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On the existence and uniqueness of the solution of a non-linear functional equation of r-th order

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Abstract. In the paper the functional equation

$$x(t) = f\big(t, x(\beta_1(t)), \ldots, x(\beta_r(t))\big)$$

is disscused. An unknown function x is defined in a metric space and takes the value in another complete metric space. The method of successive approximations is used. Under some assumptions concerning the Lipschitz coefficients of function f and the functions β_i the existence and uniqueness of solution in the suitable class of functions is established.

In the present note we consider a non-linear functional equation of the form

$$x(t) = f\big(t, x(\beta_1(t)), \ldots, x(\beta_r(t))\big) \tag{1}$$

with an unknown function x defined in a metric space (M_1, ϱ_1) having the range in a complete metric space (M_2, ϱ_2). The special cases of equation (1) were considered by several authors: see [1]–[8]. In the general case the existence and uniqueness problem for equation (1) was disscused in [7]. There the comparative method was used and the comparative equation was solved by the method of iteration. This method permits us to establish the general results concerning the problem in question but they are somewhat hidden behind the rather complicated formulas.

In the present note we prove by the Banach fixed-point theorem some results concerning the existence and uniqueness problem for equation (1). This result is more general then the corresponding results established in [1], [4], [5], [6], [8].

1. Let (M_i, ϱ_i), $i = 1, 2$, be arbitrarily fixed metric spaces and let (M_2, ϱ_2) be complete. We make

ASSUMPTION A. *Suppose that*:

1° $f\colon M_1 \times M_2^r \to M_2$, $\beta_i\colon M_1 \to M_1$, $i = 1, 2, \ldots, r$, *where r is a fixed positive integer number* (the case $r = +\infty$ is not excluded), 2° *there exist* $t_0 \in M_1$, *a function* $x_0\colon M_1 \to M_2$, $p, H \in R_+ \overset{\text{def}}{=} [0, +\infty)$ *such that*

$$\varrho_2\big(F(x_0)(t), x_0(t)\big) \leqslant H\varrho_1^p(t_0, t), \qquad t \in M_1, \tag{2}$$

where $F(x)(t)$ *denotes the right-hand side of equation* (1),

3° *there exist functions* $l_i\colon M_1 \to R_+$ *such that*

$$\varrho_2\big(f(t, x_1, \ldots, x_r), f(t, \bar{x}_1, \ldots, \bar{x}_r)\big) \leqslant \sum_{i=1}^{r} l_i(t)\,\varrho_2(x_i; \bar{x}_i) \tag{3}$$

for any $t \in M_1$ *and* $x_i, \bar{x}_i \in M_2$, $i = 1, 2, \ldots, r$.

Using the notation introduced, we define the set V_p of functions x defined in M_1 with range in M_2 such that $x \in V_p$ iff there exist $c_x \in R_+$ and

$$\varrho_2\big(x(t), x_0(t)\big) \leqslant c_x\varrho_1^p(t_0, t), \qquad t \in M_1.$$

In V_p we define the metric ϱ: for $x, y \in V_p$ we put

$$\varrho(x, y) = \inf\big\{c\colon\ \varrho_2\big(x(t), y(t)\big) \leqslant c\varrho_1^p(t_0, t),\ t \in M_1\big\}.$$

It is clear that (V_p, ϱ) is a complete metric space.

2. Now we have

THEOREM A. *If Assumption* A *is satisfied and*

$$\sup_{t \in M_2 \setminus \{t_0\}} \sum_{i=1}^{r} l_i(t)\,\frac{\varrho_1^p\big(t_0, \beta_i(t)\big)}{\varrho_1^p(t_0, t)} \overset{\text{def}}{=} a < 1, \tag{4}$$

then there exists in V_p *a unique solution* $\bar{x}$ *of equation* (1) *and it can be obtained by the method of successive approximations.*

Proof. First we observe that $F(V_p) \subset V_p$. Indeed by (2), (3) and the definition of V_p we have for $x \in V_p$

$$\begin{aligned}
\varrho_2\big(x_0(t), F(x)(t)\big) &\leqslant \varrho_2\big(x_0(t), F(x_0)(t)\big) + \varrho_2\big(F(x_0)(t), F(x)(t)\big)\\
&\leqslant H\varrho_1^p(t_0, t) + \sum_{i=1}^{r} l_i(t)\,\varrho_2\Big(x_0\big(\beta_i(t)\big), x\big(\beta_i(t)\big)\Big)\\
&\leqslant H\varrho_1^p(t_0, t) + \sum_{i=1}^{r} l_i(t)\,c\varrho_1^p\big(t_0, \beta_i(t)\big)\\
&\leqslant H\varrho_1^p(t_0, t) + \left(\sup_{t \in M_2 \setminus \{t_0\}} \sum_{i=1}^{r} l_i(t)\,\frac{\varrho_1^p\big(t_0, \beta_i(t)\big)}{\varrho_1^p(t_0, t)}\right) \times\\
&\qquad \times c\varrho_1^p(t_0, t)\\
&\leqslant (H + ac)\,\varrho_1^p(t_0, t).
\end{aligned}$$

Further we prove that F is a contraction in V_p. For $x, y \in V_p$ there exist $c \in R_+$ such that

$$\varrho_2\big(x(t), y(t)\big) \leqslant c\varrho_1^p(t_0, t), \qquad t \in M_1;$$

thus for any such c we get

$$\varrho_2\big(F(x)(t), F(y)(t)\big) \leqslant \sum_{i=1}^{r} l_i(t)\,\varrho_2\Big(x\big(\beta_i(t)\big), y\big(\beta_i(t)\big)\Big) \leqslant c\sum_{i=1}^{r} l_i(t)\,\varrho_1^p\big(t_0, \beta_i(t)\big)$$
$$\leqslant \Big(\sup_{t \in M_1 \setminus \{t_0\}} \sum_{i=1}^{r} l_i(t)\,\frac{\varrho_1^p\big(t_0, \beta_i(t)\big)}{\varrho_1^p(t_0, t)}\Big)\, c\varrho_1^p(t_0, t)$$
$$= ac\varrho_1^p(t_0, t);$$

hence in view of the definition of the metric in V_p we obtain

$$\varrho\big(F(x), F(y)\big) \leqslant a\varrho(x, y);$$

now the assertion of the theorem is implied by the Banach fixed point theorem.

3. In order to formulate another theorem we introduce

ASSUMPTION B. *Suppose that Assumption* A *is satisfied and there exist constants* $\mu \in R$, ν, L_i, $B_i \in R_+$, $i = 1, \ldots, r$, *such that*

$$l_i(t) \leqslant L_i \varrho_1^\mu(t_0, t), \qquad t \in M_1,\ i = 1, \ldots, r, \tag{5}$$

$$\varrho_1\big(t_0, \beta_i(t)\big) \leqslant B_i \varrho_1^\nu(t_0, t), \qquad t \in M_1,\ i = 1, \ldots, r. \tag{6}$$

From Theorem A we infer

THEOREM B. *If Assumption* B *is satisfied,* $\mu + \nu p = p$ *and*

$$\sum_{i=1}^{r} L_i B_i^p < 1, \tag{7}$$

then there exists in V_p *a unique solution* $\bar{x}$ *of equation* (1).

Proof. By assumptions (5)–(7) we have

$$\sup_{t \in M_1 \setminus \{t_0\}} \sum_{i=1}^{r} l_i(t)\,\frac{\varrho_1^p\big(t_0, \beta_i(t)\big)}{\varrho_1^p(t_0, t)} \leqslant \sup_{t \in M_1 \setminus \{t_0\}} \sum_{i=1}^{r} L_i \varrho_1^\mu(t_0, t)\,\frac{B_i^p \varrho_1^{\nu p}(t_0, t)}{\varrho_1^p(t_0, t)}$$
$$= \sum_{i=1}^{r} L_i B_i^p < 1;$$

now the assertion of the theorem results by Theorem A.

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