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Oscillation theorems and nodes of eigenfunctions of certain differential equations of the fourth order

by JAN BOCHENEK (Kraków)

Introduction. Let G be a bounded Jordan-measurable domain in the space E^m of m variables $X = (x_1, \ldots, x_m)$ which can be approximated by an increasing sequence of domains G_n with regular boundaries (i.e., the boundary ∂G_n of G_n is a surface of class C^1_σ; for the definition of the surface of class C^1_σ see [6], p. 132). We do not require any regularity properties of the boundary of G.

We shall consider a differential equation of the form

$$\mathscr{E}(u) - \mu u = 0, \tag{1}$$

where $\mathscr{E}(u)$ is a differential operator of the form $\mathscr{E}(u) = L_1[L_0(u)]$ and the operators $L_k(\varphi)$ $(k = 0, 1)$ are selfadjoint differential operators, i.e.,

$$L_k(\varphi) = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a^k_{ij}(X)\frac{\partial \varphi}{\partial x_j}\right] + q^k(X)\varphi \qquad (k = 0, 1),$$

μ being a real parameter. We make the following assumptions: $a^k_{ij}(X) = a^k_{ji}(X)$ $(i, j = 1, \ldots, m)$ are of class C^{3-2k} in $\bar{G}$ $(k = 0, 1)$, $q^k(X) \geqslant 0$ are of class C^{2-2k} in $\bar{G}$ $(k = 0, 1)$, and quadratic forms $\sum_{i,j=1}^{m} a^k_{ij}(X)\xi_i\xi_j$ $(k = 0, 1)$ are positive definite in $\bar{G}$.

We shall also consider the generalized boundary condition (cf. [1], [3]) which in the case where the boundary ∂G is regular may be written in the form

$$R_k(\varphi^k) = 0 \quad \text{on } \partial G \quad (k = 0, 1); \quad \varphi^0(X) = u(X), \quad \varphi^1(X) = L_0(u), \tag{2}$$

where $R_k(u) = 0$ on ∂G means

$$\frac{du}{d\nu_k} - h^k(X)u = 0 \text{ on } \partial G - \Gamma_k, \quad u = 0 \text{ on } \Gamma_k \ (k = 0, 1) \tag{3}$$

and Γ_k denote $(m-1)$-dimensional parts of ∂G (Γ_k being connected or not); in extreme cases Γ_k may be the whole boundary of G or the empty set. Here $h^k(X)$ $(k = 0, 1)$ are non-negative continuous functions in $\bar{G}$, and $d\varphi/d\nu_k$ $(k = 0, 1)$ are the transversal derivatives of φ with respect to the operators L_k $(k = 0, 1)$, respectively, i.e.,

$$\frac{d\varphi}{d\nu_k} = \sum_{i,j=1}^{m} a_{ij}^k(X) \frac{\partial\varphi}{\partial x_i} \cos(n, x_j) \qquad (k = 0, 1),$$

n being the interior normal to ∂G.

1. The oscillation properties of solutions of problem (1), (2). We make the following assumption:

ASSUMPTION A. *No solution $u(X)$ of equation* (1) *can vanish identically in any subdomain of domain G if $u(X) \not\equiv 0$ in G.*

Let $u(X) \in C^4(G) \cap L^2(G)$ be a solution of problem (1), (2) with $\mu \geqslant 0$, and $u(X) \not\equiv 0$ in G. Let us write

$$G^+ = \{X\colon\ X \in G,\ u(X) > 0\},$$
$$G^- = \{X\colon\ X \in G,\ u(X) < 0\},$$
$$G^0 = \{X\colon\ X \in G,\ u(X) = 0\}.$$

We shall prove the following

THEOREM 1. *If Γ_1 is not the empty set or if the function $h^1(X) > 0$ in $\bar{G}$, then every neighbourhood of any point $X_0 \in G^0$ contains points of G^+ and of $\bar{G}^-$, or G^0 is empty.*

Proof. Let us suppose that G^0 and G^+ are not empty sets, and let

$$U(X) = \begin{cases} u(X) & \text{for } X \in G^+, \\ 0 & \text{for } X \in \bar{G} - \bar{G}^+. \end{cases}$$

It is evident that the function $U(X)$ satisfies equation (1) for $X \in G^+$, i.e.,

$$(4) \qquad \mathscr{E}(U) - \mu U = 0 \quad \text{for } X \in G^+.$$

Since $U(X) \in L^2(G) \cap C(G)$, equation (4) may be written in the form

$$(5) \qquad \mathscr{E}(U) - \mu L_1[K(U)] = 0,$$

where K is the inverse operator for the restriction of operator L_1 to the function space $\mathscr{F}_{h^1, \Gamma_i}(G)$ [1]. The existence of operator K follows from the assumptions of Theorem 1 (cf. [3]). Equation (5) takes the form

$$L_1[L_0(U) - \mu K(U)] = 0.$$

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[1] For the definition of the function space $\mathscr{F}_{h^1, \Gamma_i}(G)$ see [1].

From this follows

$$L_0(U)-\mu K(U) = 0 \quad \text{for } X \in G^+. \tag{6}$$

As we know (cf. [3]), $K(U) \geqslant 0$ for $X \in G$. Since $\mu \geqslant 0$, if we write equation (6) in the form

$$L_0(U) = \mu K(U), \tag{7}$$

then we see that the right-hand side of (7) is non-negative.

Suppose that $X_0 \in G^0$, i.e. $U(X_0) = 0$, and that in some neighbourhood of X_0 all points different from X_0 belong to G^+. Since $U(X) > 0$, it follows that the function $U(X)$ attains its infimum equal to zero. But this is at variance with the well-known theorem of E. Hopf (cf. [5]). The case where in some neighbourhood of X_0 all points different from X_0 belong to G^- is reduced to the previous one by change of sing of the function $u(X)$.

Under the assumptions of Theorem 1 and by this theorem we have the following statements:

COROLLARY 1. *The set G^0 does not contain isolated points.*

COROLLARY 2. *The set G^0 divides the domain G.*

COROLLARY 3. *Every point $X_0 \in G^0$ is the "osicillation point" of $u(X)$, i.e., in every neighbourhood of X_0 the function $u(X)$ changes its sign.*

A simple consequence of Corollary 2 is

COROLLARY 4. *The dimension of G^0 is smaller by one than the dimension of G, i.e.,* $\dim G^0 = m-1$.

2. Eigenvalues and eigenfunctions of problem (1), (2). The object of the following considerations are some properties of zero points (nodes) of the eigenfunctions of problem (1), (2).

DEFINITION. We shall say that a real number λ is an *eigenvalue of problem* (1), (2) if there exists a function $u(X) \not\equiv 0$, $u(X) \in \mathscr{L}^2(G) \cap C^4(G)$ and satisfying the boundary condition (2) (in a generalized sense) and equation (1) with $\mu = \lambda$. This function $u(X)$ we shall call the *eigenfunction of problem* (1), (2) corresponding to the eigenvalue λ.

Besides problem (1), (2) defined in the introduction, we shall consider the equation

$$L_0(u)-\mu K(u) = 0 \tag{8}$$

and the boundary condition

$$R_0(u) = 0 \quad \text{on } \partial G, \tag{9}$$

where K in equation (8) is the operator defined in Theorem 1.

The properties of the operator K are discussed in paper [3].

The eigenvalues and eigenfunctions of problem (8), (9) are defined as in paper [2].

To avoid any mistake we shortly recall this definition. To this end we write

$$(10)\quad D(\varphi,\psi) = \int_G \Big[\sum_{i,j=1}^{m} a_{ij}^0(X)\frac{\partial\varphi}{\partial x_i}\frac{\partial\psi}{\partial x_j} + q^0(X)\varphi\psi\Big]dX + \int_{\partial G-\Gamma_0} h^0\varphi\psi\,dS,$$

$$(11)\quad H(\varphi,\psi) = \int_G \varphi K(\psi)\,dX = \big(\varphi, K(\psi)\big).$$

The bilinear forms (10) and (11) are defined in the space $\mathscr{D}$ (for the definition of the space $\mathscr{D}$ see [1]) and have all the fundamental properties mentioned in [1].

The first eigenvalue λ_1 of problem (8), (9) is defined as (comp. [1], [2], [4])

$$(12)\quad \lambda_1 = \min_{\varphi\in\mathring{\mathscr{D}}}\frac{D(\varphi)}{H(\varphi)},$$

where $\mathring{\mathscr{D}}$ is the subclass of $\mathscr{D}$ of functions φ such that $\varphi = 0$ on Γ_0 (in the generalized sense), and the first eigenfunction $u_1(X)$ is that φ at which the minimum (12) is attained.

Having defined the eigenvalues $\lambda_1,\dots,\lambda_n$ and the corresponding eigenfunctions $u_1,\dots,u_n$, we put

$$(13)\quad \lambda_{n+1} = \min_{\varphi\in\mathscr{K}_n}\frac{D(\varphi)}{H(\varphi)},$$

where $\mathscr{K}_n$ is the subclass of $\mathring{\mathscr{D}}$ consisting of the functions φ satisfying the orthogonality conditions

$$(14)\quad H(\varphi, u_i) = 0,\qquad i = 1,\dots,n,$$

and $u_{n+1}(X)$ is that $\varphi\in\mathscr{K}_n$ at which the minimum (13) is attained.

We shall need the following assumption:

HYPOTHESIS Z. *Given* (8) *and* (9) *there exists a sequence of eigenvalues of* (8), (9)

$$(15)\quad 0\leqslant\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\dots$$

and a corresponding sequence of eigenfunctions

$$(16)\quad u_1(X),\ u_2(X),\ u_3(X),\ \dots$$

which belong to $\mathscr{F}$.

In the sequel we shall use the following formula:

$$(17)\quad D(\varphi,\psi) - \int_G L_0(\varphi)\psi\,dX + \int_{\partial G-\Gamma_0}\psi\Big(\frac{d\varphi}{d\nu_0} - h^0\varphi\Big)dS = 0$$

for every $\varphi \in \mathscr{F}$ and $\psi \in \mathring{\mathscr{D}}$. The proof of formula (17) is quite similar to that of an analogous formula in [1]; see also [2] ([2]).

LEMMA 1. *The function $u(X) \not\equiv 0$ in G and $u(X) \in \mathscr{L}^2(G) \cap C^4(G)$ is an eigenfunction of problem* (1), (2) *corresponding to the eigenvalue λ if and only if the function $u(X)$ is an eigenfunction of problem* (8), (9) *corresponding to the eigenvalue λ.*

The proof of Lemma 1 is given in [3].

Lemma 1 and Hypothesis Z imply the following

COROLLARY 5. *If the functions of sequence* (16) *are of class $C^4(G)$, then these functions are eigenfunctions of problem* (1), (2) *corresponding to the eigenvalues of sequence* (15).

Under some additional assumptions sequence (16) is a complete system in $\mathscr{L}^2(G)$, and then sequence (15) contains all the eigenvalues of problem (1), (2) (comp. [3]).

It follows from the definition (variational) of $u_n(X)$, $n = 1, 2, 3, \ldots$, that

$$H(u_i, u_j) \begin{cases} = 0 & \text{if } i \neq j, \\ \neq 0 & \text{if } i = j, \end{cases} \quad i, j = 1, 2, 3, \ldots \tag{18}$$

On the other hand, as we proved in [3], the first eigenfunction $u_1(X)$ of problem (1), (2) does not vanish at any point of domain G. From this and from (18) it follows that all eigenfunctions $u_n(X)$ $(n \geqslant 2)$ change their sign in G. And since the functions are continuous, there exist zero sets in G for these functions. Because every eigenfunction $u_n(X)$ $(n \geqslant 2)$ satisfies equation (1) with $\mu = \lambda_n \geqslant 0$, it follows by Theorem 1 that for $n \geqslant 2$ the zero set G_n^0 of eigenfunction $u_n(X)$ (the nodes of the eigenfunction $u_n(X)$) has all the properties mentioned in corollaries 1, 2, 3 and 4, provided that Γ_1 is non-void or $h^1(X) > 0$ in $\bar{G}$.

In virtue of Corollary 2 the set G_n^0 $(n \geqslant 2)$ divides G into the so-called nodal domains of the eigenfunction $u_n(X)$.

3. Estimation of the number of nodal domains of the n-th eigenfunction of problem (1), (2). Under the assumption that Γ_1 is non-void or $h^1(X) > 0$ in $\bar{G}$, we have shown in section 2 that the nodes of the eigenfunction $u_n(X)$ $(n \geqslant 2)$ divide G into nodal domains. In this section we shall estimate the number of nodal domains of the function $u_n(X)$ depending on n, provided that Γ_1 is non-void or $h^1(X) > 0$ in $\bar{G}$.

At first we shall prove the following

LEMMA 2. *Under Hypothesis* Z, *the nodes of any function $u(X) \not\equiv 0$*

([2]) For the definitions of the function spaces $\mathscr{D}$, $\mathring{\mathscr{D}}$, $\mathscr{F}$ and $\mathscr{F}_{h,\Gamma}(G)$ see [1] and [2].

in G and $u(X) \in \mathscr{L}^2(G) \cap C^4(G)$ satisfying (1) *with $\mu < \lambda_n$ and boundary condition* (2) *divide G into less than n nodal domains.*

Proof. It follows from Lemma 1 that the function $u(X)$ satisfies equation (8) with $\mu < \lambda_n$ and $u(X) \in \mathscr{F}_{h^0, \Gamma_0}(G)$. Suppose the nodes of $u(X)$ divide G into subdomains $G_1, G_2, \ldots, G_n$. Put

$$(19) \qquad U_i = \begin{cases} u(X) & \text{in } G_i, \\ 0 & \text{in } \bar{G} - \bar{G}_i, \end{cases} \qquad i = 1, \ldots, n.$$

It is obvious that $U_1, \ldots, U_n$ are linearly independent in G. Put

$$F(X) = \sum_{i=1}^{n} a_i U_i,$$

where $a_1, \ldots, a_n$ are real numbers such that $a_1^2 + \ldots + a_n^2 > 0$, and the function $F(X)$ is orthogonal to $u_1, \ldots, u_{n-1}$ with respect to the functional H. The function $F(X)$ belongs to $\mathscr{K}_n$; thus by (13)

$$(20) \qquad D(F) \geqslant \lambda_n H(F).$$

On the other hand, each function U_i $(i = 1, \ldots, n)$ satisfies the equation

$$(21) \qquad L_0(U_i) = \mu K(U_i) \quad \text{for } X \in G_i, \ i = 1, \ldots, n.$$

Multiplying both sides of equation (21) by U_j $(1 \leqslant j \leqslant n)$ and then integrating over G, we have

$$(22) \qquad D(U_i, U_j) = \mu H(U_i, U_j), \quad i, j = 1, \ldots, n.$$

Further, multiplying both sides of (22) by $a_i a_j$ and summing, we get

$$(23) \qquad D(F) = \mu H(F).$$

In virtue of (20) and (23) we have

$$\lambda_n H(F) \leqslant D(F) = \mu H(F).$$

Thus $\lambda_n \leqslant \mu$, which contradicts our assumption.

LEMMA 3. *Under assumption* Z, *if $\lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+s-1} < \lambda_{n+s}$ (i.e. λ_n is an s-fold eigenvalue of* (1), (2)), *then the nodes of each function $u(X) \not\equiv 0$ in G and $u(X) \in \mathscr{F}_{h^0, \Gamma_0}(G) \cap C^4(G)$ satisfying* (1) *with $\mu = \lambda_n$ divide G into at most n domains.*

The proof of this lemma is quite similar to the proof of an analogous lemma in [1] and is omitted.

Lemmas 2 and 3 imply the following

THEOREM 2. *Under Assumption* Z, *if $N(n)$ denotes the number of* *nodal domains of the n-th eigenfunction of problem* (1), (2), *then for each n we have*

$$N(n) \leqslant n, \tag{24}$$

the equality occurring only when $\lambda_{n-1} < \lambda_n$ (see [1]).

THEOREM 3. *Under Assumption* Z, *if* $\lambda_i < \lambda_j$, *then in each nodal subdomain of* $u_i(X)$ *there are nodes of the eigenfunction* $u_j(X)$.

Proof. Let G_i be an arbitrary fixed nodal domain of $u_i(X)$, and let

$$U_i = \begin{cases} u_i & \text{for } X \in G_i, \\ 0 & \text{for } X \in \bar{G} - \bar{G}_i. \end{cases}$$

The function U_i satisfies equation (8) with $\mu = \lambda_i$ in the subdomain G_i, i.e.,

$$L_0(U_i) = \lambda K(U_i), \qquad X \in G_i. \tag{25}$$

Multiplying both sides of equation (25) by u_j and then integrating both sides of this equation over G_i, we get

$$D_i(U_i, u_j) = \lambda_i H_i(u_j, U_i) - \int_{\Gamma_i} \frac{dU_i}{d\nu_0} u_j dS, \tag{26}$$

where D_i and H_i are defined as bilinear forms D and H, respectively, by integration over the subdomain G_i, while Γ_i denotes that part of the boundary of G_i which is composed of nodes of u_i.

On the other hand, by applying (17) to u_j, U_j, we have

$$D(u_j, U_i) = \lambda_j H(U_i, u_j).$$

Since $U_i \equiv 0$ in $\bar{G} - \bar{G}_i$, the last equality takes the form

$$D_i(u_j, U_i) = \lambda_j H_i(U_i, u_j). \tag{27}$$

It is evident that $D_i(u_j, U_i) = D_i(U_i, u_j)$, whence by (27) we have $H_i(u_j, U_i) = H_i(U_i, u_j)$. From this and from (26) and (27) we get

$$(\lambda_j - \lambda_i) H_i(U_i, u_j) = -\int_{\Gamma_i} u_j \frac{dU_i}{d\nu_0} dS. \tag{28}$$

Suppose $u_i > 0$ and $u_j > 0$ in G_i. Then the left-hand side of (28) is positive. On the other hand, the transversal goes into the interior of G_i ([6], p. 163), and so $dU_i/d\nu_0 = du_i/d\nu_0 \geqslant 0$ on Γ_i. Thus the right-hand side of (28) is non-positive, whence a contradiction. Analogously, we arive at a contradiction when assuming any other possible combination of the signs of u_i and u_j. Therefore u_j has to change its sign in G_i.

Remark 1. All the results of this paper may be generalized without essential changes to the case of a more general equation of the form

$$L_1^p[L_0(u)] - \mu u = 0, \tag{29}$$

where p is any integer and L_0, L_1 are the operators defined in the introduction to this paper, with the boundary conditions

$$R_0(u) = 0 \text{ on } \partial G, \quad R_1(\varphi^k) = 0 \text{ on } \partial G \quad (k = 1, \ldots, p), \tag{30}$$

where $\varphi^k = L_0(\varphi^{k-1})$, $k = 1, \ldots, p$, $\varphi^0 = u$.

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