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On a generalization of Simpson's formula

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1. Recently S. Gołąb has proposed and solved certain interesting problems related to the well-known Simpson's formula. The object of this paper is to give a generalization of Simpson's formula by taking as our starting point a mean-value theorem of Cioranescu. We later propose some problems in the light of Gołąb's results and resolve them in the general case providing thereby at the same time a generalization of Gołąb's results.

We know the following formula, due to Cioranescu:

$$\int_a^b f(x)P_n(x)\,dx = \frac{f^{(n)}(\xi)}{(n+1)!}\int_a^b \cdots \int_a^b V_{n+1}^2\,dv_{n+1} \qquad (a<\xi<b), \tag{1}$$

where $P_n(x)$ is a Legendre polynomial over the interval (a, b), $dv_{n+1} = dx_0 dx_1 \dots dx_n$ and V_{n+1} is the Vandermonde's determinant $V_{n+1}(x_0 x_1 \dots x_n) = \operatorname{Det}(x_k \dots x_k^n)$. The problem naturally arises whether we can give an analogue of Simpson's formula for the above integral on the left.

We may take the interval $(0, h)$ instead of (a, b) without loss of generality, and let us write

$$\int_0^h f(x)P_n(x)\,dx = h^{n+1}\left\{\lambda_0 f^{(n)}(0) + \lambda_1 f^{(n)}\left(\frac{h}{2}\right) + \lambda_2 f^{(n)}(h)\right\}, \tag{2}$$

a formula easily suggested by (1). Let

$$f(x) = \sum_{i=0}^{\infty} a_i x^i .$$

Since over the interval $(0, h)$, $P_n(x)$ is given by

$$P_n(x) = \left(\frac{h}{2}\right)^n \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}\left\{\frac{4}{h^2}\left(x-\frac{h}{2}\right)^2 - 1\right\}^n .$$

We can easily find the following result:

$$(3)\qquad \int_0^h x^{n+p}P_n(x)\,dx = \frac{\{\Gamma(n+p+1)\}^2}{\Gamma(p+1)\Gamma(2n+p+2)}\,h^{n+p+1}\,.$$

From this one can easily find the coefficients $\lambda_0, \lambda_1, \lambda_2$ in (2) and we have

$$\lambda_0 = \lambda_2 = \frac{\Gamma(n+2)}{\Gamma(2n+4)} \quad \text{and} \quad \lambda_1 = 4(n+1)\frac{\Gamma(n+2)}{\Gamma(2n+4)}\,.$$

We thus establish the formula

$$(4)\qquad \int_0^h f(x)P_n(x)\,dx = \frac{\Gamma(n+2)}{\Gamma(2n+4)}\,h^{n+1}\left[f^{(n)}(0)+4(n+1)f^{(n)}\left(\frac{h}{2}\right)+f^{(n)}(h)\right]+R(h)\,,$$

where

$$(5)\qquad R(h) = -\frac{n+1}{48}\cdot\frac{\Gamma(n+3)}{\Gamma(2n+6)}\,h^{n+5}f^{(n+4)}(\xi) \qquad (0<\xi<h)\,.$$

For $n=0$, (4) and (5) give Simpson's formula and the remainder thereof.

2. Putting $T(h) = \int_0^h f(x)P_n(x)\,dx$ and

$$T_1(h;\lambda_0,\lambda_1,\lambda_2) = h^{n+1}[\lambda_0 f^{(n)}(0)+\lambda_1 f^{(n)}(\theta h)+\lambda_2 f^{(n)}(h)] \qquad (0<\theta<1)$$

we now propose to determine $\lambda_0, \lambda_1, \lambda_2$ such that $R(h) \equiv T(h) - T_1(h;\lambda_0,\lambda_1,\lambda_2)$ is of the greatest possible order of smallness. Let $f(x)$ be regular in the neighbourhood of the origin having the expansion

$$(\mathrm{A})\qquad f(h) = f(0)+a_1h+\ldots+a_nh^n+a_{n+p}h^{n+p}+ + a_{n+q}h^{n+q}+a_{n+r}h^{n+r}+a_{n+s}h^{n+s}+\ldots$$

where $a_{n+p}, a_{n+q}, a_{n+r}, a_{n+s} \neq 0$ and $1 \leqslant p < q < r < s \ldots$ We easily get the following equations to determine $\lambda_0, \lambda_1, \lambda_2$:

$$(6)\qquad \lambda_0+\lambda_1+\lambda_2 = \frac{\Gamma(n+1)}{\Gamma(2n+2)}\,,$$

$$(7)\qquad \theta^p\lambda_1+\lambda_2 = \frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\,,$$

$$(8)\qquad \theta^q\lambda_1+\lambda_2 = \frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\,.$$

Equations (7) and (8) determine λ_1, λ_2 uniquely as

$$W = \begin{vmatrix} \theta^p & 1 \\ \theta^q & 1 \end{vmatrix} = \theta^p - \theta^q > 0 \qquad (p < q)$$

and λ_0 is then determined by (6). The remainder is then of the order of smallest of h^{r+1+n}.

3. The next question naturally is: *Can we choose θ in such a way that the order of smallness of $R(h)$ is further increased?* In that case we will require, along with (6), (7), and (8), the following relation:

$$\theta^r \lambda_1 + \lambda_2 = \frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}. \tag{9}$$

The condition of consistency of (7), (8), and (9) gives

$$\Delta(\theta) \equiv \begin{vmatrix} \theta^p & 1 & \dfrac{\Gamma(p+n+1)}{\Gamma(p+2n+2)} \\ \theta^q & 1 & \dfrac{\Gamma(q+n+1)}{\Gamma(q+2n+2)} \\ \theta^r & 1 & \dfrac{\Gamma(r+n+1)}{\Gamma(r+2n+2)} \end{vmatrix} = 0. \tag{10}$$

Putting $\alpha = q-p$, $\beta = r-q$, (10) can be written thus:

$$\begin{aligned} \Delta(\theta) \equiv & \left\{\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)} - \frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\right\}\theta^{\alpha+\beta} - \\ & - \left\{\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)} - \frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}\right\}\theta^{\alpha} + \\ & + \left\{\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)} - \frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}\right\} = 0. \end{aligned} \tag{11}$$

Equation (11) is an algebraic equation in θ of degree $\alpha+\beta$. We shall show that *this equation has a unique root θ in the open interval* $(0,1)$. Since

$$\begin{aligned} \Delta'(\theta) = \theta^{\alpha-1}\Bigg[(\alpha+\beta) & \left\{\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)} - \frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\right\}\theta^{\beta} - \\ & - \alpha\left\{\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)} - \frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}\right\}\Bigg], \end{aligned}$$

we have

$$\begin{gathered} \Delta(0) > 0, \quad \Delta(1) = 0 \quad \text{and} \\ \Delta'(1) = \frac{\beta}{(p+n+1)_{n+1}} - \frac{\alpha+\beta}{(q+n+1)_{n+1}} + \frac{\alpha}{(r+n+1)_{n+1}} \end{gathered} \tag{12}$$

where $(p+n+1)_{n+1} = (p+n+1)(p+n+2)\dots(p+2n+1)$.

Now in order to prove that $\Delta'(1) > 0$, we observe that

$$\Delta'(1) = \frac{1}{(r+n+1)_{n+1}}\left\{\beta\left(1+\frac{\alpha+\beta}{p+n+1}\right)\dots\left(1+\frac{\alpha+\beta}{p+2n+1}\right)+\right.$$
$$\left.+\alpha-(\alpha+\beta)\left(1+\frac{\beta}{q+n+1}\right)\dots\left(1+\frac{\beta}{q+2n+1}\right)\right\}$$
$$=\frac{1}{(r+n+1)_{n+1}}[\beta(\alpha+\beta)\{S_1+(\alpha+\beta)S_2+\dots\}-$$
$$-(\alpha+\beta)\beta\{s_1+\beta s_2+\dots\}]$$

where S_ν is the sum of the products, taken ν at a time, of $\frac{1}{p+n+1},\dots,$ $\frac{1}{p+2n+1}$ and s_ν is the sum S_ν where p is replaced by q. Obviously $S_\nu > s_\nu$ since $q > p$, and hence we have $\Delta'(1) > 0$.

From this and from (12) it follows that $\Delta(\theta) < 0$ in a small neighbourhood of $\theta = 1$ on the left. Also the unique root of $\Delta'(\theta) = 0$ is given by

$$\theta^\beta = \frac{\alpha}{\alpha+\beta}\cdot\frac{(r+n+1)_{n+1}-(p+n+1)_{n+1}}{(q+n+1)_{n+1}-(p+n+1)_{n+1}}\cdot\frac{(q+n+1)_{n+1}}{(r+n+1)_{n+1}}. \tag{13}$$

In order to show that the expression on the right is necessarily less than unity, we write it as follows:

$$(*)\qquad \alpha\left\{\frac{1}{(p+n+1)_{n+1}}-\frac{1}{(r+n+1)_{n+1}}\right\}$$
$$\div(\alpha+\beta)\left\{\frac{1}{(p+n+1)_{n+1}}-\frac{1}{(q+n+1)_{n+1}}\right\}.$$

Now

$$\frac{1}{(p+k)_{n+1}}=\frac{1}{n!}\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{1}{p+k+i},$$

so that

$$(\alpha+\beta)\left\{\frac{1}{(p+n+1)_{n+1}}-\frac{1}{(q+n+1)_{n+1}}\right\}$$
$$=\frac{\alpha(\alpha+\beta)}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{(q+n+i+1)(p+n+i+1)}$$
$$=\frac{\alpha(\alpha+\beta)}{n!}\int_0^1\int_0^1 x^{p+n}y^{q+n}(1-xy)^n\,dx\,dy\,.$$

Hence (13) reduces to

$$\theta^\beta = \frac{I^n_{pr}}{I^n_{pq}} \tag{14}$$

where

$$I_{pq}^n = \int_0^1\int_0^1 x^{p+n}y^{q+n}(1-xy)^n\,dxdy\,.$$

Obviously $I_{pq}^n > I_{pr}^n$ where $r > q$, since

$$I_{pq}^n - I_{pr}^n = \int_0^1\int_0^1 x^{p+n}y^n(y^q-y^r)(1-xy)^n\,dxdy > 0\,.$$

We have thus proved that there is a choice of θ for which $R(h)$ is infinitely small of order $n+r+2$ at least. From the form (A) of $f(h)$ it follows that $R(h)$ will be infinitely small of order $n+s+1$ at least.

4. Let the integral

$$T(h) = \int_0^h f(x)P_n(x)\,dx$$

of Art. 2 be approximated by

$$T_2(h) = h^{n+1}\{\lambda_0 f^{(n)}(0) + \lambda_1 f^{(n)}(\theta_1 h) + \lambda_2 f^{(n)}(\theta_2 h)\}$$

where $0 < \theta_1 \leqslant \frac{1}{2} < \theta_2 \leqslant 1$. For $\theta_1 = \frac{1}{2}$, $\theta_2 = 1$, we get the same formula (2) of Art. 1, which for $n = 0$ is Simpson's formula.

We then propose the following problem:

Determine the values of $\lambda_0, \lambda_1, \lambda_2$ so that for given θ_1 and θ_2 $(0 < \theta_1 \leqslant \frac{1}{2} < \theta_2 \leqslant 1)$ the remainder is of the greatest order of smallness with respect to h.

Let $f(x)$ be regular in the neighbourhood of the origin having the development (A) of Art. 2. We easily get the following equations to determine $\lambda_0, \lambda_1, \lambda_2$:

$$\lambda_0 + \lambda_1 + \lambda_2 = \frac{\Gamma(n+1)}{\Gamma(2n+2)}, \tag{15}$$

$$\lambda_1\theta_1^p + \lambda_2\theta_2^p = \frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}, \tag{16}$$

$$\lambda_1\theta_1^q + \lambda_2\theta_2^q = \frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}. \tag{17}$$

Equations (16) and (17) determine λ_1, λ_2 uniquely, as

$$W = \begin{vmatrix} \theta_1^p & \theta_2^p \\ \theta_1^q & \theta_2^q \end{vmatrix} = \theta_1^p\theta_2^q - \theta_1^q\theta_2^p = \theta_1^p\theta_2^p(\theta_2^{q-p} - \theta_1^{q-p}) \neq 0 \quad (0 < \theta_1 < \theta_2 \leqslant 1)\,.$$

We have thus proved that:

If θ_1 and θ_2 are given numbers $(0 < \theta_1 < \theta_2 < 1)$ and λ_1, λ_2 are given by (16) *and* (17) *and λ_0 by* (15), *then $R(h)$ is of the order of smallness $r+n+1$.*

5. Naturally the following problem now arises:

Can we further raise the order of smallness of $R(h)$ by a proper choice of θ_2?

Then, along with (15), (16) and (17), the following relation must be satisfied:

$$\lambda_1\theta_1^r + \lambda_2\theta_2^r = \frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}. \tag{18}$$

The condition of consistency of (16), (17) and (18) then requires that

$$\Delta(\theta) \equiv \begin{vmatrix} \theta_1^p & \theta^p & \dfrac{\Gamma(p+n+1)}{\Gamma(p+2n+2)} \\ \theta_1^q & \theta^q & \dfrac{\Gamma(q+n+1)}{\Gamma(q+2n+2)} \\ \theta_1^r & \theta^r & \dfrac{\Gamma(r+n+1)}{\Gamma(r+2n+2)} \end{vmatrix} = 0\,. \tag{19}$$

Putting $q-p=\alpha$, $r-q=\beta$ in (19) $(\alpha, \beta \geqslant 1)$

$$\begin{aligned} \Delta(\theta) \equiv \theta^{\alpha+\beta}\left[\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)} - \theta_1^\alpha\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right] - \\ -\theta^\alpha\left[\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)} - \frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\cdot\theta_1^{\alpha+\beta}\right] + \\ +\left[\theta_1^\alpha\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)} - \theta_1^{\alpha+\beta}\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\right] = 0\,. \end{aligned} \tag{20}$$

We shall now show that this equations has a unique root θ_0 lying in $(\theta_1, 1)$. We have

$$\begin{aligned} \Delta(0) &= \theta_1^\alpha\left\{\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)} - \theta_1^\beta\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\right\} \\ &= \theta_1^\alpha\left\{\frac{\Gamma(q+n+1)}{\Gamma(r+2n+2)}\right\}[(q+n+1)_\beta - \theta_1^\beta(q+2n+2)_\beta] \\ &= \theta_1^{\alpha+\beta}\frac{\Gamma(q+n+1)}{\Gamma(r+2n+2)}\cdot(q+n+1)_\beta\left\{\left(\frac{1}{\theta_1}\right)^\beta - \left(1+\frac{n+1}{q+n+1}\right)_\beta\right\} \end{aligned} \tag{21}$$

where

$$\left(1+\frac{n+1}{q+n+1}\right)_\beta = \left(1+\frac{n+1}{q+n+1}\right)\left(1+\frac{n+1}{q+n+2}\right)\cdots\left(1+\frac{n+1}{q+n+\beta}\right).$$

Now

$$\left(\frac{1}{\theta_1}\right)^\beta > \left(1+\frac{n+1}{q+n+1}\right), \quad \text{so} \quad \Delta(0) > 0\,.$$

Also $\Delta(\theta_1)=0$ and

$$(22)\qquad \Delta(1)=\theta_1^{\alpha+\beta}\left[\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}-\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\right]- -\theta_1^{\alpha}\left[\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}-\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}\right]+ +\left[\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}-\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}\right].$$

The right-hand side is a function of θ_1 and corresponds to the left-hand side of (11), where we have shown that there exists a value θ' in $(0,1)$ such that $\Delta(1)>0$ for $\theta_1<\theta'$. Also

$$(23)\qquad \Delta'(\theta)=(\alpha+\beta)\,\theta^{\alpha+\beta-1}\left[\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}-\theta_1^{\alpha}\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right]- -\alpha\theta^{\alpha-1}\left[\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}-\theta_1^{\alpha+\beta}\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right],$$

so that

$$\Delta'(1)=\theta_1^{\alpha+\beta}\left[(\alpha+\beta)\left(\frac{1}{\theta_1}\right)^{\beta}\left\{\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}\cdot\left(\frac{1}{\theta_1}\right)^{\alpha}-\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right\}- -\alpha\left\{\left(\frac{1}{\theta_1}\right)^{\alpha+\beta}\cdot\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}-\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right\}\right]$$

and

$$\Delta'(\theta_1)=(\alpha+\beta)\,\theta_1^{2\alpha+\beta-1}\left[\left(\frac{1}{\theta_1}\right)^{\alpha}\frac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}-\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right]- -\alpha\theta_1^{2\alpha+\beta-1}\left[\left(\frac{1}{\theta_1}\right)^{\alpha+\beta}\frac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}-\frac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}\right].$$

We shall now show that

$$(24)\qquad \Delta'(1)>0 \quad\text{and}\quad \Delta'(\theta_1)<0\,.$$

The proof depends on the following inequality:

$$(25)\qquad \frac{\alpha+\beta}{\alpha}\cdot A^{\beta}>\frac{\dfrac{\Gamma(r+n+1)}{\Gamma(r+2n+2)}A^{\alpha+\beta}-\dfrac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}}{\dfrac{\Gamma(q+n+1)}{\Gamma(q+2n+2)}A^{\alpha}-\dfrac{\Gamma(p+n+1)}{\Gamma(p+2n+2)}}>\frac{\alpha+\beta}{\alpha},$$

for $A\geqslant 3$, $\alpha\geqslant 1$

or

$$\frac{A^{\beta}}{\alpha}\{(p+n+1)_{\alpha}A^{\alpha}-(p+2n+2)_{\alpha}\}$$
$$>\frac{1}{\alpha+\beta}\cdot\frac{\Gamma(q+2n+2)}{\Gamma(r+2n+2)}\{(p+n+1)_{\alpha+\beta}A^{\alpha+\beta}-(p+2n+2)_{\alpha+\beta}\}$$
$$>\frac{1}{\alpha}\{(p+n+1)_{\alpha}A^{\alpha}-(p+2n+2)_{\alpha}\}$$

or

$$\frac{A^\beta\left\{A^a-\left(1+\dfrac{n+1}{p+n+1}\right)_a\right\}}{a\left(1+\dfrac{n+1}{p+n+1}\right)_a}>\frac{A^{a+\beta}-\left(1+\dfrac{n+1}{p+n+1}\right)_{a+\beta}}{(a+\beta)\left(1+\dfrac{n+1}{p+n+1}\right)_{a+\beta}}$$
$$>\frac{A^a-\left(1+\dfrac{n+1}{p+n+1}\right)_a}{a\left(1+\dfrac{n+1}{p+n+1}\right)_a}.$$

In other words

(25′) $$A^\beta K(a)>K(a+\beta)>K(a)$$

where

$$K(a)=\frac{A^a-\left(1+\dfrac{n+1}{p+n+1}\right)_a}{a\left(1+\dfrac{n+1}{p+n+1}\right)_a}.$$

Since β is a positive integer, it sufficient to prove the result for $\beta=1$. The right-hand side of the inequality is certainly true if

(26) $$\frac{A^{a+1}-\left(1+\dfrac{n+1}{p+n+1}\right)_{a+1}}{(a+1)\left(1+\dfrac{n+1}{p+n+1}\right)_{a+1}}>\frac{A^a-\left(1+\dfrac{n+1}{p+n+1}\right)_a}{a\left(1+\dfrac{n+1}{p+n+1}\right)_a}$$

or

$$\frac{A^a}{\left(1+\dfrac{n+1}{p+n+1}\right)_a}\left[\frac{A}{(a+1)\left(1+\dfrac{n+1}{p+n+a+1}\right)}-\frac{1}{a}\right]>\frac{1}{a+1}-\frac{1}{a}.$$

Since

$$\frac{A}{1+\dfrac{n+1}{p+n+a+1}}\cdot\frac{1}{a+1}-\frac{1}{a}>\frac{1}{a+1}-\frac{1}{a},$$

the inequality (26) is proved.

In order to prove the left-hand side of inequality (25′), we must show that:

(27) $$A^{a+1}\left[1+\frac{(n+1)(a+1)}{p+n+a+1}\right]>\left(1+\frac{n+1}{p+n+1}\right)_{a+1}[(a+1)A-a].$$

We first observe that if (27) holds for $A=m$, then it also holds for $A>m$. Take

$$\Phi(A)=A^{a+1}\left[1+\frac{(n+1)(a+1)}{p+n+a+1}\right]-\left(1+\frac{n+1}{p+n+1}\right)_{a+1}[(a+1)A-a];$$

then

$$\Phi'(A) = (a+1)\left[\left\{1+\frac{(n+1)(a+1)}{p+n+a+1}\right\}A^a - \right.$$
$$\left. -\left(1+\frac{n+1}{p+n+a+1}\right)\left(1+\frac{n+1}{p+n+1}\right)_a\right] > 0\,,$$

showing that $\Phi(A)$ is an increasing function of A, and this proves our assertion. (27) holds true if we show that

$$A^{a+1} - (\overline{a+1}\,A - a)\left(1+\frac{n+1}{p+n+1}\right)_a > -\frac{a(n+1)}{p+2n+a+2}\cdot A^{a+1}\,.$$

The right-hand side being negative, it is sufficient to prove that

$$A^{a+1} - \big((a+1)A - a\big)\left(1+\frac{n+1}{p+n+1}\right)_a > 0$$

which is certaintly true, since

$$A^{a+1} > \big((a+1)A - a\big)2^a\,, \quad \text{for} \quad (A \geqslant 3\,,\ a \geqslant 3)\,.$$

The cases $a = 1$ and $a = 2$ are verified as follows: For $a = 1$, we should have

$$(28)\quad \left[A - \left(1+\frac{n+1}{p+n+1}\right)\right]^2$$
$$> \frac{n+1}{p+n+3}\left[\left(1+\frac{n+1}{p+n+1}\right)\left(1+\frac{n+1}{p+n+2}\right) - A^2\right].$$

Since the right-hand side is always negative for $A \geqslant 2$, while the left-hand side is positive, the inequality holds. For $a = 2$, we should have

$$(29)\quad A^3 - (3A-2)\left(1+\frac{n+1}{p+n+1}\right)\left(1+\frac{n+1}{p+n+2}\right) > -A^3\cdot\frac{2(n+1)}{p+2n+4}\,.$$

Proving the above for $A = 2$ implies the proof for $A \geqslant 2$, so that we should have

$$(n+1)\left[\frac{2p+3n+4}{(p+n+1)(p+n+2)} - \frac{4}{p+2n+4}\right] < 1\,,$$

which reduces to proving

$$p^3 + (6n+9)p^2 + (6n^2+18n+14) > 0\,,$$

which is always true. Hence the inequality for $a = 2$.

In these two cases we see that inequality (27) holds under less restrictive conditions $A \geqslant 2$. It appears that the inequality can be proved to be true for all $A \geqslant 2$ and $a \geqslant 1$. I have verified a few particular cases but the general proof seems to evade me.

We have thus proved the inequality (25). On examining the expressions for $\Delta_1'(1)$ and $\Delta_1'(\theta_1)$ we at once get (24). Also

$$\Delta_1'(\theta) = \theta^{a-1}\theta_1^{a+\beta}\Big[(a+\beta)(r+n+1)_{n+1}\Big(\frac{\theta}{\theta_1}\Big)^{a+\beta}\Big\{(p+n+1)_{n+1}\Big(\frac{1}{\theta_1}\Big)^{a}- -(q+n+1)_{n+1}\Big\} - a(q+n+1)_{n+1}\Big\{(p+n+1)_{n+1}\Big(\frac{1}{\theta_1}\Big)^{a+\beta}-(r+n+1)_{n+1}\Big\}\Big]$$

which vanishes at $\bar{\theta}$ given by

$$\bar{\theta} = \theta_1\left[\frac{a(q+n+1)_{n+1}\{(p+n+1)_{n+1}(1/\theta_1)^{a+\beta}-(r+n+1)_{n+1}\}}{(a+\beta)(r+n+1)_{n+1}\{(p+n+1)_{n+1}(1/\theta_1)^{a}-(q+n+1)_{n+1}\}}\right]$$

and by (25) $\theta_1 < \bar{\theta} < 1$.

Hence from (21), (22), (24) we see that there exists a unique value $\theta_0 > \bar{\theta}$ for which $\Delta_1(\theta)$ vanishes and the order of $R(h)$ is less than that of h^{n+s+1} for this choice of θ.

6. In connection with Art. 3, we have tired without success to show that the order of smallness of $R(h)$ is exactly $n+s+1$. In the case of $n=0$, this result had been proved by S. Gołąb and C. Olech [3]. For a general $n>0$, it would be necessary to show that if the order of $R(h)$ is $n+s+2$ then we are lead to a contradiction. For then we would have also

$$\bar{\lambda}_1\theta^s + \lambda_2 = 1/(s+n+1)_{n+1}$$

which combined with (7) and (8) gives

$$\begin{vmatrix} \theta^p & 1 & 1/(p+n+1)_{n+1} \\ \theta^q & 1 & 1/(q+n+1)_{n+1} \\ \theta^s & 1 & 1/(s+n+1)_{n+1} \end{vmatrix} = 0$$

and on further simplification we may write it as

$$\text{(30)}\qquad \left\{\frac{1}{(p+n+1)_{n+1}} - \frac{1}{(q+n+1)_{n+1}}\right\}\theta^{a+\gamma} - -\left\{\frac{1}{(p+n+1)_{n+1}} - \frac{1}{(s+n+1)_{n+1}}\right\}\theta^{a} + +\left\{\frac{1}{(q+n+1)_{n+1}} - \frac{1}{(s+n+1)_{n+1}}\right\} = 0$$

where $\gamma = s-q$. We must now show that (30) and (11) cannot have a common root θ.

In short, it reduces to showing that the equation

$$K\theta^x = \lambda + \frac{\mu}{(x+q+n+1)_{n+1}}$$

where

$$K = [(q+n+1)_{n+1} - (p+n+1)_{n+1}]\theta^a > 0 ,$$
$$\lambda = [(q+n+1)_{n+1}\theta^a - (p+n+1)_{n+1}] < 0$$

and

$$\mu = (p+n+1)_{n+1}(q+n+1)_{n+1}(1-\theta^a) > 0$$

has only three zeros $-a$, 0 and β.

A similar problem remains unsolved in connection with the problem treated in Art. 5 where we try to show that the order of smallness of $R(h)$ is exactly $n+s+1$. The case $n=0$ has been proved by A. Sharma [4]. As before, for a general $n>0$, we are led to a contradiction when we try to show that the order of $R(h)$ is $n+s+2$. Following the same technique, the problem is to prove that the equation

$$\begin{aligned}(s+n+1)_{n+1}\theta^{a+\gamma}\theta_1^{-\gamma}\{(p+n+1)_{n+1}\theta_1^{-a} - (q+n+1)_{n+1}\} - \\ - (q+n+1)_{n+1}\theta^a\{(p+n+1)_{n+1}\theta_1^{-a-\gamma} - (s+n+1)_{n+1}\} + \\ + (p+n+1)_{n+1}\{(q+n+1)_{n+1}\theta_1^{-\gamma} - (s+n+1)_{n+1}\} = 0\end{aligned}$$

and (20) cannot have a common root θ_0 such that $\theta_1 < \theta_0 < 1$.

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