{aligned} r(t)y''(t) &\geqslant r(t_2)y''(t_2) - \int_{t_2}^{t} q(s)y'(s)\,ds - \int_{t_2}^{t} p(s)y(s)\,ds \\ &\geqslant r(t_2)y''(t_2) + M\big(y(t) - y(t_2)\big) - y(t_2)\int_{t_2}^{t} p(s)\,ds \\ &\geqslant r(t_2)y''(t_2) - MK - y(t_2)\int_{t_2}^{t} p(s)\,ds. \end{aligned}$$

So $r(t)y''(t) \geqslant L$ for large t, where L is a positive constant. This in turn implies that $y'(t) > 0$ for large t, a contradiction. Hence $y'(t) > 0$ for large t. Consequently, $\lim_{t\to\infty} y(t)$ exists. If possible, let $\lim_{t\to\infty} y(t) = -A$, where $A > 0$. Now, integrating (1) from σ to t, where $\sigma > t_1$ is sufficiently large, we get

$$r(t)y''(t) \geqslant r(\sigma)y''(\sigma) - y(t)\int_\sigma^t p(s)\,ds.$$

Hence $y''(t) > 0$ for large t. Consequently, $y(t) > 0$ for large t, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

Let $y(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Suppose that $y'(t)$ is oscillatory (or non-negative Z-type) with consecutive zeros (or double zeros) at a and b ($t_1 \leqslant a < b$) such that $y'(t) > 0$ for $t \in (a, b)$. So there exists a $c \in (a, b)$ such that $y''(c) = 0$ and $y''(t) > 0$ for $t \in (a, c)$. Clearly, $y'(t) > y(t)/t$ for $t \in (a, c)$. Now integrating (1) from a to c, we obtain

$$0 \geqslant -r(a)y''(a) \geqslant -\int_a^c q(t)y'(t)\,dt - \int_a^c p(t)y(t)\,dt$$

$$\geqslant -\int_a^c [q(t)+tp(t)]\frac{y(t)}{t}\,dt > 0,$$

a contradiction. If possible, let $y'(t) > 0$ for large t. Proceeding as in the case $y(t) < 0$ and $y'(t) \leqslant 0$ above, we get $r(t)y''(t) \leqslant -L$ for large t, where $L > 0$. This in turn implies that ultimately $y'(t) < 0$, a contradiction. Hence $y'(t) \leqslant 0$ for large t. Consequently, $\lim_{t\to\infty} y(t)$ exists. Suppose that $\lim_{t\to\infty} y(t) = A > 0$. Now, integrating (1) from σ to t, where $\sigma \geqslant t_1$ is sufficiently large, we get $y''(t) < 0$ for large t. This implies that $y(t) < 0$ ultimately, a contradiction. Hence $\lim_{t\to\infty} y(t) = 0$.

So the theorem is completed.

EXAMPLE.

$$\left(\frac{1}{8t}y''\right)' - \left(4-\frac{1}{t}\right)y' + \frac{3}{t}y = \frac{7}{t^2} - \frac{1}{t^3} - \frac{1}{t^5}, \quad t > 1.$$

All bounded solutions of the equation tend to zero as $t \to \infty$. In particular, $y(t) = 1/t$ is such a solution.

THEOREM 4. *Let* $\int_0^\infty p(t)\,dt < \infty$, $\int_0^\infty f(t)\,dt = \infty$ *and* $\int_0^\infty \frac{dt}{r(t)} = \infty$. *If q is once continuously differentiable and such that* $q'(t) \geqslant 0$ *and* $\int_0^\infty q'(t)\,dt < \infty$, *then all solutions of* (1) *with* $\beta = 1$ *are unbounded.*

Proof. Let $y(t)$ be a bounded solution of (1) such that $|y(t)| \leqslant K$. Now, integrating (1) from $t_0 \geqslant 0$ to t, we get

$$\begin{aligned} r(t)y''(t) &= r(t_0)y''(t_0) - q(t)y(t) + q(t_0)y(t_0) + \int_{t_0}^t q'(s)y(s)\,ds \\ &\qquad - \int_{t_0}^t p(s)y^{\alpha}(s)\,ds + \int_{t_0}^t f(s)\,ds \\ &\geqslant r(t_0)y''(t_0) + Kq(t_0) + q(t_0)y(t_0) - K\int_{t_0}^\infty q'(s)\,ds \\ &\qquad - K^{\alpha}\int_{t_0}^\infty p(s)\,ds + \int_{t_0}^t f(s)\,ds. \end{aligned}$$

So, for $0 < \lambda < 1$, $y''(t) \geqslant \frac{\lambda}{r(t)} \int_{t_0}^{t} f(s)\,ds$ for large t. Since $\int_0^\infty f(t)\,dt = \infty$ and $\int_0^\infty \frac{dt}{r(t)} = \infty$ imply that

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)} \left(\int_{t_0}^{s} f(\theta)\,d\theta \right) ds = \infty,$$

we get $\lim_{t\to\infty} y'(t) = \infty$. This in turn implies that $\lim_{t\to\infty} y(t) = \infty$, a contradiction. So $y(t)$ must be unbounded.

This completes the proof of the theorem.

EXAMPLE.

$$(ty'')' - \frac{1}{t^2} y' + \frac{3}{t^9} y^3 = 12t, \qquad t > 0.$$

All solutions of this equation are unbounded. In particular, $y(t) = t^3$ is an unbounded solution.

3. In this section sufficient conditions are obtained for non-oscillation of solutions of (1) with $p(t) \leqslant 0$. We do not put any sign restriction on $q(t)$ in some of our results. We begin with the following lemma:

LEMMA 5. *Consider*

$$(r(t)z')' + q(t)z = 0, \tag{3}$$

where r and q are real-valued continuous functions on $[0, \infty)$ such that $r(t) > 0$. If $z(t)$ is a non-oscillatory solution of (3) *such that $z(t) > 0$ or < 0 for $t \in [a, \infty)$ and if u is a once continuously differentiable function on $[a, \infty)$ such that $u(b) = 0 = u(c)$, $a < b < c$, and $u(t) \not\equiv 0$ on $[b, c]$, then*

$$\int_b^c \left[r(t)(u'(t))^2 - q(t)(u(t))^2 \right] dt > 0.$$

Proof.

$$0 < \int_b^c \left[r^{1/2}(t)u'(t) - r^{1/2}(t)\frac{u(t)z'(t)}{z(t)} \right]^2 dt$$

$$= \int_b^c \left[r(t)(u'(t))^2 + r(t)\frac{u^2(t)(z'(t))^2}{z^2(t)} \right] dt - \left[\frac{r(t)z'(t)u^2(t)}{z(t)} \right]_b^c$$

$$+ \int_b^c u^2(t) \left[\frac{(r(t)z'(t))'z(t) - r(t)(z'(t))^2}{z^2(t)} \right] dt$$

$$= \int_b^c \left[r(t)(u'(t))^2 + r(t)\frac{u^2(t)(z'(t))^2}{z^2(t)} \right] dt - \int_b^c u^2(t) \left[\frac{q(t)z^2(t) + r(t)(z'(t))^2}{z^2(t)} \right] dt$$

$$= \int_b^c [r(t)(u'(t))^2 - q(t)u^2(t)]\, dt.$$

Hence the lemma.

Remark. If $q(t) \leqslant 0$, then all solutions of (3) are non-oscillatory. For $q(t) \nleqslant 0$, sufficient conditions were given by Moore [9], p 73, Wintner [9], p. 63, and Potter [9], p. 81, for the non-oscillation of all solutions of (3).

THEOREM 6. *If* (3) *admits a non-oscillatory solution, then there exists a positive increasing solution of* (1) *with* $\beta = 1$.

Proof. Let $z(t)$ be a non-oscillatory solution of (3). So there exists a $t_0 > t$ such that $z(z) > 0$ or < 0 for $t \geqslant t_0$. Let $y(t)$ be a solution of (1) with $y(a) = 0$, $y'(a) = 0$ and $y''(a) > 0$, where $a > t_0$. From the continuity of $y''(t)$ it follows that $y(t) > 0$, $y'(t) > 0$ and $y''(t) > 0$ to the right of 'a' but close to 'a'. We claim that $y'(t) > 0$ for $t > a$. Otherwise, there exists a $b > a$ such that $y'(b) = 0$ and $y'(t) > 0$ for $t \in (a, b)$. Clearly, $y(t) > 0$ for $t \in (a, b)$. Now, integrating

$$[r(t)y'(t)y''(t)]' = r(t)(y''(t))^2 - q(t)(y'(t))^2 - p(t)y^{\alpha}(t)y'(t) + f(t)y'(t)$$

from a to b, we get

$$0 > \int_a^b [r(t)(y''(t))^2 - q(t)(y'(t))^2]\, dt > 0$$

by Lemma 5, a contradiction. Hence $y'(t) > 0$ for $t > a$. This in turn implies that $y(t) > 0$, $t > a$.

Hence the theorem.

THEOREM 7. *If* (3) *admits a non-oscillatory solution, then all solution of* (1) *with* $\beta = 1$ *and* $p(t) \equiv 0$ *are non-oscillatory.*

Proof. Let $z(t)$ be a non-oscillatory solution of (3) such that $z(t) > 0$ or < 0 for $t \geqslant t_0 \geqslant 0$. Let $y(t)$ be a solution of (1). If possible, let $y(t)$ be of non-negative Z-type with consecutive double zeros at a and b ($t_0 < a < b$). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Now, integrating

$$[r(t)y'(t)y''(t)]' = r(t)(y''(t))^2 - q(t)(y'(t))^2 + f(t)y'(t) \tag{4}$$

from a to c, we obtain

$$0 > \int_a^c [r(t)(y''(t))^2 - q(t)(y'(t))^2]\, dt > 0,$$

a contradiction. Similarly, it can be shown that $y(t)$ cannot be of non-positive Z-type.

If possible, let $y(t)$ be oscillatory with consecutive zeros at a, b and a' ($t_0 < a < b < a'$) such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. Integration of (4) from c to c' yields a contradiction.

Hence $y(t)$ is non-oscillatory and this completes the proof of the theorem.

EXAMPLE. From Potter's theorem [9], Theorem 2.36, p. 81, it follows that the solutions of

$$(3e^{-3t} z')' + e^{-3t} z = 0$$

are non-oscillatory. Hence all solutions of

$$(3e^{-3t} y'')' + e^{-3t} y' = 155\, e^{2t}$$

are non-oscillatory. In particular, $y(t) = e^{5t}$ is a non-oscillatory solution of the equation.

THEOREM 8. *If* $\lim_{t \to \infty} [f(t)/p(t)] = -\infty$ *and* (3) *has a non-oscillatory solution, than all bounded solutions of* (1) *with* $\beta = 1$ *are non-oscillatory.*

Proof. Let $y(t)$ be a bounded solution of (1) such that $|y(t)| \leqslant K$. From the given assumption it follows that

$$f(t) - p(t) y^{\alpha}(t) \geqslant f(t) + K^{\alpha} p(t) > 0 \quad \text{for } t \geqslant t_0,$$

where $t_0 > 0$ is sufficiently large. So

$$\big(r(t) y''(t)\big)' + q(t) y'(t) > 0$$

for $t \geqslant t_0$. Setting $u(t) = y'(t)$, we get

$$\big(r(t) u'(t)\big)' + q(t) u(t) > 0$$

for $t \geqslant t_0$. If possible, let $y(t)$ be weakly oscillatory, that is, oscillatory or Z-type. So $u(t)$ is oscillatory. Let b and c ($t_0 < b < c$) be consecutive zeros of $u(t)$ such that $u(t) > 0$ for $t \in (b, c)$. Now, integrating

$$\big(r(t) u(t) u'(t)\big)' > r(t) \big(u'(t)\big)^2 - q(t) u^2(t)$$

from b to c, we get a contradiction because of Lemma 5. Hence the theorem.

EXAMPLE. (i) Consider

$$(t^5 y'')' - \frac{1}{t^4} y' - \frac{1}{t} y^3 = 4t + \frac{1}{t^6} - \frac{1}{t^4}, \quad t \geqslant 1. \tag{5}$$

Since all solutions of $(t^5 z')' - \frac{1}{t^4} z = 0$ are non-oscillatory, it follows from the above theorem that all bounded solutions of (5) are non-oscillatory. In particular, $y(t) = 1/t$ is a bounded non-oscillatory solution of (5).

(ii) Consider

$$(6) \qquad (t^5 y'')' + \frac{1}{t^4} y' - \frac{1}{t} y^3 = 4t - \frac{1}{t^6} - \frac{1}{t^4}, \qquad t \geqslant 1.$$

From a result due to Moore [9], p. 73, it follows that all solutions of $(t^5 z')' + \frac{1}{t^4} z = 0$ are non-oscillatory. So all bounded solutions of (6) are non-oscillatory.

Remark. Suppose that $p(t) \not\equiv 0$. We may note that Theorem 8 cannot be applied to (1) with $\beta = 1$ when $f(t) \equiv 0$.

THEOREM 9. *If* $r(t) + p(t) > 0$ *for large* t, *then a solution* $y(t)$ *of* (1) *with* $q(t) \equiv 0$ *which satisfies* $(u'')^2 + u^\alpha u' > 0$ *in any interval on which it is negative is non-oscillatory.*

Proof. Clearly $y(t)$ cannot be of non-negative Z-type. Let $y(t)$ be of non-positive Z-type with consecutive double zeros at a and b. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. Integrating

$$(7) \qquad \big(r(t) y'(t) y''(t)\big)' = r(t)\big(y''(t)\big)^2 - p(t) y^\alpha(t) y'(t) + f(t) y'(t)$$

from c to b, we get

$$0 > -\int_c^b p(t)\big[(y''(t))^2 + y^\alpha(t) y'(t)\big]\, dt > 0,$$

a contradiction.

Suppose that $y(t)$ is oscillatory with consecutive zeros at a, b and a' ($a < b < a'$) such that $y'(a) \leqslant 0$, $y'(b) \geqslant 0$, $y'(a') \leqslant 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = 0 = y'(c')$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. If $y''(b) < 0$, then we integrate (7) from c to b, and if $y''(b) \geqslant 0$, then (7) is integrated from b to c' to yield the necessary contradiction.

This completes the proof of the theorem.

EXAMPLE.

$$(2ty'')' - \frac{1}{t^2} y^3 = t^{10} - 72t^2, \qquad t \geqslant 1.$$

$y(t) = -t^4$ is an unbounded non-oscillatory solution of the equation.

THEOREM 10. *Let* $\int_0^\infty f(t)\, dt = \infty$, $\int_0^\infty \frac{dt}{r(t)} = \infty$ *and* $\int_0^\infty p(t)\, dt > -\infty$. *If* $q(t)$

is once continuously differentiable and such that $q(t) \geqslant 0$, $q'(t) \leqslant 0$ *and* $\int_0^\infty q'(t)\,dt > -\infty$, *then all solutions of* (1) *with* $\beta = 1$ *are unbounded.*

The proof is similar to that of Theorem 4.

THEOREM 11. *Let* $r(t)+p(t) > 0$ *for large* t, $\int_0^\infty \frac{dt}{r(t)} = \infty$, $\int_0^\infty p(t)\,dt > -\infty$ *and* $\int_0^\infty f(t)\,dt = \infty$. *If* $y(t)$ *is a solution of* (1) *with* $q(t) \equiv 0$ *satisfying* $(u'')^2 + u^\alpha u' > 0$ *in any interval on which it is negative, then either* $y(t) > 0$, $y'(t) > 0$ *or* $y(t) < 0$, $y'(t) \leqslant 0$ *for large* t.

Proof. From Theorem 9 it follows that $y(t)$ is non-oscillatory. So there exists a $t_0 > 0$ such that $y(t) > 0$ or < 0 for $t \geqslant t_0$. Let $y(t) > 0$ for $t \geqslant t_0$. Since $\big(r(t)y''(t)\big)' \geqslant 0$ for $t \geqslant t_0$, it is clear that $y'(t)$ cannot be oscillatory or of Z-type. If possible, let $y'(t) < 0$ for $t \geqslant t_1 \geqslant t_0$. So $\lim_{t\to\infty} y(t)$ exists and, integrating (1) from t_1 to t, we obtain

$$r(t)y''(t) \geqslant r(t_1)y''(t_1) - y^\alpha(t)\int_{t_1}^{t} p(s)\,ds + \int_{t_1}^{t} f(s)\,ds.$$

Hence $\lim_{t\to\infty} r(t)y''(t) = \infty$. Consequently, $y'(t) > 0$ for large t, a contradiction. So $y'(t) > 0$ for large t.

Let $y(t) < 0$ for $t \geqslant t_0$. From

$$\begin{aligned}\big(r(t)y'(t)y''(t)\big)' &= r(t)\big(y''(t)\big)^2 - p(t)y^\alpha(t)y'(t) + f(t)y'(t)\\ &\geqslant -p(t)\big[\big(y''(t)\big)^2 + y^\alpha(t)y'(t)\big] + f(t)y'(t)\\ &\geqslant f(t)y'(t),\end{aligned}$$

for $t \geqslant t_0$, it is clear that $y'(t)$ cannot be oscillatory or of non-negative Z-type. If possible, let $y'(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Integrating (1) from t_1 to t, we get

$$r(t)y''(t) \geqslant r(t_1)y''(t_1) - y^\alpha(t_1)\int_{t_1}^{t} p(s)\,ds + \int_{t_1}^{t} f(s)\,ds.$$

So $y''(t) > 0$ for large t. This in turn implies that $y(t) > 0$ for large t, a contradiction. So $y'(t) \leqslant 0$ for large t.

This completes the proof of the theorem.

THEOREM 12. *Suppose that the conditions of Theorem* 11 *are satisfied. If* $y(t)$ *is a solution of* (1) *with* $q(t) \equiv 0$ *satisfying* $(u'')^2 + u^\alpha u' > 0$ *in any interval on which it is negative, then* $y(t)$ *is unbounded. In fact,* $\lim_{t\to\infty} |y(t)| = \infty$.

The proof is straightforward and hence it is omitted.

References

[1] J. H. Barrett, *Oscillation theory of ordinary linear differential equations*, Advances in Math. 3 (1969), 415–509.

[2] L. Erbe, *Oscillation, nonoscillation and asymptotic behaviour for third order nonlinear differential equations*, Ann. Mat. Pura Appl. 110 (1976), 373–393.

[3] M. Hanan, *Oscillation criteria for third order linear differential equations*, Pacific J. Math. 11 (1961), 919–944.

[4] J. W. Heidel, *Qualitative behaviour of solutions of a third order nonlinear differential equation*, ibidem 27 (1968), 507–526.

[5] A. C. Lazer, *The behaviour of solutions of the differential equation* $y'''+p(x)y'+q(x)y=0$, ibidem 17 (1966), 435–466.

[6] J. L. Nelson, *A stability theorem for a third order non-linear differential equation*, ibidem 24 (1968), 341–344.

[7] N. Parhi, *Nonoscillatory behaviour of solutions of nonhomogeneous third order differential equations*, Applicable Analysis 12 (1981), 273–285.

[8] –, S. Parhi, *Oscillation and nonoscillation theorems for nonhomogeneous third order differential equations*, Bulletin of Institute of Mathematics Academia Sinica 11 (1983), 125–139.

[9] C. A. Swanson, *Comparison and oscillation theory of linear differential equations*, New York 1968.

[10] P. Waltman, *Oscillation criteria for third order nonlinear differential equations*, Pacific J. Math. 18 (1966), 385–389.

DEPARTMENT OF MATHEMATICS, BERHAMPUR UNIVERSITY
BERHAMPUR, ORISSA, INDIA

Reçu par la Rédaction 1984.10.27