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On some condition for a submanifold of Euclidean space to be a sphere

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Abstract. For a connected submanifold N of a connected Riemannian manifold M the following conditions are considered:

(I) $d_M(p,q) = d_M(p_1,q_1)$ for any $p,q,p_1,q_1 \in N$ such that $d_N(p,q) = d_N(p_1,q_1)$,

(II) for any $p,q,p_1,q_1 \in N$, $d_M(p,q) = d_M(p_1,q_1)$ if and only if $d_N(p,q) = d_N(p_1,q_1)$,

where d_M and d_N denote the distance functions for M and N, respectively.

The following statements are proved:

1. If (I) holds for a compact submanifold N of co-dimension one in a Euclidean space M, then N is a sphere.

2. Consider the n-dimensional real projective space P^n with the Riemannian metric induced by the projection $S^n \to P^n$. Then P^n can be embedded isometrically into the Euclidean space E^r, $r = \frac{1}{2}n(n+3)$, as a submanifold satisfying (II). This submanifold is contained in some sphere in E^r and all its geodesics are flat circles of the same radius in E^r.

The examples given in the paper show that (I) and (II) are not equivalent and that there exists a wide class of totally geodesic submanifolds in Riemannian manifolds with property (II).

1. Introduction. Let M be a connected Riemannian manifold and N its connected submanifold. We denote the distances in M and N by d_M and d_N, respectively.

(i) The pair (M, N) is said *to have property* (I) if for any $p, q, p_1, q_1 \in N$ such that $d_N(p,q) = d_N(p_1,q_1)$ we have also $d_M(p,q) = d_M(p_1,q_1)$.

(ii) The pair (M, N) is said *to have property* (II) if for any $p, q, p_1, q_1 \in N$, $d_N(p,q) = d_N(p_1,q_1)$ if any only if $d_M(p,q) = d_M(p_1,q_1)$.

The above conditions are obviously satisfied for an $(n-1)$-dimensional sphere contained in the n-dimensional Euclidean space. It will be shown in Section 2 that a compact and connected submanifold of co-dimension one in a Euclidean space with property (I) is isometric to a sphere, namely it is a sphere. The examples given in Section 3 show that conditions (I) and (II) are not equivalent and that there exist totally geodesic, compact and connected submanifolds of arbitrary co-dimension in Riemannian manifolds which have property (II) and are not isometric to a sphere. Section 3 contains also the proof of the theorem stating that the n-dimensional projective space ($n > 1$) can be embedded in the $\frac{1}{2}n(n+3)$-dimensional Euclidean space and in the $[\frac{1}{2}n(n+3)-1]$-dimensional sphere as a compact submanifold with property (II) which is neither totally geodesic nor isometric to a sphere.

2. A condition for a compact submanifold of co-dimension one in a Euclidean space to be a sphere. An affine space A over a finite-dimensional real vector space V is a set A with a free and transitive action on A of the additive group of V. For affine spaces A_i over V_i ($i = 1, 2$) a mapping F: $A_1 \to A_2$ is called *affine* if there exists a linear mapping f: $V_1 \to V_2$ such that $F(a+v) = F(a)+f(v)$ for $a \in A_1$, $v \in V_1$. The mapping f is uniquely determined by F and is called its *linear part*. An affine space A over V is called *Euclidean* if a scalar product is chosen for V. In this case A becomes a metric space, the distance of $p, q \in A$ being given by $|q-p|$, where $q-p$ is the unique $v \in V$ with $q = p+v$. We shall use the obvious equality

$$2u\cdot v = |u|^2+|v|^2-|u-v|^2 \tag{1}$$

for a scalar product and the induced norm.

We shall need the following

LEMMA 1. *If A is a Euclidean affine space over V, X_1 and X_2 are subsets of A and h is an isometry of X_1 onto X_2, then h can be extended to an affine isometry of A onto itself.*

Proof. Let A_i over V_i be the affine subspace of A spanned by X_i ($i = 1, 2$). Choose $p_0, \ldots, p_k \in X_1$ such that $p_1-p_0, \ldots, p_k-p_0$ form a basis of V_1 and let H be the affine mapping of A_1 into A_2 with a linear part f such that $H(p_0) = h(p_0)$ and $f(p_j-p_0) = h(p_j)-h(p_0)$, $j = 1, \ldots, k$. Clearly $H(p_j) = h(p_j)$, $j = 1, \ldots, k$. The mapping f preserves the norms of the vectors of the above basis of V_1 and the norms of their differences, and so in view of (1) it preserves their scalar products and hence all scalar products in V_1. Thus $\dim V_1 \leqslant \dim V_2$. The same reasoning for h^{-1}: $X_2 \to X_1$ shows that also $\dim V_2 \leqslant \dim V_1$; hence H is an affine isometry of A_1 onto A_2. For any $p \in X_1$ it follows from (1) that the vectors $h(p)-h(p_0)$ and $H(p)-h(p_0)$ give the same scalar product with $h(p_j)-h(p_0)$, $j = 1, \ldots, k$; hence their difference $h(p)-H(p)$ is orthogonal to each vector of a basis of V_2 and $h(p) = H(p)$. Thus H is an extension of h. By completing some orthonormal bases $e_1, \ldots, e_k$ of V_1 and $f(e_1), \ldots, f(e_k)$ of V_2 to orthonormal bases of V we can easily extend h to an affine isometry of A onto itself. Q.E.D.

Using Lemma 1, we shall prove

PROPOSITION 1. *If N is a compact, connected, $(n-1)$-dimensional submanifold of class C^2 of the n-dimensional Euclidean space E^n such that the pair (E^n, N) has property* (I), *then N is a sphere.*

Proof. Let L_i be geodesics of N with arc length parameter, s_i real numbers $(i = 1, 2)$. Choose $r > 0$ such that L_i restricted to the segment $[s_i - r, s_i + r]$ is minimal, $i = 1, 2$. The assignment

$$L_1([s_1-r, s_1+r]) \ni L_1(s_1+h) \to L_2(s_2+h) \in L_2([s_2-r, s_2+r])$$

is an isometry of $L_1([s_1-r, s_1+r])$ onto $L_2([s_2-r, s_2+r])$ in the sense of d_N, since $d_N\big(L_2(s_2+h_1), \ L_2(s_2+h_2)\big) = |h_1-h_2| = d_N\big(L_1(s_1+h_1), L_1(s_1+h_2)\big)$ for $|h_1|, |h_2| \leqslant r$. In view of (I) this is also an isometry of these sets in the sense of Euclidean distance.

By Lemma 1 we can find a linear orthogonal mapping F: $E^n \to E^n$ and a vector $a \in E^n$ such that $L_2(s_2+h) = F\big(L_1(s_1+h)\big)+a$ for $|h| \leqslant r$. Hence $|\ddot{L}_2(s_2)| = |\ddot{L}_1(s_1)|$. Since the numbers s_i and geodesics L_i have been arbitrary, we see that the curvature of each geodesic of N is constant and equal for all geodesics. At any point of N the normal curvatures in all directions are equal, since they are just the curvatures of the corresponding geodesics. Hence each point of N is umbilic. In virtue of Theorem 2.1 of [2], p. 128, N is a sphere (the proof given there for $n = 3$ is valid for arbitrary n). Q.E.D.

3. Some remarks about submanifolds satisfying condition (II) in the case of arbitrary co-dimension. Condition (II) of Section 1 obviously implies condition (I). The following example shows that the inverse implication fails in general.

EXAMPLE 1. Define an analytic mapping f: $E^1 \to E^4$ by

$$f(t) = \left(\frac{1}{\sqrt{5}}\cos t, \frac{1}{\sqrt{5}}\sin t, \frac{1}{\sqrt{5}}\cos 2t, \frac{1}{\sqrt{5}}\sin 2t\right).$$

For each t we have $f(t+2\pi) = f(t)$, $|df/dt| = 1$, $|f(t)|^2 = \frac{2}{5}$, and f restricted to $[0, 2\pi)$ is one-one, and so f defines an isometric embedding of the unit circle into E^4 and into the sphere S^3 in E^4 with centre at the origin and radius $\sqrt{\frac{2}{5}}$. Let $N = f(E^1)$. The distance in N of two points of N is the smallest number of the form $|t_1 - t_2|$, where $f(t_1)$ and $f(t_2)$ are those points. The obvious equality

$$|f(t_1)-f(t_2)|^2 = \tfrac{4}{5} - \tfrac{2}{5}\cos|t_1-t_2| - \tfrac{2}{5}\cos 2|t_1-t_2|$$

shows that $|p-q|^2 = \frac{4}{5} - \frac{2}{5}\cos d_N(p, q) - \frac{2}{5}\cos 2d_N(p, q)$ for $p, q \in N$; hence the pair (E^4, N) has property (I). Now define the function H: $[0, \pi] \to E^1$

by $H(r) = \frac{4}{5} - \frac{2}{5}\cos r - \frac{2}{5}\cos 2r$. We have

$$|p-q|^2 = H\big(d_N(p,q)\big)$$

for $p, q \in N$. To prove that the pair (E^4, N) does not satisfy condition (II) it is sufficient to show that H is not one-one. It is easy to verify that $H(\frac{1}{2}\pi) = \frac{6}{5} = H(\frac{2}{3}\pi)$. Since the pair (E^4, S^3) has property (II) and N is a submanifold of S^3, it follows that also the pair (S^3, N) satisfies condition (I), but not (II).

The next example shows that even property (II) for a pair (M, N), where N is a compact submanifold of positive co-dimension in M, does not in general imply that N is isometric to a sphere. Namely, there is a wide class of totally geodesic submanifolds with this property.

EXAMPLE 2. Let M_1 and M_2 be connected Riemannian manifolds and $q \in M_2$. Then $M_1 \times \{q\}$ is a submanifold of $M_1 \times M_2$ in a natural manner. Let $(p_1, q), (p_2, q) \in M_1 \times \{q\}$ and a curve L join them in $M_1 \times M_2$, where $L(s) = \big(L_1(s), L_2(s)\big)$. Let $L_0(s) = \big(L_1(s), q\big)$. The curve L_0 joins (p_1, q) to (p_2, q) in $M_1 \times \{q\}$ and the canonical orthogonal decomposition $\dot{L}(s) = \dot{L}_1(s) + \dot{L}_2(s)$ shows that $|\dot{L}_0(s)| = |\dot{L}_1(s)| \leqslant |\dot{L}(s)|$. Hence the distance of (p_1, q) and (p_2, q) in $M_1 \times \{q\}$ is equal to their distance in $M_1 \times M_2$ and also equal to the distance of p_1 and p_2 in M_1. Thus $M_1 \times \{q\}$ is a totally geodesic submanifold of $M_1 \times M_2$, isometric to M_1, and the pair $(M_1 \times M_2, M_1 \times \{q\})$ has property (II). The manifold $M_1 \times \{q\}$ may be chosen quite arbitrarily; in particular, it need not be homeomorphic to a sphere.

Now we shall prove that the assumption of co-dimension one in Proposition 1 cannot be omitted. First we define a mapping G and prove three lemmas.

We fix a positive integer n and set $r = \frac{1}{2}n(n+3)$. Using the equality $r = \binom{n+1}{2} + n$, we shall index the first $\binom{n+1}{2}$ coordinates in the Euclidean space E^r by pairs of integers (i, j), where $1 \leqslant i < j \leqslant n+1$, and the last n coordinates in E^r by integers k, $1 \leqslant k \leqslant n$. Using this convention, we define an analytic mapping $G\colon E^{n+1} \to E^r$ by

$$G(x^1, \ldots, x^{n+1}) = \begin{cases} \dfrac{x^i x^j}{2}, & 1 \leqslant i < j \leqslant n+1, \\[2ex] \dfrac{(x^1)^2 + \ldots + (x^k)^2 - k(x^{k+1})^2}{2\sqrt{2k(k+1)}}, & 1 \leqslant k \leqslant n. \end{cases}$$

LEMMA 2. *If $x, y \in E^{n+1}$ and $|x| = |y|$, then $G(x) = G(y)$ if and only if $x = y$ or $x = -y$.*

Proof. Let $x = (x^1, \ldots, x^{n+1})$, $y = (y^1, \ldots, y^{n+1})$, $G(x) = G(y)$ and $(x^1)^2 + \ldots + (x^{n+1})^2 = (y^1)^2 + \ldots + (y^{n+1})^2$. By comparing the last coordinate in $G(x)$ and $G(y)$, we see that $(x^{n+1})^2 = (y^{n+1})^2$ and $(x^1)^2 + \ldots + (x^n)^2 = (y^1)^2 + \ldots + (y^n)^2$. Furthermore, the comparison of the last but one coordinate in $G(x)$ and $G(y)$ gives $(x^n)^2 = (y^n)^2$ and $(x^1)^2 + \ldots + (x^{n-1})^2 = (y^1)^2 + \ldots + (y^{n-1})^2$. Continuing this reasoning, we obtain $|x^i| = |y^i|$, $x^i x^j = y^i y^j$, $1 \leqslant i \leqslant j \leqslant n+1$, which immediately implies our assertion. Q.E.D.

LEMMA 3. *G maps each circle in E^{n+1} with centre at the origin and radius 2 onto a circle in E^r with radius 1, passed twice, i.e., for any $a, b \in E^{n+1}$ with $|a| = |b| = 2$ and $a \cdot b = 0$ there exist $c, v, w \in E^r$ with $|v| = |w| = 1$ and $v \cdot w = 0$ such that for any real number t we have*

$$G(\cos t \cdot a + \sin t \cdot b) = c + \cos 2t \cdot v + \sin 2t \cdot w. \tag{2}$$

Proof. Let $a = (a^1, \ldots, a^{n+1})$, $b = (b^1, \ldots, b^{n+1})$. We have

$$(a^1)^2 + \ldots + (a^{n+1})^2 = (b^1)^2 + \ldots + (b^{n+1})^2 = 4, \qquad a^1 b^1 + \ldots + a^{n+1} b^{n+1} = 0. \tag{3}$$

Let us set

$$c = \begin{cases} \dfrac{a^i a^j + b^i b^j}{4}, & 1 \leqslant i < j \leqslant n+1, \\[2ex] \dfrac{(a^1)^2 + \ldots + (a^k)^2 - k(a^{k+1})^2 + (b^1)^2 + \ldots + (b^k)^2 - k(b^{k+1})^2}{4\sqrt{2k(k+1)}}, & 1 \leqslant k \leqslant n, \end{cases}$$

$$v = \begin{cases} \dfrac{a^i a^j - b^i b^j}{4}, & 1 \leqslant i < j \leqslant n+1, \\[2ex] \dfrac{(a^1)^2 + \ldots + (a^k)^2 - k(a^{k+1})^2 - (b^1)^2 - \ldots - (b^k)^2 + k(b^{k+1})^2}{4\sqrt{2k(k+1)}}, & 1 \leqslant k \leqslant n, \end{cases}$$

$$w = \begin{cases} \dfrac{a^i b^j + a^j b^i}{4}, & 1 \leqslant i < j \leqslant n+1, \\[2ex] \dfrac{a^1 b^1 + \ldots + a^k b^k - k a^{k+1} b^{k+1}}{2\sqrt{2k(k+1)}}, & 1 \leqslant k \leqslant n. \end{cases}$$

Equality (2) follows immediately from the definition of G and from the identities $1 + \cos 2t = 2\cos^2 t$, $1 - \cos 2t = 2\sin^2 t$, $\sin 2t = 2\cos t \cdot \sin t$. To compute $|v|$, $|w|$ and $v \cdot w$ we shall make use of the definition of Euclidean norm and scalar product, collecting terms of the same type together and using the obvious equality

$$\sum_{s=k}^{n} \frac{1}{s(s+1)} = \sum_{s=k}^{n} \left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{1}{k} - \frac{1}{n+1}.$$

Thus we have

$$|v|^2 = \sum_{i<j} [(a^i)^2(a^j)^2 + (b^i)^2(b^j)^2]\left[\frac{1}{16} - \frac{j-1}{16j(j-1)} + \sum_{s=j}^{n} \frac{1}{16s(s+1)}\right] +$$

$$+ \sum_{k=1}^{n+1} [(a^k)^4 + (b^k)^4]\left[\frac{(k-1)^2}{32k(k-1)} + \sum_{s=k}^{n} \frac{1}{32s(s+1)}\right] - \sum_{i<j} \frac{a^i b^i a^j b^j}{8} -$$

$$- \sum_{k=1}^{n+1} (a^k)^2(b^k)^2\left[\frac{(k-1)^2}{16k(k-1)} + \sum_{s=k}^{n} \frac{1}{16s(s+1)}\right] -$$

$$- \sum_{i<j} (a^i)^2(b^j)^2\left[\frac{-(j-1)}{16j(j-1)} + \sum_{s=j}^{n} \frac{1}{16s(s+1)}\right] -$$

$$- \sum_{j<i} (a^i)^2(b^j)^2\left[\frac{-(i-1)}{16i(i-1)} + \sum_{s=i}^{n} \frac{1}{16s(s+1)}\right]$$

$$= \sum_{i<j} \frac{n}{16(n+1)} [(a^i)^2(a^j)^2 + (b^i)^2(b^j)^2] + \sum_{k=1}^{n+1} \frac{n}{32(n+1)} [(a^k)^4 + (b^k)^4] -$$

$$- \sum_{i<j} \frac{a^i b^i a^j b^j}{8} - \sum_{k=1}^{n+1} \frac{n}{16(n+1)} (a^k)^2(b^k)^2 + \sum_{i\neq j} \frac{(a^i)^2(b^j)^2}{16(n+1)}$$

$$= \frac{n}{32(n+1)} \sum_{i,j=1}^{n+1} [(a^i)^2(a^j)^2 + (b^i)^2(b^j)^2] - \sum_{i\neq j} \frac{a^i b^i a^j b^j}{16} -$$

$$- \sum_{k=1}^{n+1} \frac{(a^k)^2(b^k)^2}{16} + \sum_{k=1}^{n+1} \frac{(a^k)^2(b^k)^2}{16(n+1)} + \sum_{i\neq j} \frac{(a^i)^2(b^j)^2}{16(n+1)}$$

$$= \frac{n}{32(n+1)} \{[(a^1)^2 + \ldots + (a^{n+1})^2]^2 + [(b^1)^2 + \ldots + (b^{n+1})^2]^2\} -$$

$$- \frac{a^1 b^1 + \ldots + a^{n+1} b^{n+1}}{16} +$$

$$+ \frac{1}{16(n+1)} [(a^1)^2 + \ldots + (a^{n+1})^2][(b^1)^2 + \ldots + (b^{n+1})^2]$$

so $|v|^2 = 1$ in view of (3). In the same manner we compute

$$|w|^2 = \sum_{i\neq j} \frac{(a^i)^2(b^j)^2}{16} + \sum_{k=1}^{n+1} (a^k)^2(b^k)^2\left[\frac{(k-1)^2}{8k(k-1)} + \sum_{s=k}^{n} \frac{1}{8s(s+1)}\right] +$$

$$+ \sum_{i<j} a^i b^i a^j b^j \left[\frac{1}{8} + \frac{-(j-1)}{4j(j-1)} + \sum_{s=j}^{n} \frac{1}{4s(s+1)}\right]$$

$$= \sum_{i\neq j} \frac{(a^i)^2(b^j)^2}{16} + \sum_{k=1}^{n+1} \frac{n}{8(n+1)}(a^k)^2(b^k)^2 +$$

$$+ \sum_{i<j} \frac{n-1}{8(n+1)} a^i b^i a^j b^j$$

$$= \sum_{i\neq j} \frac{(a^i)^2(b^j)^2}{16} + \sum_{k=1}^{n+1} \frac{(a^k)^2(b^k)^2}{16} +$$

$$+ \sum_{k=1}^{n+1} \frac{n-1}{16(n+1)}(a^k)^2(b^k)^2 + \sum_{i\neq j} \frac{n-1}{16(n+1)} a^i b^i a^j b^j$$

$$= \tfrac{1}{16}[(a^1)^2 + \ldots + (a^{n+1})^2][(b^1)^2 + \ldots + (b^{n+1})^2] +$$

$$+ \frac{n-1}{16(n+1)}(a^1 b^1 + \ldots + a^{n+1} b^{n+1})^2;$$

hence $|w|^2 = 1$ in virtue of (3). Similarly we have

$$v\cdot w = \sum_{i<j} [(a^i)^2 - (b^i)^2] a^j b^j \left[\frac{1}{16} - \frac{j-1}{16j(j-1)} + \sum_{s=j}^{n} \frac{1}{16s(s+1)}\right] +$$

$$+ \sum_{i<j} [(a^j)^2 - (b^j)^2] a^i b^i \left[\frac{1}{16} - \frac{j-1}{16j(j-1)} + \sum_{s=j}^{n} \frac{1}{16s(s+1)}\right] +$$

$$+ \sum_{k=1}^{n+1} [(a^k)^2 - (b^k)^2] a^k b^k \left[\frac{(k-1)^2}{16k(k-1)} + \sum_{s=k}^{n} \frac{1}{16s(s+1)}\right]$$

$$= \sum_{i\neq j} \frac{n}{16(n+1)}[(a^i)^2 - (b^i)^2] a^j b^j + \sum_{k=1}^{n+1} \frac{n}{16(n+1)}[(a^k)^2 - (b^k)^2] a^k b^k$$

$$= \frac{n}{16(n+1)}[(a^1)^2 + \ldots + (a^{n+1})^2 - (b^1)^2 - \ldots - (b^{n+1})^2] \times$$

$$\times (a^1 b^1 + \ldots + a^{n+1} b^{n+1}),$$

and from (3) it follows that $v\cdot w = 0$, which completes the proof. Q.E.D.

LEMMA 4. *For each $x\in E^{n+1}$ we have*

$$|G(x)|^2 = \frac{n}{8(n+1)}|x|^4.$$

Proof. Let $x = (x^1, \ldots, x^{n+1})$. Then

$$|G(x)|^2 = \sum_{i<j}(x^i)^2(x^j)^2\left[\frac{1}{4}-\frac{j-1}{4j(j-1)}+\sum_{s=j}^{n}\frac{1}{4s(s+1)}\right]+ \\ +\sum_{k=1}^{n+1}(x^k)^4\left[\frac{(k-1)^2}{8k(k-1)}+\sum_{s=k}^{n}\frac{1}{8s(s+1)}\right]$$

$$= \sum_{i<j}\frac{n}{4(n+1)}(x^i)^2(x^j)^2+\sum_{k=1}^{n+1}\frac{n}{8(n+1)}(x^k)^4$$

$$= \frac{n}{8(n+1)}[(x^1)^2+\ldots+(x^{n+1})^2]^2 = \frac{n}{8(n+1)}|x|^4,$$

as desired. Q.E.D.

Now let S^n denote the sphere in E^{n+1} with centre at the origin and radius 2 and let $P^n = \{\{v, w\} \mid v, w\in S^n, v+w = 0\}$ be the projective space obtained from S^n by identifying pairs of antipodes. Let p be the canonical projection of S^n onto P^n, i.e., $p(v) = \{v, -v\}\in P^n$ for $v\in S^n$. Then p is locally diffeomorphic with respect to the natural analytic structures of S^n and P^n. Moreover, there exists a unique Riemannian metric on P^n such that p is an isometric mapping (the metric transferred by p from S^n onto P^n is well defined since the central symmetry is an isometry of S^n onto itself). This metric will be called the *natural metric* of the projective space.

THEOREM 1. *Let $n > 1$ and $r = \frac{1}{2}n(n+3)$. Then the projective space P^n with its natural metric can be embedded isometrically into E^r and into some $(r-1)$-dimensional sphere contained in E^r as a non-totally geodesic analytic submanifold with property* (II). *All geodesics of this submanifold are circles of the same radius.*

Proof. We shall consider the mapping $G\colon E^{n+1}\to E^r$ restricted to the sphere S^n. The set $N = G(S^n)$ is a compact and connected subset of E^r. In view of Lemma 2 there exists a one-one continuous mapping $\bar{G}$ of P^n onto N such that $\bar{G}\circ p = G$. $\bar{G}$ is a homeomorphism since P^n is compact. The structure transferred by $\bar{G}$ from P^n makes N into an analytic manifold such that $\bar{G}\colon P^n\to N$ is an analytic diffeomorphism. Hence $G = \bar{G}\circ p\colon S^n\to N$ is analytic and locally diffeomorphic. Each vector tangent to S^n is tangent to a great circle, and so in virtue of Lemma 3 the mapping $G\colon S^n \to E^r$ is isometric. Hence the inclusion mapping $i\colon N \to E^r$ is analytic and regular, i.e., N is a compact analytic submanifold of E^r. Moreover, $G\colon S^n \to N$ is isometric and $\bar{G}\colon P^n \to N$ is an isometry, N being considered with the metric induced from E^r. The geodesics of N are just the G-images of the great circles of S^n. By Lemma 3 all geodesics of N are circles with radius 1. In view of Lemma 2 two distinct geodesics of N may have at most one point in common, and so each half-circle with radius 1, which is a geodesic of N, is minimal. Thus the distance in N of two points of N is the length of the shorter arc of any circle with radius 1 joining these points (any pair of points of N can be joined by such a circle). Hence the pair (E^r, N) has property (II). By Lemma 4 N is a submanifold of the sphere S with centre at the origin and radius $\sqrt{\frac{2n}{n+1}}$. The pairs (E^r, S) and (E^r, N) both satisfy condition (II), and hence so does the pair (S, N). N is not totally geodesic either in E^r, as it contains no straight line, or in S, since the geodesics of N and those of S are circles of distinct radii ($\sqrt{\frac{2n}{n+1}} > 1$ for $n > 1$). Q.E.D.

Remark. James in [1] describes, for each n, an embedding of the n-dimensional projective space into E^{2n}, and even into E^{2n-1} whenever n is odd and $n \geqslant 3$. However, his embeddings do not satisfy the assertion of Theorem 1.

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References

[1] I. M. James, *Some embeddings of projective spaces*, Proc. Cambridge Phil. Soc. 55 (1959), p. 294–298.

[2] T. J. Willmore, *An introduction to differential geometry*, Clarendon Press, Oxford 1959.

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