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On the existence of invariant densities for Markov operators

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Abstract. A new sufficient condition for the existence of invariant densities for Markov operator is proved. It generalizes a result of E. Straube [6] concerning the existence of invariant measures for point to point transformations.

The aim of this paper is to give a new sufficient condition for the existence of invariant densities for Markov operators and to show an application to the problem of the existence of invariant measures. We follow the idea of E. Straube [6] but our criterion is valid for arbitrary Markov operators and is not restricted to finite measure spaces. The main results, an existence theorem for Markov operators is proved in Section 1. In Section 2 we show an application concerning a class of piecewise convex transformation of the unit interval [0, 1] into itself. This gives a generalization of the known results of A. Rényi [5], A. Lasota [2], [4] and A. Lasota and J. Yorke [3].

1. Let (X, Σ, m) be a σ-finite measure space. A linear operator P of the space $L_1 = L_1(X, \Sigma, m)$ into itself is called a *Markov operator* if it satisfies the following two conditions:

(a) $$Pf \geqslant 0 \qquad \text{for } f \geqslant 0,\ f \in L_1;$$

(b) $$\|Pf\| = \|f\| \qquad \text{for } f \geqslant 0,\ f \in L_1.$$

Given a Markov operator P, we denote by U the operator adjoint to P. The elements of the set

$$D = \{f \in L_1 \colon f \geqslant 0,\ \|f\| = 1\}$$

will be called *densities*. A density f is called *invariant* if $Pf = f$.

Let a Markov operator P and an element $f \in D$ be given. We say that the pair (P, f) satisfies *condition* (C) if:

(C) There exists a set A with $0 < m(A) < \infty$ and a number $\delta > 0$ such that for every measurable subset E of A

$$\limsup_{k \to \infty} \int_{E \cup (X \setminus A)} P^k f \, dm < 1 \qquad \text{whenever } m(E) < \delta.$$

THEOREM 1. *Let $P\colon L_1 \to L_1$ be a Markov operator. If there exists a density f such that (P, f) satisfies condition* (C), *then P has an invariant density nonvanishing on A.*

Proof. Define

$$\lambda(h) = \operatorname{Lim} \int_X P^k f\, h\, dm \quad \text{for } h \in L_\infty, \tag{1.1}$$

where Lim is a Banach limit [1]. Thus $\lambda \in L_\infty^*$ is positive. Using the classical decomposition theorem for finitely additive measures [7] we can write λ in the form $\lambda = \lambda_m + \lambda_p$, where λ_m is the maximal countably additive linear operator on L_∞ such that $\lambda_m \leqslant \lambda$. Let g be the density of the measure $B \mapsto \lambda_m(\mathbf{1}_B)$; that is,

$$\lambda_m(\mathbf{1}_B) = \int_B g\, dm \quad \text{for } B \in \Sigma.$$

We are going to show that g is a nontrivial fixed point of P. From (1.1) it follows that

$$\lambda_m(Uh) \leqslant \lambda(Uh) = \lambda(h) \quad \text{for } h \in L_\infty.$$

Since $\lambda_m \circ U$ is a countably additive function, we have $\lambda_m \circ U \leqslant \lambda_m$. Moreover,

$$\lambda_m(U\mathbf{1}_A) + \lambda_m(U\mathbf{1}_{X\setminus A}) = \lambda_m(U\mathbf{1}_X) = \lambda_m(\mathbf{1}_X) = \lambda_m(\mathbf{1}_A) + \lambda_m(\mathbf{1}_{X\setminus A})$$

for $A \in \Sigma$ and consequently $\lambda_m \circ U = \lambda_m$. Choose a set $B \in \Sigma$. Then

$$\int_X \mathbf{1}_B Pg\, dm = \int_X U\mathbf{1}_B g\, dm = \lambda_m(U\mathbf{1}_B) = \lambda_m(\mathbf{1}_B) = \int_X g\, \mathbf{1}_B\, dm,$$

which proves that g is a fixed point of P.

Now we are going to prove that $\lambda_m \neq 0$. Assume the contrary. Then $\lambda'(B) = \lambda(\mathbf{1}_B)$, $B \subset A$, $B \in \Sigma$, is a purely finitely additive function of sets; that is, for every measure μ the condition $\mu \leqslant \lambda'$ implies $\mu = 0$. Further, in this case there exists (see [7]) a decreasing sequence of sets $\{E_n\}$ such that $E_n \subset A$, $\lim_{n\to\infty} m(E_n) = 0$ and $\lambda'(E_n) = \lambda'(A)$. Consequently

$$\begin{aligned}\operatorname{Lim} \int_{E_n \cup (X\setminus A)} P^k f\, dm &= \lambda(\mathbf{1}_{E_n \cup (X\setminus A)}) \\ &= \lambda(\mathbf{1}_{E_n}) + \lambda(\mathbf{1}_{X\setminus A}) = \lambda(\mathbf{1}_A) + \lambda(\mathbf{1}_{X\setminus A}) = 1\end{aligned}$$

for all n, which contradicts to condition (C). Thus $g \neq 0$ and the function $g/\|g\|$ is an invariant density for P.

Remark 1. If $m(X) < \infty$, then for every density f condition (C) is equivalent to the following condition:

(C)′ There exists a number $\delta > 0$ such that for every measurable set E

$$\limsup_{k\to\infty} \int_E P^k f\, dm < 1 \quad \text{if} \quad m(E) < \delta.$$

Remark 2. Assume that a Markov operator P has an invariant density g and that f is another density satisfying

$$m(\operatorname{supp} f \cap \operatorname{supp} g) > 0. \tag{1.2}$$

Then (P, f) satisfies condition (C).

Proof. Define $B = \operatorname{supp} f \cap \operatorname{supp} g$. Choose a constant $c > 0$ such that the set

$$G = \{x \in B \colon f(x) \leqslant cg(x)\}$$

has positive measure. Since $\|\mathbf{1}_{X \setminus G} f\| < 1$, there exists a measurable set A such that $m(A) < \infty$ and

$$\alpha := c \int_{X \setminus A} g\, dm + \|\mathbf{1}_{X \setminus G} f\| < 1.$$

Further, since $\alpha < 1$ and g is integrable, there exist two numbers β, $\delta > 0$ such that $\alpha + \beta < 1$ and

$$\int_E g\, dm \leqslant \beta/c \quad \text{if} \quad m(E) < \delta.$$

Now, for every set $E \subset A$ such that $m(E) < \delta$ we have

$$\int_{E \cup (X \setminus A)} P^k f\, dm = \int_{E \cup (X \setminus A)} P^k(\mathbf{1}_{X \setminus G} f)\, dm + \int_{E \cup (X \setminus A)} P^k(\mathbf{1}_G f)\, dm$$
$$\leqslant \|\mathbf{1}_{X \setminus G} f\| + c \int_{E \cup (X \setminus A)} g\, dm \leqslant \alpha + \beta.$$

Since $\alpha + \beta < 1$, this completes the proof.

Corollary 1. *Let $P\colon L_1 \to L_1$ be a Markov operator. Assume that for a certain density f the sequence $\{P^n f\}$ can be written in the form $P^n f = f_n + r_n$, where $\{f_n\}$ is weakly precompact and $\|r_n\| \leqslant \alpha < 1$ for almost all n. Then the operator P has an invariant density.*

Proof. Choose $\varepsilon = (1-\alpha)/3$. Since $\{f_n\}$ is weakly precompact, there exists a set $A \in \Sigma$ with $m(A) < \infty$ and a $\delta > 0$ such that

$$\int_{X \setminus A} f_n\, dm \leqslant \varepsilon \quad \text{and} \quad \int_E f_n\, dm \leqslant \varepsilon \quad \text{if} \quad m(E) < \delta.$$

Now we have

$$\int_{E \cup (X \setminus A)} P^n f\, dm \leqslant \int_E f_n\, dm + \int_{X \setminus A} f_n\, dm + \int_X r_n\, dm \leqslant 2\varepsilon + \alpha < 1$$

whenever $m(E) < \delta$ and $\|r_n\| \leqslant \alpha$. Thus condition (C) is evidently satisfied.

2. Denote by m the Lebesgue measure on the interval $[0, 1]$. A measurable transformation S of $[0, 1]$ into itself is called *nonsingular* if the condition $m(A) = 0$ implies $m(S^{-1}(A)) = 0$. For any nonsingular transformation S we define the Frobenius–Perron operator by the formula

$$Pf(x) = \frac{d}{dx} \int\limits_{S^{-1}([0,x])} f(s)\,ds.$$

It is well known that P is a Markov operator. Moreover, if f is a fixed point of P, then the measure

$$\mu(A) = \int_A f\,dm$$

satisfies the condition $\mu(A) = \mu(S^{-1}(A))$, which means that μ is invariant under S. This property of the Frobenius–Perron operator enables us to use Theorem 1 in proving the existence of invariant measures for point to point transformations.

We say that a function φ: $[a, b] \to R$ is *convex* if it satisfies

$$\varphi(\alpha x + (1-\alpha) y) \leqslant \alpha\varphi(x) + (1-\alpha)\varphi(y)$$

for $x, y \in [a, b]$ and $0 \leqslant \alpha \leqslant 1$.

Let $\{[a_k, b_k]\}_{k=1}^{n,\infty}$ be an at most countable sequence of closed intervals such that

$$a_1 = 0, \quad 0 \leqslant a_k < b_k \leqslant 1, \quad \sum_k (b_k - a_k) = 1$$

and for $j \neq k$ the intersection $(a_j, b_j) \cap (a_k, b_k)$ is empty. Consider a function S: $[0, 1] \to [0, 1]$ and denote by S_k the restriction of S to the interval $[a_k, b_k]$. We will assume that S satisfies the following conditions:

(1) $S(a_k) = 0$;

(2) S_k are convex;

(3) $\lambda = 1/S'(0) < 1$;

(4) There exists a number $d > 0$ such that

$$K = \sum_{j=0}^{\infty} \sum_{k \geqslant 2} (1/a_k)[\psi_k(\lambda^j d) - a_k] < 1,$$

where

$$\psi_k(x) = \begin{cases} S_k^{-1}(x) & \text{for } x \in S_k([a_k, b_k]), \\ b_k & \text{for } x \in [0, 1] \setminus S_k([a_k, b_k]). \end{cases}$$

PROPOSITION 1. *If S: $[0, 1] \to [0, 1]$ satisfies conditions* (1)–(4), *then S has an absolutely continuous normalized invariant measure.*

Proof. The Frobenius–Perron operator corresponding to S can be written in the form

$$Pf(x) = \sum_k \psi_k'(x) f(\psi_k(x)). \tag{2.1}$$

The functions ψ_k are increasing, continuous and differentiable except on at most countable set of points. Denote by M the set of all integrable decreasing functions. Since ψ'_k are decreasing, from (2.1) it follows that $P(M) \subset M$. For $f \in M$, $f \geqslant 0$, $\|f\| = 1$ we have

$$1 \geqslant \int_0^x f(s)\,ds \geqslant x f(x)$$

and consequently $f(x) \leqslant 1/x$. Hence, for $f \in M$ we have

$$\int_0^x Pf(s)\,ds = \int_0^{\psi_1(x)} f(s)\,ds + \sum_{k \geqslant 2} \int_{a_k}^{\psi_k(x)} f(s)\,ds$$

$$\leqslant \int_0^{\lambda x} f(s)\,ds + \sum_{k \geqslant 2} (1/a_k)\,[\psi_k(x) - a_k]$$

and by induction

$$\int_0^d P^n f(s)\,ds \leqslant \int_0^{\lambda^n d} f(s)\,ds + K. \tag{2.2}$$

Define $f_n = \mathbf{1}_{[d,1]} P^n \mathbf{1}$ and $r_n = \mathbf{1}_{[0,d]} P^n \mathbf{1}$. The functions $\{f_n\}$ are bounded by the constant $1/d$, hence the sequence $\{f_n\}$ is weakly precompact. Choose α such that $K < \alpha < 1$. From (2.2), for n large enough, we have

$$\|r_n\| = \int_0^d P^n \mathbf{1}(s)\,ds \leqslant \alpha.$$

In view of Corollary 1 this completes the proof.

Remark 3. Observe that the inequality

$$\sum_k 1/S'(a_k) < \infty \tag{4)'$$

implies condition (4). In fact, we have

$$K = \sum_{j=0}^{\infty} \sum_{k \geqslant 2} (1/a_k)\,[\psi_k(\lambda^j d) - a_k]$$

$$\leqslant (1/b_1) \sum_{k \geqslant 2} \sum_{j=0}^{\infty} \lambda^j d/S'(a_k)$$

$$\leqslant (d/b_1(1-\lambda)) \sum_{k \geqslant 2} 1/S'(a_k),$$

which implies that $K < 1$ for sufficiently small d. In the case when inequality (4)′ is satisfied the existence of an invariant measure for piecewise convex transformations was proved by A. Lasota and J. Yorke [2]–[4]. However, it is easy to construct a simple transformation for which condition (4) is satisfies but (4)′ is not.

EXAMPLE 1. Define S: $[0, 1] \to [0, 1]$ by the formula

$$S(x) = \begin{cases} 2x & \text{for } 0 \leqslant x < 1/2, \\ (x-1/2)^2 & \text{for } 1/2 \leqslant x \leqslant 1. \end{cases}$$

An elementary calculation shows that $K = 2\sqrt{2d}/(\sqrt{2}-1)$ and consequently $K < 1$ for d small enough. From Proposition 1 it follows that the transformation S has an absolutely continuous invariant measure.

References

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