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Remarks on minimax solutions

by V. LAKSHMIKANTHAM and S. LEELA (Calgary, Canada) *

1. A classical result of Perron is that the maximum solution of a scalar differential equation can be obtained as the least upper bound of the family of under-functions $m(t)$ which satisfy the inequality $m'(t) \leqslant g(t, m(t))$ and the same initial condition.

Similar arguments hold for infinite systems of differential inequalities, provided the maximum solution for a single equation is already known [3] and an abstract extension of this method has been given in [2].

The existence of minimax solutions for finite systems of differential equations has been considered in [1]. In this note we modify the abstract approach of [2] to suit the situation of minimax solutions. As an application, we deduce the existence of minimax solutions for an infinite system of differential equations, thus extending the results of [1].

2. Let E_1 and F_1 be two partially ordered sets with the partial ordering $\leqslant$. We use the same symbol of order relation viz. '$\leqslant$' for both the sets. Assume that the following conditions hold:

(2.1) $$x, y, z \in E_1, \quad x \leqslant y, \; y \leqslant z \quad \text{imply} \quad x \leqslant z;$$

(2.2) $$x, y \in E_1, \quad x \leqslant y, \; y \leqslant x \quad \text{imply} \quad x = y;$$

(2.3) $$\bar{x}, \bar{y}, \bar{z} \in F_1, \quad \bar{x} \leqslant \bar{y}, \; \bar{y} \leqslant \bar{z} \quad \text{imply} \quad \bar{x} \leqslant \bar{z};$$

(2.4) $$\bar{x} \in F_1, \quad \text{then} \quad \bar{x} \leqslant \bar{x}.$$

Corresponding to the sets E_1, F_1, let us consider E_2, F_2, two partially ordered sets with the dual order relation, denoted by the symbol $\geqslant$ and satisfying conditions (2.1*) to (2.4*) analogous to (2.1) to (2.4). We shall use u, v, w and $\bar{u}, \bar{v}, \bar{w}$ to denote elements belonging to E_2 and F_2 respectively.

Let the operators P_1, P_2 be defined on E_1, E_2 taking values in F_1, F_2 respectively. Let the functions Q_1, Q_2 be defined on $E_1 \times E_1 \times E_2$, $E_1 \times E_2 \times E_2$ taking values in F_1, F_2 respectively.

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Consider the simultaneous equations

$$(2.5)\qquad \begin{aligned} P_1(x) &= Q_1(x, x, u)\,, \\ P_2(u) &= Q_2(x, u, u)\,. \end{aligned}$$

By a *solution* ω of (2.5) we shall mean the ordered pair (x, u), $x \in E_1$, $u \in E_2$ which satisfies the equation (2.5) simultaneously.

A solution $r = (\xi, \eta)$ of (2.5) is said to be a *minimax solution* if, for every solution $\omega = (x, u)$ of (2.5), the relations

$$x \leqslant \xi, \qquad u \geqslant \eta$$

are satisfied.

We shall say that Q_1, Q_2 possess a *mixed quasimonotone property* when the following conditions are valid: $y_1, y_2 \in E_1$, $y_1 \leqslant y_2$ imply that

$$(2.6)\qquad Q_1(x, y_1, u) \leqslant Q_1(x, y_2, u), \qquad x \in E_1,\ u \in E_2;$$

$$(2.7)\qquad Q_2(y_1, u, v) \geqslant Q_2(y_2, u, v), \qquad u, v \in E_2;$$

Also for $u_1, u_2 \in E_2$, $u_1 \geqslant u_2$ imply that

$$(2.8)\qquad Q_1(x, y, u_1) \leqslant Q_1(x, y, u_2), \qquad x, y \in E_1;$$

$$(2.9)\qquad Q_2(y, u_1, v) \geqslant Q_2(y, u_2, v), \qquad y \in E_1,\ v \in E_2\,.$$

We define the sets

$$(2.10)\qquad U_1 = [x \in E_1\colon\ P_1(x) \leqslant Q_1(x, x, u),\ u \in E_2];$$

$$(2.11)\qquad U_2 = [u \in E_2\colon\ P_2(u) \geqslant Q_2(x, u, u),\ x \in E_1]\,.$$

We now give below the theorem which proves the existence of the minimax solution of the equations (2.5).

THEOREM 1. *Let P_1, P_2, Q_1, Q_2 be as defined above. Suppose that Q_1, Q_2 have the mixed quasimonotone property. Assume further that there exist two functions Z_1, Z_2 defined on E_1, E_2 such that $Z_1(E_1) \subset E_1$, $Z_2(E_2) \subset E_2$, satisfying the conditions:*

$$(2.12)\qquad \begin{aligned} P_1\big(Z_1(x)\big) &= Q_1\big(Z_1(x), x, u\big), \qquad u \in E_2\,, \\ P_2\big(Z_2(u)\big) &= Q_2\big(x, u, Z_2(u)\big), \qquad x \in E_1; \end{aligned}$$

and

$$(2.13)\qquad \begin{aligned} P_1(x) &\leqslant Q_1(x, y, u), \qquad y \in E_1\,, \\ P_2(u) &\geqslant Q_2(y, v, u), \qquad v \in E_2; \end{aligned}$$

together imply that

$$x \leqslant Z_1(y), \qquad u \geqslant Z_2(v)\,.$$

Let the sets U_1, U_2 defined in (2.10) *and* (2.11) *be non-empty. Then,*

$$(2.14)\qquad Z_1(E_1) \subset U_1, \qquad Z_2(E_2) \subset U_2\,.$$

Moreover, the existence of $(\sup U_1, \inf U_2)$ *implies the existence of* $\big(\sup Z_1(U_1), \inf Z_2(U_2)\big)$ *and vice versa. Also,* $\sup U_1 = \sup Z_1(U_1)$, $\inf U_2 = \inf Z_2(U_2)$ *and*

$$r = (\sup U_1, \inf U_2)$$

is the minimax solution of (2.5).

We wish to reduce our theorem to Theorem 1 of [2] with the following interpretation. Consider the set $E = E_1 \times E_2$ in which the partial ordering is as follows:

Let $\xi = (x, u)$, $\eta = (y, v)$ belong to E, where $x, y \in E_1$ and $u, v \in E_2$. Then, $\xi \leqslant \eta$ implies $x \leqslant y$ and $u \geqslant v$.

Let $P(\xi) = \big(P_1(x), P_2(u)\big)$ and $Q(\xi, \eta) = \big(Q_1(x, y, v), Q_2(y, v, u)\big)$, where P and Q are defined on E and $E \times E$ respectively, taking values in $F = F_1 \times F_2$. The partial ordering for the set F is the same as for E.

Defining the function $Z(\xi) = \big(Z_1(x), Z_2(u)\big)$ and the set $U = [\xi \in E\colon P(\xi) \leqslant Q(\xi, \xi)]$, we find that the assumptions of Theorem 1 imply that P, Q defined as above satisfy the requirements of Theorem 1 of [2] and the maximal solution of $P(\xi) = Q(\xi, \xi)$ is the minimax solution of (2.5). The theorem is proved.

3. In this section, we give a more general theorem on minimax solutions. As an application of Theorem 1, it is natural to consider the existence of the minimax solution for the system

$$\begin{aligned} u_i' &= f_i(t, u_1, u_2, \ldots, u_p;\ v_1, v_2, \ldots, v_q), \quad & u_i(t_0) &= u_i^0, \quad & 1 \leqslant i \leqslant p\,, \\ v_j' &= g_j(t, u_1, u_2, \ldots, u_p;\ v_1, v_2, \ldots, v_q), \quad & v_j(t_0) &= v_j^0, \quad & 1 \leqslant j \leqslant q\,, \end{aligned} \tag{3.1}$$

where p, q are arbitrary (may be infinite). The existence of minimax solutions for finite systems of differential equations follows when p, q are both finite. The minimax solutions for infinite systems are covered by the other choices of p and q (either p or q, or both may be infinite). In case p, q are both infinite, the functions f_i and g_j in (3.1) are to be interpreted as $f_i(t, u_1, u_2, \ldots; v_1, v_2, \ldots)$ and $g_j(t, u_1, u_2, \ldots; v_1, v_2, \ldots)$ respectively.

Let the functions f_i and g_j satisfy the following assumptions:

(3.2) $f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ and $g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ are defined for $t \in [t_0, t_0 + a]$ and arbitrary $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$;

(3.3) there exist constants M_i and M_j such that

$$\left.\begin{aligned} |f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)| &\leqslant M_i\,, \\ |g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)| &\leqslant M_j\,, \end{aligned}\right. \qquad t \in [t_0, t_0 + a]\,,$$

for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$;

(3.4) the functions f_i and g_j are continuous in the sense, that

$$t^\nu \to t, \quad u_i^\nu \to u_i, \quad v_j^\nu \to v_j \quad (1 \leqslant i \leqslant p, \quad 1 \leqslant j \leqslant q)$$

imply

$$f_i(t^\nu, u_1^\nu, u_2^\nu, \ldots, u_p^\nu; v_1^\nu, v_2^\nu, \ldots, v_q^\nu) \to f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q),$$

$$g_j(t^\nu, u_1^\nu, u_2^\nu, \ldots, u_p^\nu; v_1^\nu, v_2^\nu, \ldots, v_q^\nu) \to g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q);$$

(3.5) the functions f_i and g_j possess mixed quasimonotone property, viz., for each $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$,

(i) $f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ is monotonic non-decreasing in $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_p$ and monotonic non-increasing in $v_1, v_2, \ldots, v_q$;

(ii) $g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ is monotonic non-increasing in $u_1, u_2, \ldots, u_p$ and monotonic non-decreasing in $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_q$.

Now, we have the following:

THEOREM 2. *Assume that the functions f_i and g_j satisfy the conditions* (3.2) *to* (3.5). *Then, there exists a minimax solution $(u_i^*(t), v_j^*(t))$ of* (3.1) *on* $[t_0, t_0 + a]$. *Further, if $m_i(t), n_j(t)$ are continuous functions defined on $[t_0, t_0 + a]$ satisfying the inequalities*

$$m_i(t_0) \leqslant u_i^0, \qquad n_j(t_0) \geqslant v_j^0;$$

$$D^+ m_i(t) \leqslant f_i\big(t, m_1(t), m_2(t), \ldots, m_p(t); n_1(t), n_2(t), \ldots, n_q(t)\big);$$

$$D^+ n_j(t) \geqslant g_j\big(t, m_1(t), m_2(t), \ldots, m_p(t); n_1(t), n_2(t), \ldots, n_q(t)\big),$$

then,

$$m_i(t) \leqslant u_i^*(t), \qquad n_j(t) \geqslant v_j^*(t), \qquad t \in [t_0, t_0 + a].$$

Proof. Let $\{\varphi_i(t)\}, \{\psi_j(t)\}$ be two sequences of continuous functions on $[t_0, t_0 + a]$ such that

$$\begin{aligned} \varphi_i(t) &\leqslant u_i^0 + M_i(t - t_0), \quad & i = 1, 2, \ldots, p, \\ \psi_j(t) &\leqslant v_j^0 + M_j(t - t_0), \quad & j = 1, 2, \ldots, q, \end{aligned} \qquad t \in [t_0, t_0 + a].$$

Denote

$$\{\varphi_i(t)\} = \big(\varphi_1(t), \varphi_2(t), \ldots, \varphi_p(t)\big) = y,$$

$$\{\psi_j(t)\} = \big(\psi_1(t), \psi_2(t), \ldots, \psi_q(t)\big) = v.$$

Let E_1, E_2 stand for systems of sequences $y = \{\varphi_i(t)\}$, $v = \{\psi_j(t)\}$. If $y_1 = \{\varphi_i^{(1)}(t)\}$, $y_2 = \{\varphi_i^{(2)}(t)\}$ are two sequences such that if $y_1, y_2 \in E_1$, then $y_1 \leqslant y_2$ implies $\varphi_i^{(1)}(t) \leqslant \varphi_i^{(2)}(t)$ on $[t_0, t_0 + a]$ and for each i. Similarly for $v_1, v_2 \in E_2$, $v_1 \geqslant v_2$ means that $\psi_j^{(1)}(t) \geqslant \psi_j^{(2)}(t)$ on $[t_0, t_0 + a]$ and for each j.

Let F_1 and F_2 stand for the systems of sequences of continuous functions on $[t_0, t_0+a]$ which take values in the real extended line, the order relations in F_1, F_2 being the same as those in E_1, E_2 respectively. One can verify that the conditions (2.1) to (2.4) and (2.1*) to (2.4*) are satisfied. Let us now define the operators P_1, P_2 and the functions Q_1, Q_2 as follows:

$$P_1(y) = \{D^+\varphi_i(t)\}; \qquad P_2(v) = \{D^+\psi_j(t)\};$$

$$Q_1(x, y, v) = f_i\big(t, \varphi_1(t), \ldots, \varphi_{i-1}(t), x_i(t), \varphi_{i+1}(t), \ldots, \varphi_p(t); \psi_1(t), \ldots, \psi_q(t)\big);$$

$$Q_2(y, v, u) = g_j\big(t, \varphi_1(t), \ldots, \varphi_p(t); \psi_1(t), \ldots, \psi_{j-1}(t), u_j(t), \psi_{j+1}(t), \ldots, \psi_q(t)\big).$$

Clearly the functions Q_1, Q_2 satisfy the mixed quasimonotone property.

For any pair of sequences $\{\varphi_i(t)\}, \{\psi_j(t)\}$, define

$$\bar{f}_i(t, r) = f_i\big(t, \varphi_1(t), \ldots, \varphi_{i-1}(t), r_i, \varphi_{i+1}(t), \ldots, \varphi_p(t); \psi_1(t), \ldots, \psi_q(t)\big);$$

$$\bar{g}_j(t, \varrho) = g_j\big(t, \varphi_1(t), \ldots, \varphi_p(t); \psi_1(t), \ldots, \psi_{j-1}(t), \varrho_j, \psi_{j+1}(t), \ldots, \psi_q(t)\big).$$

Let, for each i, $r_i(t)$ be the maximal solution of

$$r' = \bar{f}_i(t, r), \qquad r_i(t_0) = u_i^0,$$

and, for each j, $\varrho_j(t)$ be the minimal solution of

$$\varrho' = \bar{g}_j(t, \varrho), \qquad \varrho_j(t_0) = v_j^0.$$

Since the functions f_i and g_j satisfy (3.3), the existence of $r_i(t)$ and $\varrho_j(t)$ on $[t_0, t_0+a]$ is ensured. Let us now define the functions Z_1, Z_2 by

$$Z_1(\{\varphi_i(t)\}) = \{r_i(t)\}, \qquad Z_2(\{\psi_j(t)\}) = \{\varrho_j(t)\}.$$

From the basic theorem on differential inequalities, it follows that the functions Z_1, Z_2 satisfy (2.13). Moreover, the sets U_1, U_2 are non-empty since $u_i^0 - M_i(t-t_0) \in U_1$ and $v_j^0 + M_j(t-t_0) \in U_2$, because of (3.3). Further,

$$|r_i'(t) \leqslant M_i \quad \text{and} \quad |\varrho_j'(t)| \leqslant M_j.$$

Therefore, the family of functions $\{r_i(t)\}, \{\varrho_j(t)\}$ are equicontinuous and uniformly bounded. This proves that

$$\sup Z_1(U_1) = \{\sup r_i(t)\}, \qquad \inf Z_2(U_2) = \{\inf \varrho_j(t)\}$$

are continuous on $[t_0, t_0+a]$.

The assertion of Theorem 2 now follows from Theorem 1.

References

[1] L. P. Burton and W. M. Whyburn, *Minimax solutions of ordinary differential systems*, Proc. Amer. Math. Soc. 3 (1952), pp. 794-803.

[2] W. Mlak, *Note on maximal solutions of differential equations*, Contributions to Differential Equations, 1 (4) (1963), pp. 461-465.

[3] — and C. Olech, *Integration of infinite systems of differential inequalities*, Ann. Polon. Math. 13 (1963), pp. 105-112.

UNIVERSITY OF CALGARY
CALGARY, ALBERTA, CANADA

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