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On Hermite expansion of x_+^p

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Abstract. We give an expansion of function x_+^p into the Hermite series.

1. Following [1], by Hermite functions we mean the functions defined by the equality

$$h_n(x) = (-1)^n\left(\sqrt{2\pi}\, n!\right)^{-1/2} e^{x^2/4}\left(e^{-x^2/2}\right)^{(n)}.$$

It is proved in [1] that any tempered distribution f can be expanded into the Hermite series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n h_n,$$

where $c_n = (f, h_n)$. The convergence of the series is understood in the sense of tempered distributions.

In [1] there are expansions of 1, $\delta(x)$, $\operatorname{sgn} x$ and $1/x$. Furthermore, in [2], [3] and [4], $1/|x|$, x^k for $k = 0, 1, \ldots, \ln|x|$ and $\delta^{(k)}(x)$ are expanded into Hermite series.

In this paper we expand x_+^p into the Hermite series. Here, by the symbol x_+^p we mean the function defined by the formula

$$x_+^p = \begin{cases} x^p & \text{for } x > 0, \\ 0 & \text{for } x \leqslant 0. \end{cases}$$

We note that x_+^p is a tempered distributions for any real $p \geqslant 0$.

2. In this section we are concered with finding the Hermite coefficient $a_{p,n}$ of x_+^p. In this case we have:

$$a_{p,n} = (x_+^p, h_n) = \int_0^{\infty} x^p h_n(x)\, dx.$$

We are going to prove the following recurrence equation:

$$(1) \qquad \sqrt{(n+1)(n+2)}\; a_{p,n+2} - (2p+1)a_{p,n} - \sqrt{n(n-1)}\; a_{p,n-2} = 0$$

for $n = 0, 1, 2, \ldots$ To obtain (1) we need the following formulae:

$$xh_n = \sqrt{n+1}\; h_{n+1} + \sqrt{n}\; h_{n-1}, \tag{2}$$

and

$$-2h'_n = \sqrt{n+1}\; h_{n+1} - \sqrt{n}\; h_{n-1}, \tag{3}$$

for $n = 0, 1, 2, \ldots$ (see [1]). According to [1], we assume that if one of the factors of the product ab is 0, then the product is taken to be 0, even if the second factor is not defined. Multiplying (2) by x^p and integrating over $(0, \infty)$ we get

$$a_{p+1,n} = \sqrt{n+1}\; a_{p,n+1} + \sqrt{n}\; a_{p,n-1}. \tag{4}$$

Similarly, multiplying (3) by x^{p+1} and integrating over $(0, \infty)$, we get after simple calculations

$$2(p+1)a_{p,n} = \sqrt{n+1}\; a_{p+1,n+1} - \sqrt{n}\; a_{p+1,n-1}. \tag{5}$$

Equalites (4) and (5) hold for $n = 0, 1, 2, \ldots$ From (4) have:

$$a_{p+1,n+1} = \sqrt{n+2}\; a_{p,n+2} + \sqrt{n+1}\; a_{p,n} \tag{6}$$

and

$$a_{p+1,n-1} = \sqrt{n}\; a_{p,n} + \sqrt{n-1}\; a_{p,n-2}. \tag{7}$$

From (6), (7) and (5) we obtain (1).

To apply formula (1) we need to know $a_{p,0}$ and $a_{p,1}$. By simple calculations we get

$$a_{p,0} = \left(\sqrt{2\pi}\right)^{-1/2} 2^p\, \Gamma\left(\frac{p+1}{2}\right). \tag{8}$$

By (4) we have $a_{p+1}, 0 = a_{p,1}$. Hence

$$a_{p,1} = \left(\sqrt{2\pi}\right)^{-1/2} 2^{p+1}\, \Gamma\left(\frac{p+2}{2}\right).$$

Instead of using formula (1) as it stands, we simplify notation by introducing polynomials $W_n(x)$ such that $W_0(x) = W_1(x) = 1$ and

$$W_{n+2}|x| = x\, W_n(x) + n(n-1)\, W_{n-2}(x), \tag{9}$$

for $n = 0, 1, 2, \ldots$ We note that $W_2(x) = x$, $W_3(x) = x$, $W_4(x) = x^2+2$, $W_5(x) = x^2+6, \ldots$ Generally, for even n, $W_n(x)$ are of degree $n/2$ and, for odd n, $W_n(x)$ are of degree $(n-1)/2$. By using the polynomials $W_n(x)$, $a_{p,n}$ can be expressed as follows:

$$a_{p,n} = \left(\sqrt{2\pi}\; n!\right)^{-1/2} 2^p\, \Gamma\left(\frac{p+1}{2}\right) W_n(2p+1) \tag{10}$$

for n even, and

$$a_{p,n} = \left(\sqrt{2\pi}\; n!\right)^{-1/2} 2^{p+1}\, \Gamma\left(\frac{p+2}{2}\right) W_n(2p+1) \tag{11}$$

for n odd.

We shall prove (10) by induction. A proof of (11) is similar and is omitted here. Instead of (10) we write equivalenty:

$$a_{p,2k} = C_p\big((2k)!\big)^{-1/2}W_{2k}(2p+1), \tag{10'}$$

where $C_p = \big(\sqrt{2\pi}\big)^{-1/2}2^p\Gamma\left(\dfrac{p+1}{2}\right)$ and $k = 0,1,2,\ldots$ From (9) we see that (10′) holds for $k=0$ and from (1) and (8) we get that (10′) holds for $k=1$. Assume that (10′) holds for l, with $1 < l \leqslant k$. By (1) we get:

$$\sqrt{(2k+1)(2k+2)}\;a_{p,2(k+1)} = (2p+1)a_{p,2k}+\sqrt{2k(2k-1)}\;a_{p,2(k-1)}.$$

Hence, by the induction assumption we obtain after simple calculations

$$\sqrt{(2k+1)(2k+2)}\;a_{p,2(k+1)} = C_p\big((2k)!\big)^{-1/2}(2p+1)\,W_{2k}(2p+1)+ \\ +2k(2k-1)\,W_{2k-2}(2p+1). \tag{12}$$

By (9) we get

$$W_{2(k+1)}(2p+1) = (2p+1)\,W_{2k}(2p+1)+2k(2k-1)\,W_{2k-2}(2p+1).$$

Hence and from (12) we have

$$a_{p,2(k+1)} = C_p\big((2k+2)!\big)^{-1/2}\,W_{2(k+1)}(2p+1).$$

Thus, by induction formula (10′) holds for $k = 0,1,2,\ldots$ This completes the proof of (10).

In this way we have proved the folloving:

THEOREM. *For any $p \geqslant 0$ the Hermite coefficients $a_{p,n}$ for the function x_+^p are given by the formulae*

$$a_{p,n} = \big(\sqrt{2\pi}\,n!\big)^{-1/2}2^p\Gamma\left(\frac{p+1}{2}\right)W_n(2p+1),$$

for n even and

$$a_{p,n} = \big(\sqrt{2\pi}\,n!\big)^{-1/2}2^{p+1}\Gamma\left(\frac{p+2}{2}\right)W_n(2p+1),$$

for n odd, where $W_n(x)$ are polynomials such that $W_0(x) = W_1(x) = 1$ and

$$W_{n+2}(x) = xW_n(x)+n(n-1)\,W_{n-2}(x)$$

for $n = 0,1,2,\ldots$

Applying the theorem we shall find the Hermite coefficient when $p=0$ and $p=\frac{1}{2}$.

Note that if $p=0$, then x_+^p is the Heaviside function $H(x)$. We have

$$H(x) = \sqrt[4]{\frac{\pi}{2}}\left(h_0 + \frac{1}{\sqrt{2!}}h_2 + \frac{3}{\sqrt{4!}}h_4 + \ldots\right) + \\ + \frac{2}{\sqrt[4]{2\pi}}\left(h_1 + \frac{1}{\sqrt{3!}}h_3 + \frac{7}{\sqrt{5!}}h_5 + \ldots\right),$$

$$\sqrt{x} = \sqrt[4]{\frac{\pi}{2}}\,\Gamma(\tfrac{3}{4})\left(h_0 + \frac{2}{\sqrt{2!}}h_2 + \frac{6}{\sqrt{4!}}h_4 + \ldots\right) + \\ + \sqrt[4]{\frac{\pi}{2}}\,\tfrac{1}{2}\Gamma(\tfrac{1}{4})\left(h_1 + \frac{2}{\sqrt{3!}}h_3 + \frac{10}{\sqrt{5!}}h_5 + \ldots\right);$$

here we assume that $\sqrt{x} = 0$ for $x < 0$.

References

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