ANNALES
POLONICI MATHEMATICI
XXXIV (1977)

Connections of order r

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Abstract. In this note we generalize the following classical theorem:

If Γ is a linear connection in M, i.e., if Γ is a connection in the bundle LM of linear frames, then Γ defines a mapping

$$\nabla^{\Gamma}\colon \mathscr{X}(M)\times\mathscr{X}(M)\ni(v,w)\to\nabla_v^{\Gamma}w\in\mathscr{X}(M),$$

where $\mathscr{X}(M)$ is the $C^{\infty}(M)$-module of vector fields on M, such that

(1) $$\nabla_v^{\Gamma}(w_1+w_2) = \nabla_v^{\Gamma}w_1+\nabla_v^{\Gamma}w_2,$$

(2) $$\nabla_{f_1x_1+f_2v_2}^{\Gamma}(w) = f_1\nabla_{v_1}^{\Gamma}w+f_2\nabla_{v_2}^{\Gamma}w,$$

(3) $$\nabla_v^{\Gamma}(fw) = f\nabla_v^{\Gamma}w+w\,\partial_v f,$$

(∂_v denotes the derivation in the direction of v).

Conversely, if $\Phi\colon \mathscr{X}(M)\times\mathscr{X}(M)\to\mathscr{X}(M)$ satisfies (1)–(3), *then there is one and only one linear connection Γ in M such that $\Phi = \Delta^{\Gamma}$.*

Let H_n^r be a vector space of all r-jets (at 0) of functions $f\colon R^n\to R$ such that $f(0) = 0$. H_n^r is an associative algebra with given by the formula: $j^r(f)*j^r(f') = j^r(ff')$. $J^r(M)$ be the set of all mappings $A\colon L^rM\to H_n^r$ ($n = \dim M$, L^rM denotes the bundle of frames of order r) such that $A\circ R_\xi = \lambda_{\xi-1}\circ A$ for all $\xi\in L_n^r$, where $\lambda_{\xi-1}\colon H_n^r\to H_n^r$ is a natural left translation on H_n^r. Our main theorem is the following:

THEOREM. *If Γ is a connection of order r in M, i.e., if Γ is a connection in L^rM, then Γ defines a mapping*

$$\nabla^{\Gamma}\colon \mathscr{X}(M)\times J^r(M)\ni(v,A)\to\nabla_v^{\Gamma}A\in J^r(M)$$

such that

(1) $$\nabla_{f_1v_1+f_2v_2}^{\Gamma}A = f_1\nabla_{v_1}^{\Gamma}A+f_2\nabla_{v_2}^{\Gamma}A,$$

(2) $$\nabla_v^{\Gamma}(A_1+A_2) = \nabla_v^{\Gamma}A_1+\nabla_v^{\Gamma}A_2,$$

(3) $$\nabla_v^{\Gamma}(fA) = (\partial_v f)A+f\nabla_v^{\Gamma}A,$$

(4) $$\nabla_v^{\Gamma}(A_1*A_2) = (\nabla_v^{\Gamma}A_1)*A_2+A_1*(\nabla_v^{\Gamma}A_2).$$

Conversely, if $\Phi\colon \mathscr{X}(M)\times J^rM\to J^r(M)$ satisfies (1)–(4), *then there is one and only one connection Γ of order r in M such that $\Phi = \nabla^{\Gamma}$.*

Introduction. In the theory of linear connections on M the following theorem is well known:

THEOREM. *If Γ is a linear connection on M, i.e., if Γ is a connection in the bundle LM of linear frames, then Γ defines a mapping*

$$\nabla^\Gamma\colon \mathscr{X}(M)\times\mathscr{X}(M)\ni(v, w)\to\nabla^\Gamma_v w\in\mathscr{X}(M),$$

where $\mathscr{X}(M)$ is the $C^\infty(M)$-module of vector fields on M (of the class C^∞), which satisfies there conditions:

$$\nabla^\Gamma_{f_1v_1+f_2v_2}w = f_1\nabla^\Gamma_{v_1}w+f_2\nabla^\Gamma_{v_2}w, \tag{1}$$

$$\nabla^\Gamma_v(w_1+w_2) = \nabla^\Gamma_v w_1+\nabla^\Gamma_v w_2, \tag{2}$$

$$\nabla^\Gamma_v(fw) = (\partial_v f)w+f\nabla^\Gamma_v w, \tag{3}$$

for all $v, w, v_1, v_2, w_1, w_2\in\mathscr{X}(M)$ and $f, f_1, f_2\in C^\infty(M)$.

Conversely, if $\Phi\colon \mathscr{X}(M)\times\mathscr{X}(M)\to\mathscr{X}(M)$ is a mapping satisfying conditions (1)–(3), *then there is one and only one linear connection Γ on M such that $\Phi = \nabla^\Gamma$* (see, for example, [4], Proposition 2.8, p. 123 and Proposition 7.5, p. 143).

The above properties permit us to define linear connections on M as mappings Φ satisfying conditions (1)–(2). A theory of linear connections based on such a definition (see for example [3]) is, in general, simpler than a theory which considers linear connections as a p special case of general connections defines in a principal fibre bundle (see [4]).

The aim of my note is to give an analogous characterization of connections of order r, i.e., connections in the bundle L^rM of r-frames. Theorem 2.6 is the main theorem of this note.

In this note differentiability means always differentiability of the class C^∞. If M is a manifold and $x\in M$, then T_xM denotes the space tangent to M at x. T_xM is the set of equivalence classes of curves $\gamma\colon (-\varepsilon, +\varepsilon)\to\to M$ such that $\gamma(0) = x$, with respect to $\sim$, where

$$\gamma\sim\gamma'\Leftrightarrow \exists(U, \varphi)\in\text{atl } M,\quad x\in U,\quad \frac{d}{dt}(\varphi\circ\gamma)(0) = \frac{d}{dt}(\varphi\circ\gamma')(0).$$

$TM = \bigcup_x T_xM$ denotes the tangent bundle.

If $f\colon M\to N$ is a mapping of the class C^∞ and $x\in M$, then

$$d_xf\colon T_xM\ni[\gamma]\to[f\circ\gamma]\in T_{f(x)}N$$

denotes the linear homomorphism induced by f ($[\gamma]$ denotes the class of a curve γ), and $df\colon TM\to TN$ is a homomorphism of vector bundles such that $(df)|T_xM = d_xf$.

$C^\infty(M)$ denotes the ring of differentiable functions $M\to R$ and $\mathscr{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M. Each vector field $v\in\mathscr{X}(M)$ defines a R-linear mapping $\partial_v\colon C^\infty(M)\to C^\infty(M)$ such that $\partial_v(fg)$

$= (\partial_v f)g + f(\partial_v g)$. For a function $f \in C^\infty(M)$, $\partial_v f$ is defined as

$$\partial_v f\colon M \ni x \to \frac{d}{dt}(f \circ \gamma_x)(0) \in R,$$

where $v_x = [\gamma_x]$.

1. A covariant derivation of geometric objects. Suppose we are given a principal fibre bundle $P(M, G)$, a manifold F and an action (on the eft) of the structural group G on F. We denote by $\lambda_\xi\colon F \to F$ a left translation on F. Now we can construct a fibre bundle

$$E = E(P, M, G, F, \{\lambda_\xi\})$$

lassociated with $P(M, G)$, with standard fibre F and the action λ_ξ of G on F. We recall only that (see [4], p. 54–55)

$$E = (P \times F)/G,$$

where the action (on the right) of G on $P \times F$ is defined as follows:

$$(p, f)\cdot \xi = \big(p \cdot \xi, \lambda_{\xi^{-1}}(f)\big).$$

For $(p, f) \in P \times F$ we denote by $\langle p, f\rangle$ the equivalence class of (p, f) in E. It is easy to verify the following lemma (see [1], Lemma 1):

LEMMA 1.1. *There is a one-to-one correspondence between sections of* $E = E(P, M, G, F, \{\lambda_\xi\})$ *and mappings* $A\colon P \to F$ *such that*

$$A \circ R_\xi = \lambda_{\xi^{-1}} \circ A \quad \text{for all } \xi \in G,$$

where R_ξ *is a right translation of* G *on* P. *If a section* $\sigma\colon M \to E$ *and a mapping* $A\colon P \to F$ *are associated, then for all* $p \in P$

$$\sigma\big(\pi(p)\big) = \langle p, A(p)\rangle.$$

The above lemma permits us to formulate

DEFINITION 1.2. A mapping $A\colon P \to F$ is called a *geometric object of the type* (F, λ_ξ), or shortly an (F, λ_ξ)*-object*, on $P(M, G)$ if

$$A \circ R_\xi = \lambda_{\xi^{-1}} \circ A,$$

for all $\xi \in G$. The family $\{\lambda_\xi\}$ is called the *transformation formula of* A.

Now we shall define the covariant derivative of a geometric object.

Suppose we are given a connection Γ in $P(M, G)$ and a vector field $v\colon M \to TM$ on M. We denote by H_v^Γ the horizontal lift of v, i.e., $H_v^\Gamma\colon P \to TP$ is a vector field on P uniquely defined by the conditions:

(i) $H_v(p) \in \Gamma_p$ = the space of horizontal vectors at the point p.

(ii) $d\pi \circ H_v = v \circ \pi$, where $\pi\colon P \to M$ is the projection.

Now we state

DEFINITION 1.3. Let $A\colon P \to M$ be an (F, λ_ξ)-object on $P(M, G)$:

$$\nabla_v^\Gamma A = dA \circ H_v^\Gamma\colon P \to TF$$

is called a *covariant derivative of* A in the direction of v and with respect to the connection Γ (see [1]).

It is easy to verify (see [1]) that

$$(\nabla_v^\Gamma A)\circ R_\xi = d\lambda_{\xi-1}\circ \nabla_v^\Gamma A.$$

This means that

PROPOSITION 1.4. *If* $A\colon P\to F$ *is an* (F, λ_ξ)*-object on* $P(M, G)$*, then* $\nabla_v^\Gamma A$ *is a* $(TF, d\lambda_\xi)$*-object on* $P(M, G)$.

In this note we shall consider only the following special case. Namely, we suppose that F is a vector space (of finite dimension) and $\lambda_\xi\colon F\to F$ is a linear isomorphism for all $\xi\in G$. Let $J_P(F, \lambda_\xi)$ denote the set of all (F, λ_ξ)-objects on $P(M, G)\cdot J_P(F, \lambda_\xi)$ admits a natural $C^\infty(M)$-module structure. Namely, if $A_1, A_2\in J_P(F, \lambda_\xi)$ and $f_1, f_2\in C^\infty(M)$, then

$$(f_1A_1+f_2A_2)(p) = f_1\big(\pi(p)\big)A_1(p)+f_2\big(\pi(p)\big)A_2(p),$$

$p\in P$. This structure is "natural" in the sense that if σ_1, σ_2 are sections of the vector bundle E associated with A_1 and A_2, respectively (see Lemma 1.1), then $f_1\sigma_1+f_2\sigma_2$ is associated with $f_1A_1+f_2A_2$.

If F is a vector space, then there is a natural isomorphism (of vector bundles) $\varphi_F\colon TF\to F\times F$ ($F\times F$ denotes the trivial bundle over F with the fibre F). φ_F and φ_F^{-1} are defined by the formulas

(1.5) $$\varphi_F([\gamma]) = \left(\gamma(0), \left(\frac{d}{dt}\gamma\right)(0)\right),$$

(1.6.1) $$\varphi_F^{-1}(a, b) = [\gamma],$$

where

(1.6.2) $$\gamma\colon R\to F, \quad \gamma(t) = a+tb.$$

Next it is easy to show the following diagram commutes:

(1.7) $$\begin{array}{ccc} TF & \xleftarrow{\;d\lambda_\xi\;} & TF \\ {\scriptstyle\varphi_F}\downarrow & & \downarrow{\scriptstyle\varphi_F} \\ F\times F & \xleftarrow{\;\lambda_\xi\times\lambda_\xi\;} & F\times F \end{array}$$

Let $\varrho = \varrho_F\colon F\times F\to F$ be the canonical projection onto the second factor, and for an (F, λ_ξ)-object A on $P(M, G)$ we write

(1.8) $$\tilde{\nabla}_v^\Gamma A = \varrho_F\circ\varphi_F\circ \nabla_v^\Gamma A\colon P\to F.$$

It is clear that

(1.9) $$\nabla_v^\Gamma A = \varphi_F^{-1}\circ(A, \tilde{\nabla}_v^\Gamma A).$$

From diagram (1.8) it follows that

$$\begin{aligned}(\tilde{\nabla}_v^\Gamma A)\circ R_\xi &= \varrho_F\circ\varphi_F\circ(\nabla_v^\Gamma A)\circ R_\xi = \varrho_F\circ\varphi_F\circ d\lambda_{\xi-1}\circ\nabla_v^\Gamma A\\ &= \varrho_F\circ(\lambda_{\xi-1}\times\lambda_{\xi-1})\circ\varphi_F\circ\nabla_v^\Gamma A = \lambda_{\xi-1}\circ\varrho_F\circ\varphi_F\circ\nabla_v^\Gamma A\\ &= \lambda_{\xi-1}\circ\tilde{\nabla}_v^\Gamma A,\end{aligned}$$

that is, we prove (see [1], [4]):

PROPOSITION 1.10. *If F is a vector space and λ_ξ is a linear automorphism of F for all $\xi\in G$, then for each (F,λ_ξ)-object A on P, $\tilde{\nabla}_v^\Gamma A$ is also an (F,λ_ξ)-object. Furthermore, the mapping*

$$\tilde{\nabla}_v^\Gamma:\ \mathscr{X}(M)\times J_P(F,\lambda_\xi)\ni(v,A)\to\tilde{\nabla}_v^\Gamma A\in J_P(F,\lambda_\xi)$$

satisfies the conditions

$$\tilde{\nabla}^\Gamma_{vf_1v_1+f_2v_2}A = f_1\tilde{\nabla}^\Gamma_{v_1}A+f_2\tilde{\nabla}^\Gamma_{v_2}A, \tag{1.10.1}$$

$$\tilde{\nabla}_v^\Gamma(A_1+A_2) = \tilde{\nabla}_v^\Gamma A_1+\tilde{\nabla}_v^\Gamma A_2, \tag{1.10.2}$$

$$\tilde{\nabla}_v^\Gamma(fA) = (\partial_v f)A+f\tilde{\nabla}_v^\Gamma A, \tag{1.10.3}$$

for all $v, v_1, v_2\in\mathscr{X}(M), f, f_1, f_2\in C^\infty(M)$ and $A, A_1, A_2\in J_P(F,\lambda_\xi)$.

The above proposition means that, under the hypothesis about F and λ_ξ, $\tilde{\nabla}^\Gamma$ defines a connection in a vector bundle $E = E(P, M, G, F, \lambda_\xi)$.

The operator $\tilde{\nabla}^\Gamma$ is also called a *convariant derivative.*

2. Connection of order r. The main theorem. Let F_n^r denote the set of all r-jets (at 0) of functions φ: $R^n\to R^n$ such that $\varphi(0) = 0$. F_n^r is a manifold diffeomorphic with R^N, where $N = n\left[\binom{n+r}{r}-1\right]$. F_n^r admits a semigroup structure defined by

$$j^r(\varphi)j^r(\psi) = j^r(\varphi\circ\psi).$$

Let L_n^r be the set of all invertible elements of F_n^r, i.e.,

$$L_n^r = \{j^r(\varphi):\ \text{Jacobian of } \varphi \text{ at } 0 \text{ is not } 0\}.$$

L_n^r is an open subset of F_n^r and it is a Lie group called a *differential r-group.* Using the standard method we can define a bundle L^rM over M with the structural group L_n^r, where $n = \dim M$. It is called the (*principal fibre*) *bundle* of r-frames.

Let H_n^r be the set of all r-jets (at 0) of functions f: $R^n\to R$ such that $f(0) = 0$. H_n^r is a vector space of dimension $L = \binom{n+r}{r}-1$. The group L_n^r acts on the left on H_n^r as follows: a left translation h_ξ: $H_n^r\to H_n^r$ is given by the formula

$$h_\xi\big(j^r(f)\big) = j^r(f\circ\varphi), \tag{2.1}$$

where $\xi^{-1} = j^r(\varphi)$. It is clear that h_ξ is a linear isomorphism.

DEFINITION 2.2. (H^r_n, h_ξ)-objects on L^rM are called *r-jet fields*, or shortly *r-jets*, on M.

Let us remarks that 1-jet fields are identified in a natural way with covector fields.

Let $J^r(M)$ be the set of all r-jet fields on M. Since H^r_n is a vector space and h_ξ is linear, $J^r(M)$ is a $C^\infty(M)$-module. We can also define an associative Abelian algebra structure on H^r_n. Namely, we set

$$j^r(f)*j^r(g) = j^r(fg), \tag{2.3}$$

where $fg\colon R^n \to R$, $(fg)(u^1, \ldots, u^n) = f(u^1, \ldots, u^n)g(u^1, \ldots, u^n)$. It is clear that for all $\xi \in L^r_n$ the mapping $h_\xi\colon H^r_n \to H^r_n$ is R-linear and

$$h_\xi(t_1 * t_2) = h_\xi(t_1) * h_\xi(t_2). \tag{2.4.1}$$

Furthermore, the mapping

$$*\colon H^r_n \times H^r_n \ni (t_1, t_2) \to t_1 * t_2 \in H^r_n \tag{2.4.2}$$

is R-linear and symmetric. Formula (2.4.1) permits us to define on $J^r(M)$ a structure of an associative Abelian algebra (over the ring $C^\infty(M)$). For $A_1, A_2 \in J^r(M)$ we set

$$A_1 * A_2\colon L^rM \ni p \to A_1(p) * A_2(p) \in H^r_n.$$

We now prove the following proposition:

PROPOSITION 2.5. *If Γ is a connection of order r on M (i.e., a connection in L^rM), then the operator*

$$\tilde{\nabla}^\Gamma\colon \mathscr{X}(M) \times J^r(M) \ni (v, A) \to \tilde{\nabla}^\Gamma_v A \in J^r(M)$$

satisfies the following conditions:

$$\tilde{\nabla}^\Gamma_{f_1v_1+f_2v_2} A = f_1 \tilde{\nabla}^\Gamma_{v_1} A + f_2 \tilde{\nabla}^\Gamma_{v_2} A, \tag{2.5.1}$$

$$\tilde{\nabla}^\Gamma_v (A_1 + A_2) = \tilde{\nabla}^\Gamma_v A_1 + \tilde{\nabla}^\Gamma_v A_2, \tag{2.5.2}$$

$$\tilde{\nabla}^\Gamma_v (fA) = (\partial_v f) A + f \tilde{\nabla}^\Gamma_v A, \tag{2.5.3}$$

$$\tilde{\nabla}^\Gamma_v (A_1 * A_2) = (\tilde{\nabla}^\Gamma_v A_1) * A_2 + A_1 * (\tilde{\nabla}^\Gamma_v A_2), \tag{2.5.4}$$

for all $v, v_1, v_2 \in \mathscr{X}(M)$, $f, f_1, f_2 \in C^\infty(M)$ and $A_1, A, A_2 \in J^r(M)$.

Proof. Conditions (2.5.1)–(2.5.3) follow from Proposition 1.10. We need only to prove the last condition. In order to do this, let $p \in L^rM$ and $H_v(p) = [\gamma]$, $\gamma\colon (-\varepsilon, +\varepsilon) \to P$, $\gamma(0) = p$. Now, by (1.8), (1.5) and Definition 1.3, we have

$$\big(\tilde{\nabla}^\Gamma_v (A_1 * A_2)\big)(p) = \frac{d}{dt}\,[(A_1 * A_2) \circ \gamma](0).$$

Since $((A_1 * A_2)\circ\gamma)(t) = A_1(\gamma(t)) * A_2(\gamma(t))$, we have

$$(2.4.1)\quad (\tilde{\nabla}_v^{\Gamma}(A_1 * A_2))(p) = \lim_{t\to 0} \frac{1}{t}\left[A_1(\gamma(t)) * A_2(\gamma(t)) - A_1(p) * A_2(p)\right]$$

$$= \lim_{t\to 0}\left\{\left[\frac{1}{t} A_1(\gamma(t)) - A_1(p)\right] * A_2(\gamma(t))\right\} +$$

$$+ A_1(p) * \left\{\lim_{t\to 0}\frac{1}{t}\left[A_2(\gamma(t)) - A_2(p)\right]\right\}$$

$$= \frac{d}{dt}(A_1\circ\gamma)(0) * A_2(p) + A_1(p) * \frac{d}{dt}(A_2\circ\gamma)(0)$$

$$= (\tilde{\nabla}_v^{\Gamma} A_1)(p) * A_2(p) + A_1(p) * (\tilde{\nabla}_v^{\Gamma} A_2)(p).$$

In this note we prove the following theorem:

THEOREM 2.6 (the main theorem). *If*

$$\Phi\colon \mathscr{X}(M)\times J^r(M)\to J^r(M)$$

is a mapping satisfying conditions (2.5.1)–(2.5.4), *then there is one and only one connection Γ of order r (i.e., a connection in L^rM) such that $\Phi = \tilde{\nabla}^{\Gamma}$.*

We shall prove the above theorem in Section 4. We end up this section with some remarks.

Let us remark that $J^1(M) = \mathscr{X}^*(M)$ is the dual module to $\mathscr{X}(M)$, i.e., $J^1(M)$ is the $C^\infty(M)$-module of covector fields on M. Next, for all $t_1, t_2 \in H_n^1$,

$$t_1 * t_2 = 0,$$

because $t_1 = j^1(\varphi)$, $t_2 = j^1(\psi)$, $\varphi(u^1, \ldots, u^n) = a_i u^i$, $\psi(u^1, \ldots, u^n) = b_i u^i$, and hence $t_1 * t_2 = j^1(\varphi\psi) = 0$, car $(\varphi\psi)(u^1, \ldots, u^n) = a_i b_j u^i u^j$.

For this reason condition (2.5.4) is equivalent to the equality $\Phi(v, 0) = 0$ for all $v \in \mathscr{X}(M)$, but this follows from (2.5.2). Thus, for $r = 1$, Theorem 2.6 is equivalent to the following one:

THEOREM 2.7. *If*

$$\Phi\colon \mathscr{X}(M)\times\mathscr{X}^*(M)\to\mathscr{X}^*(M)$$

is a mapping satisfying conditions (2.5.1)–(2.5.3), *then there is one and only one linear connection Γ on M such that $\Phi = \nabla$.*

This is a well-known classical theorem in the theory of linear connections. Usually it is formulated for a mapping $\mathscr{X}(M)\times\mathscr{X}(M)\to\mathscr{X}(M)$ (see Proposition 7.5, p. 143, in [4]), but using the same method we can formulate this theorem for a mapping $\mathscr{X}(M)\times\mathscr{X}^*(M)\to\mathscr{X}^*(M)$. Thus linear connections on M can be defined as mappings $\mathscr{X}(M)\times\mathscr{X}^*(M)\to\mathscr{X}^*(M)$ or as mappings $\mathscr{X}(M)\times\mathscr{X}(M)\to\mathscr{X}(M)$ satisfying (2.5.1)–(2.5.3) (see, for example, [3]). Our theorem permits us to define connections of order $r, r \geqslant 2$, on M as mappings $\mathscr{X}(M) \times J^r(M) \to J^r(M)$ satisfying (2.5.1)–(2.5.4).

3. Some properties of H_n^r. We denote by I_r the set of all multi-indices $a = (a_1, \ldots, a_n)$ such that each $a_i \geqslant 0$ and $1 \leqslant a_1 + \ldots + a_n \leqslant r$.

We introduce the following notations.

If $a \in I_r$ we write $|a| = a_1 + \ldots + a_n$ and $a! = a_1! \ldots a_n!$. For $a, \beta \in I_r$ we write $a \leqslant \beta$ if $a_i \leqslant \beta_i$ for $i = 1, \ldots, n$, and $a + \beta = (a_1 + \beta_1, \ldots, a_n + \beta_n)$. If $a \geqslant \beta$, then $a - \beta = (a_1 - \beta_1, \ldots, a_n - \beta_n)$ and

$$\binom{a}{\beta} = \binom{a_1}{\beta_1} \ldots \binom{a_n}{\beta_n}.$$

By 1_j we denote the multi-index $(0, \ldots, \overset{j}{1}, \ldots, 0)$.

For each $a \in I_r$ we define the element $e^a = e^{a_1 \ldots a_n}$ of H_n^r given by

(3.1) $$e^a = j^r(\varphi^a), \quad \varphi^a\colon (R^n, 0) \to (R, 0),\ \varphi^a(u^1, \ldots, u^n) = \frac{1}{a!} (u^1)^{a_1} \ldots (u^n)^{a_n}.$$

If $\beta \in I_r$, then D_β denotes the operator

$$D_\beta = \frac{\partial^{|\beta|}}{(\partial u^1)^{\beta_1} \ldots (\partial u^n)^{\beta_n}}.$$

Let us remark that

$$(D_a \varphi^\beta)(0) = \delta^a_\beta = \begin{cases} 1 & \text{if } a_1 = \beta_1, \ldots, a_n = \beta_n, \\ 0 & \text{in the other case.} \end{cases}$$

It is clear that

$$\{e^a\}_{1 \leqslant |a| \leqslant r}$$

is a canonical base of H_n^r; thus, for each $t \in H_n^r$ there is one and only one system $\{t_a\}_{1 \leqslant |a| \leqslant r}$ of real numbers such that

(3.3) $$t = \sum_{a \in I_r} t_a e^a = t_a e^a.$$

(We shall also use the summation convection for multi-indices $a \in I_r$ — the symbol $X_a Y^a$ always means $\sum_{a \in I_r} X_a Y^a$.) We prove the following proposition:

PROPOSITION 3.4. *In H_n^r we have*

$$e^a * e^\beta = \begin{cases} \binom{a+\beta}{a} e^{a+\beta} & \text{if } |a| + |\beta| \leqslant r, \\ 0 & \text{if } |a| + |\beta| > r. \end{cases}$$

Proof. By definition, $e^{\alpha} * e^{\beta} = j^r(\varphi^{\alpha}\varphi^{\beta})$, where

$$(\varphi^{\alpha}\varphi^{\beta})(u^1, \ldots, u^n) = \varphi^{\alpha}(u^1, \ldots, u^n)\varphi^{\beta}(u^1, \ldots, u^n)$$
$$= \frac{1}{\alpha!}\frac{1}{\beta!}(u^1)^{\alpha_1+\beta_1} \ldots (u^n)^{\alpha_n+\beta_n}$$
$$= \binom{\alpha+\beta}{\alpha}\varphi^{\alpha+\beta}(u^1, \ldots, u^n),$$

and hence we obtain our assertion.

Let us remark that we can identify H_n^r with the space $V_1 \times \ldots \times V_r$, where

$$V_s = \{(x_{\alpha})_{|\alpha|=s} \in R^{l_s}\}$$

and the dimension l_2 depends on n and s. Now, if $|\alpha| = s$, then e^{α} identifies with

$$\big(0, \ldots, 0, (\delta^{\alpha}_{\beta})_{|\beta|=s}, 0, \ldots, 0\big).$$

We can also identify F_n^r with $\Omega_1 \times \ldots \times \Omega_r$, where

$$\Omega_s = \{(\xi^i_{\alpha})_{i=1,\ldots,n;|\alpha=s} \in R^{nl}\}.$$

Now L_n^r is identified with

$$GL(n, R) \times \Omega_2 \times \ldots \times \Omega_r,$$

because $|\alpha| = 1$ implies $\alpha = 1_j$ for some j, and hence $\Omega_1 = R^{n^2}$ is the semi-group of square matrices. Since L_n^r is an open subset of F_n^r and F_n^r can be considered as a vector space, we have (natural) identifications

$$\mathscr{L}(L_n^r) = T_e L_n^r = T_e F_n^r = F_n^r = \Omega_1 \times \ldots \times \Omega_r,$$

where e is the unity of the group L_n^r.

Let $\{E_i^{\alpha}\}_{i=1,\ldots,n;1\leqslant|\alpha|\leqslant r}$ be a canonical base of F_n^r, i.e., $E_i^{\alpha} = j^r(\psi_i^{\alpha})$, where ψ_i^{α}: $(R^n, 0) \to (R^n, 0)$ and

$$\psi_i^{\alpha}(u^1, \ldots, u^n) = \Big(0, \ldots, 0, \underbrace{\frac{1}{\alpha!}(u^1)^{\alpha_1} \ldots (u^n)^{\alpha_n}}_{(i)}, 0, \ldots, 0\Big).$$

We now show a technical lemma for the proof of our main theorem.

PROPOSITION 3.6. *We fix* $e^{1_i} \in H_n^r$.

(a) *If* $f^{(1)i}$: $GL(n, R) \to H_n^r$

$$f^{(1)i}(A) = h_{(A,0,\ldots,0)^{-1}}(e^{1_i}),$$

then

$$d_I f^{(1)i}(E_k^{1_j}) = \delta^i_k e^{1_j},$$

for $i, j, k = 1, \ldots, n$, *where* $I = [\delta^i_j]$ *is the unit matrix.*

(b) *If* $f^{(p)i}\colon \Omega_p \to H^r_n$, *where* $p \geqslant 2$,

$$f^{(p)i}(z) = h_{(I,0,\ldots,0,\underset{(p)}{z},0,\ldots,0)^{-1}}(e^{1i}),$$

then

$$d_0 f^{(p)i}(E^a_j) = \delta^i_j e^a,$$

for $i, j = 1, \ldots, n$ *and* $a \in I_r$ *such that* $|a| = p$.

Proof. (a) Let $A = [a^i_j]$. Thus $(A, 0, \ldots, 0) = j^r(\psi)$, where $\psi = (\psi^1, \ldots, \psi^n)$ and

$$\psi^i(u^1, \ldots, u^n) = a^i_j u^j.$$

From (2.1) and (3.1), we have $f^{(1)i}(A) = j^r(\varphi^{1i} \circ \psi)$, where

$$(\varphi^{1i} \circ \psi)(u^1, \ldots, u^n) = a^i_j u^j,$$

and hence

$$f^{(1)i}(A) = (a^i_j, 0, \ldots, 0) \in H^r_n = V_1 \times \ldots \times V_r.$$

If we note

$$f^{(1)i}(A) = \big(f^{(1)i}_{a^1}(A), \ldots, f^{(1)i}_{a^r}(A)\big) \in H^r_n = V_1 \times \ldots \times V_r,$$

where $a^s \in I_r$ and $|a^s| = s$, then

$$f^{(1)i}_{1_j}(A) = a^i_j, \qquad F^{(1)i}_{a^s}(A) = 0 \qquad \text{for } s \geqslant 2,$$

and hence

$$(d_I f^{(1)i})(E^{1j}_k) = \left(\frac{\partial}{\partial a^k_j} f^{(1)i}_a\right)(I) e^a = \sum_{s=1}^{r} \left(\frac{\partial}{\partial a^k_j} f^{(1)i}_{1_s}(I)\right) e^{1_s} = \delta^i_k \delta^j_s e^{1_s} = \delta^i_k e^{1j}.$$

(b) In this case, let $z = (z^i_a)_{i=1,\ldots,n;|a|=p} \in \Omega_p$. Now $(I, 0, \ldots, z, \ldots, 0) = r^r(\psi)$, where $\psi = (\psi^1, \ldots, \psi^n)$ and

$$\psi^i(u^1, \ldots, u^n) = u^i + \sum_{|a|=p} \frac{1}{a!} z^i_a (u^1)^1 \ldots (u^n)^n,$$

and hence $f^{(p)i}(z) = j^r(\varphi^{1i} \circ \psi)$, where

$$(\varphi^{1i} \circ \psi)(u^1, \ldots, u^n) = \psi^i(u^1, \ldots, u^n)$$
$$= u^i + \sum_{|a|=p} \frac{1}{a!} z^i (u^1)^1 \ldots (u^n)^n.$$

Thus, if we note

$$f^{(p)i}(z) = \big(f^{(p)i}_{a^1}(z), \ldots, f^{(p)i}_{a^r}(z)\big) \in H^r_n = V_1 \times \ldots \times V_r,$$

where $a^s \in I_r$ and $|a^s| = s$, then we obtain

$$f^{(p)i}_{1_j}(z) = \delta^i_j, \quad f^{(p)i}_{a}(z) = z^i \quad \text{if} \quad |a| = p,$$

$$f^{(p)i}_{a}(z) = 0 \quad \text{if} \quad |a| \neq 1 \quad \text{and} \quad |a| \neq p.$$

From the above formulas, for $j = 1, \ldots, n$, and $|a| = p$, it follows that

$$d_0 f^{(p)i}(E^a_j) = \left(\frac{\partial}{\partial z^j_a} f^{(p)i}_\beta\right)(0)e^\beta = \sum_{|\beta|=p} \left(\frac{\partial}{\partial z^j_a} f^{(p)i}_\beta\right)(0)e^\beta = \delta^i_j \delta^a_\beta e^\beta = \delta^i_j e^a.$$

4. Proof of the main theorem.

PROPOSITION 4.1. *Let Γ_1 and Γ_2 be two connections of order r on M. If for each $v \in \mathcal{X}(M)$ and for each $A \in J^r(M)$, $\tilde{\nabla}^{\Gamma_1}_v A = \tilde{\nabla}^{\Gamma_2}_v A$, then $\Gamma_1 = \Gamma_2$.*

Proof. We fix a vector field $v \in \mathcal{X}(M)$ and a point p in $L^r(M)$, and let

$$H^{\Gamma_1}_v(p) = [\gamma_1], \quad H^{\Gamma_2}_v(p) = [\gamma_2], \quad \gamma_1, \gamma_2 \colon (-\varepsilon, +\varepsilon) \to L^r M,$$

$\gamma_1(0) = \gamma_2(0) = p$. The condition $\tilde{\nabla}^{\Gamma_1}_v A = \tilde{\nabla}^{\Gamma_2}_v A$ implies that for each $A \in J^r(M)$

$$\frac{d}{dt}(A \circ \gamma_1)(0) = \frac{d}{dt}(A \circ \gamma_2)(0). \tag{4.1.1}$$

Let $\psi = (\pi, \varphi) \colon (L^r M) | U \to U \times L^r_n$ be a trivialization of $L^r M$ in some neighbourhood U of the point $x = \pi(p)$. Since $d\pi([\gamma_1]) = v_{\pi(p)} = d\pi([\gamma_2])$, condition (4.1.1) is equivalent to the following one:

$$\left(\frac{\partial A'}{\partial \xi^\omega}\right)\big(\varphi(\gamma_1(0))\big) \frac{d(\varphi^\omega \circ \gamma_1)}{dt}(0) = \left(\frac{\partial A'}{\partial \xi^\omega}\right)\big(\varphi(\gamma_2(0))\big) \frac{d(\varphi^\omega \circ \gamma_2)}{dt}(0), \tag{4.1.2}$$

where $A' \colon L^r_n \ni \xi \to (A \circ \psi^{-1})\big(\pi(p), \xi\big) \in H^r_n$ and $\xi = (\xi^\omega)$ is a system of (natural) coordinates in L^r_n. Since (4.1.2) is satisfied for all A' such that $A'(\xi\eta) = (h_{\eta^{-1}} \circ A')(\xi)$, it follows that for any trivialization $\psi = (\pi, \varphi)$ of $L^r M$ in a neighbourhood of $x = \pi(p)$ we have

$$\frac{d}{dt}(\varphi \circ \gamma_1)(0) = \frac{d}{dt}(\varphi \circ \gamma_2)(0),$$

i.e., $[\gamma_1] = [\gamma_2]$. The condition $H^{\Gamma_1}_v = H^{\Gamma_2}_v$ for all $v \in \mathcal{X}(M)$ immediately implies $\Gamma_1 = \Gamma_2$.

The above proposition shows the uniqueness of Γ in our main theorem (Theorem 2.6). To finish the proof of Theorem 2.6 we must show the existence of Γ. We shall prove this in a few steps.

PROPOSITION 4.2. *If* $\Phi\colon \mathscr{X}(M)\times J^r(M)\to J^r(M)$ *satisfies conditions* (2.5.1)–(2.5.3) *and* U *is an open subset of* M, *then*

(a) *if* $v_1, v_2 \in \mathscr{X}(M)$ *are such that* $v_1|U = v_2|U$, *then for all* $A\in J^r(M)$

$$\Phi(v_1, A)|\pi^{-1}(U) = \Phi(v_2, A)|\pi^{-1}(U),$$

(b) *if* $A_1, A_2\in J^r(M)$ *are such that* $A_1|\pi^{-1}(U) = A_2|\pi^{-1}(U)$, *then for all* $v\in\mathscr{X}(M)$

$$\Phi(v, A_1)|\pi^{-1}(U) = \Phi(v, A_2)|\pi^{-1}(U).$$

Proof. (a) From (2.5.1) it is sufficient to show that

$$v|U = 0 \Rightarrow \Phi(v, A)|\pi^{-1}(U) = 0.$$

Let $p\in\pi^{-1}(U)$. There is an $f\in C^\infty(M)$ such that $f(\pi(p)) = 1$ and $f(x) = 0$ for $x\notin U$. Since $fv = 0$ on L^rM, we have $0 = \Phi(fv, A) = f\Phi(v, A)$, and hence $\Phi(v, A)(p) = 0$.

(b) Analogously, from (2.5.2), it is sufficient to show that

$$A|\pi^{-1}(U) = 0 \Rightarrow \Phi(v, A)|\pi^{-1}(U) = 0.$$

Let $p\in\pi^{-1}(U)$. There is an open neighbourhood V of $\pi(p)$ such that $\overline{V}\subset U$ and $\overline{V}$ is compact. Now there is an $f\in C^\infty(M)$ such that $f(x) = 1$ for $x\in\overline{V}$ and $f(x) = 0$ for $x\in\overline{U}$, and hence, from (2.5.3) and from $fA = 0$ on L^rM, we obtain

$$0 = \Phi(v, fA) = (\partial_v f)A + f\Phi(v, A).$$

Since $f = 1$ on V, we have $\partial_v f = 0$ on V, and hence $\Phi(v, A)(p) = 0$. The proof is now complete.

If Φ satisfies conditions (2.5.1)–(2.5.3), then the above proposition permits us to define the mapping

$$\Phi_U\colon \mathscr{X}(U)\times J^r(U)\to J^r(U), \tag{4.3}$$

where U is open in M, uniquely determined by the condition

$$\Phi_U(v|U, A|\pi^{-1}(U)) = \Phi(v, A)|\pi^{-1}(U) \tag{4.3'}$$

for all $v\in\mathscr{X}(M)$ and $A\in J^r(M)$. We shall often write Φ instead of Φ_U.

PROPOSITION 4.4. *If* $\Phi\in\mathscr{X}(M)\times J^r(M)\to J^r(M)$ *satisfies* (2.5.1)–(2.5.4), *then* Φ_U *also satisfies those conditions.*

The proof is trivial.

Now let us fix a trivialization $\psi\colon L^rM|U\to U\times L^r_n$. For $a\in I_r$ we define $A^a\in J^r(U)$ by the following condition:

$$(A^a\circ\psi^{-1})(x, e) = e^a, \tag{4.5}$$

where e^a is defined by (3.1) and e is the unit element of L^r_n. It is easy to see that $\{A^a\}_{1\leqslant|a|\leqslant r}$ is a base of $J^r(U)$, i.e., for each $A\in J^r(U)$ there is one and only one system of functions $f_a\in C^\infty(U)$, $1\leqslant|a|\leqslant r$, such that

$$A = f_a A^a. \tag{4.6}$$

$\{A^a\}_{1\leqslant|a|\leqslant r}$ is a base of $J_r(U)$ associated with the trivialization ψ, and functions f_a are called *coordinates of A* with respect to ψ.

PROPOSITION 4.7. *If $\Phi_1,\Phi_2\colon \mathscr{X}(M)\times J^r(M)\to J^r(M)$ are two mappings satisfying* (2.5.1)–(2.5.4), *$\psi\colon L^rM\,|\,U\to U\times L_n^r$ is a trivialization and $e_1,\ldots,e_n$ is a base of $\mathscr{X}(U)$ (considered as $C^\infty(U)$-module), then the equality*

$$(\Phi_1)_U(e_k, A^a) = (\Phi_2)_U(e_k, A^a),$$

for $k=1,\ldots,n$ and $a\in I_r$ such that $|a|=1$, implies $(\Phi_1)_U=(\Phi_2)_U$.

Proof. From Proposition 3.4, for $a=(a_1,\ldots,a_n)\in I_r$, we have

$$A^{a+1_j} = \frac{1}{a_j+1} A^a * A^{1_j}$$

and next, from (2.5.4), we obtain

$$(\Phi_p)_U(e_k, A^{a+1_j}) = \frac{1}{a_j+1}\{(\Phi_p)_U(A^a)*A^{1_j}+A^a*(\Phi_p)_U(A^{1_j})\}$$

for $p=1,2$. Hence, since $|a+1_j|=|a|+1$ and $|a|=1$ if and only if $a=1_j$ for some j, the equality $(\Phi_1)_U(e_k,A^a)=(\Phi_2)_U(e_k,A^a)$ for a such that $|a|=1$ implies by induction

$$(\Phi_1)_U(e_k, A^a) = (\Phi_2)_U(e_k, A^a)$$

for all $a\in I_r$ and $k=1,\ldots,n$. Now from (2.5.1)–(2.5.3) and (4.6) we obtain our assertion.

PROPOSITION 4.8. *If $\Phi\colon \mathscr{X}(M)\times J^r(M)\to J^r(M)$ satisfies* (2.5.1)–(2.5.4), *then for each open sufficiently small subset U of M there is one and only one connection Γ in $L^rM\,|\,U$ such that $\Phi_U=\tilde{\nabla}^\Gamma$.*

Proof. Let U be so small that $L^rM\,|\,U$ admits a trivialization $\psi\colon L^rM\,|\,U\to U\times L_n^r$ and $\mathscr{X}(U)$ admits a base $e_1,\ldots,e_n$. Let functions $-\Gamma^i_{ka}\in C^\infty(U)$, $i,k=1,\ldots,n$, $a\in I_r$, be coordinates of $\Phi_U(e_k,A^{1_i})$ with respect to ψ, i.e.,

$$\Phi_U(e_k, A^{1_i}) = -\Gamma^i_{ka}A^a \tag{4.8.1}$$

(let us recall that A^a is defined by (4.5)). Now we can define on U an $\mathscr{L}(L_n^r)$-valued 1-form

$$\omega_0(e_k) = \Gamma^i_{ka}E^a_i, \tag{4.8.2}$$

where $\{E_i^a\}_{i=1,\dots,n;1\leq|a|\leq r}$ is a base of L_n^r given by (3.5). Now ω_0 defines a connection Γ in $L^rM\,|\,U$ with the connection form ω such that

$$\sigma^*\omega = \omega_0,$$

where $\sigma\colon U\to L^rM$, $\sigma(x) = \psi^{-1}(x, e)$ is a section (see Proposition 1.4, p. 66 in [4]). We want to show that $\Phi_U = \tilde{\nabla}^\Gamma$. By Proposition 4.7 it is sufficient to show

$$(\Phi_U)(e_k, A^a) = \tilde{\nabla}^\Gamma_{e_k}A^a$$

for $k = 1, \dots, n$ and a such that $|a| = 1$, and since $\Phi_U(e_k, A^a)$ and $\tilde{\nabla}^\Gamma_{e_k}A^a$ are r-jet fields on U, the above condition is equivalent to the following one:

$$\Phi_U(e_k, A^a)\big(\varphi^{-1}(x, e)\big) = (\tilde{\nabla}^\Gamma_{e_k}A^a)\big(\psi^{-1}(x, e)\big) \tag{4.8.3}$$

for $k = 1, \dots, n$ and a such that $|a| = 1$.

To prove the last condition we remark that if we note

$$d\psi\Big(H^\Gamma_{e_k}\big(\psi^{-1}(x, e)\big)\Big) = e_k\oplus w,$$

where $w\in T_eL_n^r = F_n^r$ (see Section 3), then

$$\begin{aligned}0 &= \omega\big(d\psi^{-1}(e_k\oplus w)\big) = \omega\big(d\psi^{-1}(e_k\oplus 0)\big)+\omega\big(d\psi^{-1}(0\oplus w)\big)\\ &= \omega\big(d\sigma(e_k)\big)+w = (\sigma^*\omega)(e_k)+w = \omega_0(e_k)+w,\end{aligned}$$

and hence, by (4.8.2)

$$w = -\Gamma^i_{ka}E_i^a.$$

Thus, for $a = 1_j$, we obtain

$$\begin{aligned}(\tilde{\nabla}^\Gamma_{e_k}A^{1_j})\big(\psi^{-1}(x, e)\big) &= d_{\psi^{-1}(x,e)}A^{1_j}\Big(H_{e_k}\big(\psi^{-1}(x, e)\big)\Big)\\ &= d_{(x,e)}(A^{1_j}\circ\psi^{-1})\big(e_k\oplus(-\Gamma^i_{ka}E_i^a)\big).\end{aligned}$$

Since

$$(A^{1_j}\circ\psi^{-1})(x, \xi) = h_{\xi^{-1}}\Big(A^{1_j}\big(\psi^{-1}(x, e)\big)\Big) = h_{\xi^{-1}}(e^{1_j}),$$

if we note

$$f^j\colon L_n^r\to H_n^r,\quad f^j(\xi) = h_{\xi^{-1}}(e^{1_j}),$$

then from Leibniz's Formula (see Proposition 1.4, p. 11, in [4]) we obtain

$$\begin{aligned}(\tilde{\nabla}^\Gamma_{e_k}A^{1_j})\big(\psi^{-1}(x, e)\big) &= -d_ef^j(\Gamma^i_{ka}E_i^a)\\ &= -d_ef^j\Big(\sum_{p=1}^{r}\sum_{|a|=p}\Gamma^i_{ka}E_i^a\Big)\\ &= -\sum_{p=1}^{r}\sum_{|a|=p}\Gamma^i_{ka}d_{x_p}f^{(p)j}(E_i^a),\end{aligned}$$

where $f^{(p)j}$ are just as in Proposition 3.6 and

$$x_p = \begin{cases} I \in GL(n, R) & \text{if } p = 1, \\ 0 \in \Omega_p & \text{if } p \geqslant 2. \end{cases}$$

Now, using Proposition 3.6, we obtain

$$(\tilde{\nabla}^{\Gamma}_{e_k} A^{1j})\big(\psi^{-1}(x, e)\big) = -\Gamma^j_{ka} e^a.$$

Since

$$\begin{aligned} \Phi(e_k, A^{1j})\big(\psi^{-1}(x, e)\big) &= -\Gamma^j_{ka} A^a\big(\psi^{-1}(x, e)\big) \\ &= -\Gamma^j_{ka} e^a, \end{aligned}$$

condition (4.8.3) is proved. The uniqueness of Γ follows immediately from Proposition 4.1.

Proof of the main theorem (Theorem 2.6). Let

$$\psi_\varkappa \colon L^r M \mid U_\varkappa \to U_\varkappa \times L^r_n, \qquad \varkappa \in J,$$

be a family of trivialization such that $\{U_\varkappa\}_{\varkappa \in J}$ is an open covering of M. By Proposition 4.8 there is a connection $\Gamma_\varkappa$ in $L^r M \mid U_\varkappa$ such that

$$\Phi_{U_\varkappa} = \tilde{\nabla}^{\Gamma_\varkappa}.$$

Since $(\Phi_{U_\iota})_{U_\iota \cap U_\varkappa} = \Phi_{U_\iota \cap U_\varkappa}$ and $(\tilde{\nabla}^{\Gamma_\varkappa})_{U_a \cap U_\varkappa} = \tilde{\nabla}^{\Gamma_a | \pi^{-1}(U_a \cap U_\varkappa)}$, we have,

$$\tilde{\nabla}^{\Gamma_\iota | \pi^{-1}(U_\iota \cap U_\varkappa)} = \tilde{\nabla}^{\Gamma_\varkappa | \pi^{-1}(U_\iota \cap U_\varkappa)},$$

and hence, by Proposition 4.1, $\Gamma_\iota | \pi^{-1}(U_\iota \cap U_\varkappa) = \Gamma_\varkappa | \pi^{-1}(U_\iota \cap U_\varkappa)$. This implies that there is a connection Γ in $L^r M$ such that $\Gamma | \pi^{-1}(U_\varkappa) = \Gamma_\varkappa$. It is clear that $\Phi = \tilde{\nabla}^{\Gamma}$. The uniqueness of Γ follows from Proposition 4.1. The proof is now complete.

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Reçu par la Rédaction le 2. 12. 1974