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Concerning solutions of an exterior boundary-value problem for a system of non-linear parabolic equations

by P. Besala (Gdańsk)

In paper [2] we have considered Fourier's first problem in an unbounded domain for the system of equations

$$\frac{\partial u_s}{\partial t} = F_s\left(x, t, u_i, \frac{\partial u_s}{\partial x_j}, \frac{\partial^2 u_s}{\partial x_j \partial x_k}\right), \tag{0.1}$$

$$s, i = 1, \ldots, n, \quad j, k = 1, \ldots, m, \quad x = (x_1, \ldots, x_m).$$

We have proved in it certain uniqueness and existence theorems for solutions of this problem as well as a theorem concerning inequalities between the solutions of two systems of the form (0.1). In the existence theorem we have assumed the existence of a solution of a suitable problem in bounded domains. Using the same method as in [2] we discuss in the present paper theorems analogous to the mentioned above but concerning an exterior non-linear boundary value problem for (0.1) ([1]). In proving these theorems we refer to fragments of the proofs in [2].

Theorem I of this paper constitutes a generalization of a well known theorem proved by M. Krzyżański [3] for one linear equation with bounded coefficients and for a linear boundary condition. Theorems analogous to I and II have been established by J. Szarski [6], [7], [8] in domains whose intersections with planes $t = \text{const}$ are bounded. W. Mlak [5] has proved a theorem similar to II for strong inequalities in a bounded domain.

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§ 1. Let Δ be a bounded and closed domain of the m-dimensional Eulicdean space $\mathcal{E}^m$ of the variables $x_1, \ldots, x_m$ and let S be its complementary domain. As in [3], the boundary $F\Delta$ of Δ is assumed to be represented by the equation

$$\Gamma(x) = 0, \tag{1.1}$$

([1]) These results without the proofs have been included in [1] (part 2).

where $\Gamma(x)$, $x = (x_1, \dots, x_m)$, is a function with continuous and bounded derivatives of second order in the domain S, while it is of class C^1 in the closure $\bar{S}$ and satisfies the condition

$$|\operatorname{grad} \Gamma(x)| \geqslant \Gamma_0 > 0 . \tag{1.2}$$

We define

$$D^h = S \times (0, h\rangle , \quad \sigma^h = F\Delta \times (0, h\rangle , \quad 0 < h \leqslant +\infty . \tag{1.3}$$

The functions $F_s(x, t, y_i, z_j, z_{jk})$ appearing on the right-hand side of (0.1) are assumed to be defined for $(x, t) \in D^h$ and arbitrary y_i, z_j, z_{jk} $(i = 1, \dots, n;\ j, k = 1, \dots, m)$. The following definition given by J. Szarski is used here: the s-th equation of (0.1) is said to be parabolic with respect to a sequence of functions $w_1(x, t), \dots, w_n(x, t)$ of class C^1 if for every two systems of numbers $z_{jk}, \bar{z}_{jk}$ $(j, k = 1, \dots, m)$, $z_{jk} = z_{kj}$, $\bar{z}_{jk} = \bar{z}_{kj}$, such that the quadratic form

$$\sum_{j,k=1}^{m} (z_{jk} - \bar{z}_{jk}) \lambda_j \lambda_k$$

is non-positive for arbitrary $\lambda_1, \dots, \lambda_m$, the inequality

$$F_s\left(x, t, w_i(x, t), \frac{\partial w_s(x, t)}{\partial x_j}, z_{jk}\right) - F_s\left(x, t, w_i(x, t), \frac{\partial w_s(x, t)}{\partial x_j}, \bar{z}_{jk}\right) \leqslant 0$$

is satisfied. If every equation of the system (0.1) is parabolic with respect to a solution $u_i(x, t)$ $(i = 1, \dots, n)$ of this system, such a solution is called a parabolic one.

Let $\varphi_i(x)$ $(i = 1, \dots, n)$ be arbitrarily given functions defined and continuous for $x \in S$ while $G_s(x, t, y_1, \dots, y_n)$ $(s = 1, \dots, n)$ are defined for $(x, t) \in \sigma^h$ and arbitrary $y_1, \dots, y_n$.

For every $(x, t) \in \sigma^h$ and every s $(s = 1, \dots, n)$ let l_s be a straight half-line entering the interior of D^h (at point (x, t)) and parallel to the plane $t = 0$. The existence of a positive constant γ_0 such that

$$\cos(l_i, n_0) \geqslant \gamma_0 \quad (i = 1, \dots, n) \quad \text{for} \quad (x, t) \in \sigma^h \tag{1.4}$$

is assumed, n_0 being the normal to σ^h directed to the interior of D^h.

The problem which we will call (F) consists in finding a parabolic solution $u_i(x, t)$ $(i = 1, \dots, n)$ of (0.1), regular in D^h [2] possessing the derivatives du_s/dl_s at points of σ^h, fulfilling the initial condition

$$u_i(x, 0) = \varphi_i(x) \quad (i = 1, \dots, n) \quad \text{for} \quad x \in S \tag{1.5}$$

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[2] I.e. continuous in the closure $\overline{D^h}$ of D^h, possessing the derivative $\dfrac{\partial u_s}{\partial t}$ and continuous derivatives $\dfrac{\partial u_s}{\partial x_j}, \dfrac{\partial^2 u_s}{\partial x_j \partial x_k}$ $(s = 1, \dots, n;\ j, k = 1, \dots, m)$ in D^h .

and the boundary condition

$$(1.6) \quad \frac{du_s}{dl_s} + G_s(x, t, u_1, \ldots, u_n) = 0 \quad (s = 1, \ldots, n) \quad \text{for} \quad (x, t) \in \sigma^h .$$

We will say that the function $F_s(x, t, y_i, z_j, z_{jk})$ satisfies the $(\mathcal{L})$-condition if there exist positive constants $L_0, \ldots, L_4$ such that for arbitrary y_i, z_j, z_{jk}; $\bar{y}_i, \bar{z}_j, \bar{z}_{jk}$ $(i = 1, \ldots, n; j, k = 1, \ldots, m)$, $y_s \geqslant \bar{y}_s$, we have the inequality

$$F_s(x, t, y_i, z_j, z_{jk}) - F_s(x, t, \bar{y}_i, \bar{z}_j, \bar{z}_{jk})$$
$$\leqslant L_0 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| + (L_1|x| + L_2) \sum_{j=1}^{m} |z_j - \bar{z}_j| + (L_3|x|^2 + L_4) \sum_{i=1}^{n} |y_i - \bar{y}_i| ,$$

where, as usually, $|x| = \big(\sum_{i=1}^{m} x_i^2\big)^{1/2}$. Further, the function $G_s(x, t, y_1, \ldots, y_n)$ is said to satisfy the $(\bar{\mathcal{L}})$-condition if there is a positive constant L such that for arbitrary $y_i, \bar{y}_i$ $(i = 1, \ldots, n)$, $y_s \geqslant \bar{y}_s$, the inequality

$$G_s(x, t, y_1, \ldots, y_n) - G_s(x, t, \bar{y}_1, \ldots, \bar{y}_n) \leqslant L \sum_{i=1}^{n} |y_i - \bar{y}_i|$$

holds.

By $E_2(M, K)$ or shortly E_2 we denote the class of functions $\psi(x, t)$, defined in an unbounded domain, for which there exist positive constants M, K such that the inequality

$$|\psi(x, t)| \leqslant M \exp(K|x|^2)$$

is satisfied in this domain.

§ 2. THEOREM I. *If the functions F_s and G_s $(s = 1, \ldots, n)$ fulfil the conditions $(\mathcal{L})$ and $(\bar{\mathcal{L}})$ respectively, then the problem* (F) *has no more than one solution of class E_2 in the domain D^h.*

Proof. Let us take two arbitrary solutions $u_i^{(1)}$ and $u_i^{(2)}$ of the problem (F) belonging to E_2 in D^h. We shall show that

$$u_i \overset{\text{df}}{=} u_i^{(1)} - u_i^{(2)} \equiv 0 \quad (i = 1, \ldots, n) .$$

From the definition of class E_2 it follows that there exist $M, K_0 > 0$ such that

$$(2.1) \qquad |u_i| \leqslant M \exp(K_0|x|^2) \quad (i = 1, \ldots, n) \quad \text{for} \quad (x, t) \in D^h .$$

It may be assumed that

$$(2.2) \qquad \Gamma(x) \equiv |x| \quad \text{for} \quad |x| > R_0 ,$$

where R_0 is the radius of a sphere $|x| = R_0$ situated in $\mathcal{E}^m$ and containing the boundary $F\Delta$ in its interior. From that and from the assumptions of § 1 it follows that there exist $A, B > 0$ such that

$$(2.3)\qquad \Big|\sum_{j=1}^{m} \Gamma'_{x_j}(x)\Big| \leqslant A\,, \quad \Big|\sum_{j,k=1}^{m} \Gamma''_{x_j x_k}(x)\Big| \leqslant B \quad \text{in } S\,.$$

We shall make use of the function

$$(2.4)\qquad H(x, t; K) = \exp\left\{\frac{K[\Gamma(x) - p(K)]^2}{1-\mu(K)t} + \nu(K)t\right\}, \qquad K > K_0\,,$$

constructed by M. Krzyżański [3], where

$$(2.5)\qquad p(K) = \frac{nL+1}{2K\Gamma_0\gamma_0}\,,$$

$$(2.6)\qquad \mu(K) = 4KL_0A^2 + 2L_1A + \frac{nL_3+1}{K}\,,$$

and

$$(2.7)\qquad \nu(K) = \max\left\{\frac{(KL_0B + KL_1Ap + L_2pn)^2 + 2KL_0A^2 + L_3p^2n + L_4n + 1}{\gamma^2}\,,\right.$$
$$\left.\frac{[KL_0B + KA(L_1R_0 + L_2)]^2 + 2KL_0A^2 + n(L_3R_0^2 + L_4) + 1}{\gamma^2}\right\},$$

γ being arbitrarily choosen from the interval $(0, 1)$.

First, we shall give the proof of the theorem for the part $D^{h_0} = S \times \times (0, h_0\rangle$ of D^h, where

$$(2.8)\qquad h_0 = \frac{1-\gamma}{\mu(K)}\,.$$

Write

$$(2.9)\qquad \mathcal{F}H(x, t; K)$$
$$= L_0 \sum_{j,k=1}^{m} \left|\frac{\partial^2 H}{\partial x_j \partial x_k}\right| + (L_1|x| + L_2) \sum_{j=1}^{m} \left|\frac{\partial H}{\partial x_j}\right| + (L_3|x|^2 + L_4)nH - \frac{\partial H}{\partial t}\,.$$

We shall show that the function $H(x, t; K)$ satisfies the following inequality

$$(2.10)\qquad \mathcal{F}H(x, t; K) \leqslant -H(x, t; K) \quad \text{for} \quad (x, t) \in D^{h_0}\,.$$

Indeed, (2.8) yields:

$$(2.11)\qquad 0 < \gamma \leqslant 1 - \mu t \leqslant 1\,.$$

Taking advantage of (2.4), (2.3) and (2.11) we get

$$(2.12)\qquad \mathcal{F}H \leqslant \frac{H}{(1-\mu t)^2}\{4K^2L_0A(\Gamma - p)^2 + 2KL_0A^2 + 2KBL_0|\Gamma - p| +$$
$$+ (L_1|x| + L_2)2KA|\Gamma - p| + (L_3|x|^2 + L_4)n - \mu K(\Gamma - p)^2 - \nu\gamma^2\}\,.$$

If $|x| > R_0$, then in view of (2.2), the inequality $|x| \leqslant \big||x|-p\big|+p$ and (2.6) we obtain

$$\mathcal{F}H \leqslant \frac{H}{(1-\mu t)^2}\Big\{-\big[\big||x|-p\big|-(KL_0B+pKL_1A+npL_3)\big]^2+ \\ +(KL_0B+pKL_1A+npL_3)^2+2KL_0A^2+np^2L_3+nL_4-\nu\gamma^2\Big\}\,.$$

Hence by (2.7) we get (2.10).

If $|x| \leqslant R_0$, then from (2.6) and (2.12) it follows that

$$\mathcal{F}H \leqslant \frac{H}{(1-\mu t)^2}\Big\{-\big[|\Gamma-p|-\big(KL_0B+KA(L_1R_0+L_2)\big)\big]^2+ \\ +\big(KL_0B+KA(L_1R_0+L_2)\big)^2+2KL_0A^2+n(L_3R_0^2+L_4)-\nu\gamma^2\Big\}\,.$$

By virtue of (2.7) we see that now inequality (2.10) is also verified. Further, we substitute

$$(2.13)\qquad u_i^{(1)} = v_i^{(1)}H(x,t;K)\,,\qquad u_i^{(2)} = v_i^{(2)}H(x,t;K)\qquad (i=1,\ldots,n)\,.$$

Putting $v_i = v_i^{(1)} - v_i^{(2)}$ we have

$$(2.14)\qquad u_i = v_iH(x,t;K)\,.$$

Let us choose an increasing sequence $\{R_a\}$, $R_a > R_0$, $R_a \to \infty$ as $a\to\infty$. The part of the boundary of domain D^h lying on the plain $t=0$ will be denoted by S^0. Furthermore, denote by $D_a^{h_0}$ and S_a^0 the parts of domains D^{h_0} and S^0 respectively contained inside the cylindric surface $\sum_a$ with the equation $|x| = R_a$. Put $C_a^{h_0} = D^{h_0}\sum_a$. We will consider the sequence with the terms

$$A_a = \max_{(i)}\max_{(x,t)\in \overline{D_a^{h_0}}} |v_i(x,t)|\qquad (a=1,2,\ldots)\,.$$

From (2.14) it follows that the theorem will be proved if we show that $A_a = 0$ for any a. Observe that the sequence $\{A_a\}$ is non decreasing and $A_a \geqslant 0$ for every a. Therefore it suffices to prove that $A_a\to 0$ as $a\to\infty$. For this purpose notice that for any a there exists an integer i_a and a point $(x_a, t_a)\in \overline{D_a^{h_0}}$ such that $A_a = |v_{i_a}(x_a,t_a)|$. A priori the following cases are possible:

1) $(x_a,t_a)\in S_a^0$, 2) $(x_a,t_a)\in D_a^{h_0}$, 3) $(x_a,t_a)\in \sigma^{h_0}$, 4) $(x_a,t_a)\in C_a^{h_0}$.

Evidently in case 1) $v_{i_a}(x_a,t_a) = 0$ and so $A_a = 0$. In case 2) we have either 2′) $v_{i_a}(x_a,t_a) = 0$ or 2″) $v_{i_a}(x_a,t_a) > 0$ or 2‴) $v_{i_a}(x_a,t_a) < 0$. Case 2′) means that $A_a = 0$. If 2″) holds, then the same reasoning as in the proof of theorem I of [2] leads to the inequality

$$(2.15)\qquad 0 \leqslant \frac{\partial v_{i_a}(x_a,t_a)}{\partial t}H(x_a,t_a;K) \leqslant v_{i_a}(x_a,t_a)\mathcal{F}H(x_a,t_a;K)\,,$$

and thus, by (2.10), we prove the impossibility of case 2''). Case 2''') may be, as in [2], reduced to 2'') by the substitution $\bar{v}_i = -v_i$.

In case 3) we have: 3') $v_{i_a}(x_a, t_a) = 0$, which means that $A_a = 0$ or 3'') $v_{i_a}(x_a, t_a) > 0$ or 3''') $v_{i_a}(x_a, t_a) < 0$. If 3'') holds, then

$$\frac{dv_{i_a}(x_a, t_a)}{dl_{i_a}} \leqslant 0 , \tag{2.16}$$

since in the contrary case there would exist such a point (x_0, t_0) lying in the interior of D^{h_0} that $v_{i_a}(x_0, t_0) > v_{i_a}(x_a, t_a)$, and this contradicts the definition of $v_{i_a}(x_a, t_a)$.

On the other hand, by (1.6), (2.13) and (2.14), we obtain

$$\frac{dv_{i_a}}{dl_{i_a}} H + v_{i_a}\frac{dH}{dl_{i_a}} + G_{i_a}(x_a, t_a, v_1^{(1)} H, \ldots, v_n^{(1)} H) - G_{i_a}(x_a, t_a, v_1^{(2)} H, \ldots, v_n^{(2)} H) = 0 ,$$

whence, according to the $(\bar{\mathcal{L}})$-condition, we get

$$-\frac{dv_{i_a}}{dl_{i_a}} H \leqslant v_{i_a}\frac{dH}{dl_{i_a}} + L \sum_{i=1}^{n} |v_i| H \leqslant v_{i_a}\left(\frac{dH}{dl_{i_a}} + LHn\right) . \tag{2.17}$$

The sign of $\Gamma(x)$ in equation (1.1) may be chosen so that

$$\frac{\partial \Gamma}{\partial x_i} = |\operatorname{grad} \Gamma(x)| \cos(x_i, n_0) . \tag{2.18}$$

Therefore, on account of (1.1), (1.2), (1.4), (2.5) and (2.17) we derive

$$\begin{aligned} -\frac{dv_{i_a}}{dl_{i_a}} H &\leqslant v_{i_a} H\left(\frac{-2Kp}{1-\mu t_a} |\operatorname{grad} \Gamma| \cos(n_0, l_{i_a}) + nL\right) \\ &\leqslant v_{i_a} H(-2Kp\Gamma_0\gamma_0 + nL) = -v_{i_a} H < 0 , \end{aligned}$$

which contradicts (2.16). Repeating the same reasoning for the function $\bar{v}_{i_a} \overset{\mathrm{df}}{=} -v_{i_a}$ one may show that case 3''') is also impossible. Therefore, in each case 1)-3) we have $A_a = 0$.

In case 4), by (2.1) and (2.14), we obtain

$$|v_{i_a}(x_a, t_a)| \leqslant \frac{M \exp(K_0 |x_a|^2)}{\exp\left\{\dfrac{K(\Gamma - p)^2}{1-\mu t_a} + \nu t_a\right\}} = \frac{M \exp(K_0 R_a^2)}{\exp\left\{\dfrac{K(R_a - p)^2}{1-\mu t_a} + \nu t_a\right\}} \to 0 \quad \text{as} \quad a \to \infty .$$

From the previous considerations it follows that the last inequality holds for all cases 1)-4). Consequently $A_a \to 0$ as $a \to \infty$, q.e.d.

If $h > h_0$, then the change of the variable $t = \tilde{t} + jh_0$ $(j = 1, 2, \ldots)$ enables us to prove the theorem successively for the parts of D^h contained in the zones: $jh_0 \leqslant t \leqslant (j+1)h_0$.

§ 3. Let us now take a system of functions $\chi_s(y_1, \ldots, y_n, \tau)$ $(s = 1, \ldots, n)$, where τ denotes a sequence of variables different from $y_1, \ldots, y_n$. We will say that the function χ_s fulfils the (W)-condition with regard to the variables $y_1, \ldots, y_n$ if for $y_i \leqslant \bar{y}_i$, $i \neq s$, $y_s = \bar{y}_s$, we have

$$\chi_s(y_1, \ldots, y_n, \tau) \leqslant \chi_s(\bar{y}_1, \ldots, \bar{y}_n, \tau) .$$

THEOREM II. *If*

1° $u_i^{(1)}(x, t)$, $u_i^{(2)}(x, t)$ $(i = 1, \ldots, n)$ *are regular solutions of class* E_2 *in* D^h *of the systems of equations*

$$\frac{\partial u_s^{(1)}}{\partial t} = F_s^{(1)}\left(x, t, u_i^{(1)}, \frac{\partial u_s^{(1)}}{\partial x_j}, \frac{\partial^2 u_s^{(1)}}{\partial x_j \partial x_k}\right) \quad (s = 1, \ldots, n), \tag{3.1}$$

$$\frac{\partial u_s^{(2)}}{\partial t} = F_s^{(2)}\left(x, t, u_i^{(2)}, \frac{\partial u_s^{(2)}}{\partial x_j}, \frac{\partial^2 u_s^{(2)}}{\partial x_j \partial x_k}\right) \quad (s = 1, \ldots, n), \tag{3.2}$$

respectively,

2° *for each* s $(s = 1, \ldots, n)$ *the equation with index* s *of system* (3.1) *is parabolic with respect to the sequence* $u_i^{(1)}(x, t)$, *or the equation with the same index of system* (3.2) *is parabolic with respect to the sequence* $u_i^{(2)}(x, t)$,

3° *for each* s *the function* $F_s^{(1)}$ *or* $F_s^{(2)}$ *satisfies the* (W)*-condition with respect to the variables* $y_1, \ldots, y_n$ *and the* $(\mathcal{L})$*-condition,*

4° $F_s^{(1)}(x, t, y_i, z_j, z_{jk}) \leqslant F_s^{(2)}(x, t, y_i, z_j, z_{jk})$ $(s = 1, \ldots, n)$ *in the domain of existence of these functions,*

5° $u_i^{(1)}(x, 0) \leqslant u_i^{(2)}(x, 0)$ $(i = 1, \ldots, n)$ *for* $x \in S$,

6° *the functions* $u_i^{(1)}(x, t)$ *and* $u_i^{(2)}(x, t)$ *fulfil the boundary conditions*

$$\frac{du_s^{(1)}}{dl_s} + G_s^{(1)}(x, t, u_1^{(1)}, \ldots, u_n^{(1)}) = 0 \quad (s = 1, \ldots, n) \quad \textit{for} \quad (x, t) \in \sigma^h , \tag{3.3}$$

$$\frac{du_s^{(2)}}{dl_s} + G_s^{(2)}(x, t, u_1^{(2)}, \ldots, u_n^{(2)}) = 0 \quad (s = 1, \ldots, n) \quad \textit{for} \quad (x, t) \in \sigma^h , \tag{3.4}$$

respectively,

7° $G_s^{(1)}(x, t, y_1, \ldots, y_n) \leqslant G_s^{(2)}(x, t, y_1, \ldots, y_n)$ $(s = 1, \ldots, n)$ *for* $(x, t) \in \sigma^h$, $-\infty < y_i < +\infty$,

8° *for each* s $(s = 1, \ldots, n)$ *the function* $G_s^{(1)}$ *or* $G_s^{(2)}$ *fulfils condition* $(\bar{\mathcal{L}})$ *and condition* (W) *with regard to the variables* $y_1, \ldots, y_n$,
then the inequalities

$$u_i^{(1)}(x, t) \leqslant u_i^{(2)}(x, t) \quad (i = 1, \ldots, n)$$

are satisfied everywhere in D^h.

Proof. The functions $u_i^{(1)}$ and $u_i^{(2)}$ belong to the class E_2 in D^h; therefore there exist $M, K_0 > 0$ such that (2.1) is satisfied for $u_i \overset{\mathrm{df}}{=} u_i^{(1)} - u_i^{(2)}$.

We shall give the proof of theorem II for the domain D^{h_0} defined in the proof of theorem I. If $h > h_0$, then the change of the variable $t = \tilde{t} + jh_0$ $(j = 1, 2, \ldots)$ enables us to prove the theorem for the whole domain D^h. Furthermore, we shall restrict ourselves to the case where all the equations of (4.1) are parabolic (with respect to $u_i^{(2)}(x, t)$), all the functions $F_s^{(2)}$ satisfy the $(\mathcal{L})$ and the (W)-conditions and likewise all the functions $G_s^{(2)}$ satisfy the $(\overline{\mathcal{L}})$ and the (W)-conditions. The reasoning in the other cases is similar.

We retain here the notation introduced in the proof of theorem I. Using transformations (2.13) and (2.14) we denote

$$\bar{A}_a = \max_{(i)} \max_{(x,t) \in \overline{D_a^{h_0}}} v_i(x, t) .$$

The theorem will be proved if we show that $\bar{A}_a \leqslant 0$ for any a. For every a there exist an index i_a and a point $(x_a, t_a) \in \overline{D^{h_0}}$ such that $\bar{A}_a = v_{i_a}(x_a, t_a)$. Consider the cases:

$$1)\ (x_a, t_a) \in S_a^0 , \quad 2)\ (x_a, t_a) \in D_a^{h_0} , \quad 3)\ (x_a, t_a) \in \sigma^{h_0} , \quad 4)\ (x_a, t_a) \in C_a^{h_0} .$$

In case 1) we infer by assumption 5° and relation (2.14), that $\bar{A}_a \leqslant 0$. In case 2) we have: 2′) $v_{i_a}(x_a, t_a) \leqslant 0$ or 2″) $v_{i_a}(x_a, t_a) > 0$. 2′) means that $\bar{A}_a \leqslant 0$. When 2″) holds, then the same reasoning as in the proof of theorem II of [2] leads us to inequality (2.15), where $\mathcal{F}H$ is defined by (2.9) and, as was shown in the proof of theorem I, $\mathcal{F}H \leqslant -H < 0$. This contradiction shows that case 2″) is impossible. In case 3) the inequality $v_{i_a}(x_a, t_a) \leqslant 0$ means that $\bar{A}_a \leqslant 0$. Suppose now that $v_{i_a}(x_a, t_a) > 0$. From the definition of $v_{i_a}(x_a, t_a)$ it follows that inequality (2.16) is satisfied. On the other hand, by (3.3), (3.4), (2.13) and (2.14) we get

$$(3.5) \quad \frac{dv_{i_a}}{dl_{i_a}} H + v_{i_a} \frac{dH}{dl_{i_a}} + G_{i_a}^{(1)}(x_a, t_a, v_1^{(1)}H, \ldots, v_n^{1}H) - \\ - G_{i_a}^{(2)}(x_a, t_a, v_1^{(2)}H, \ldots, v_n^{(2)}H) = 0 .$$

By virtue of assumption 7°

$$(3.6) \quad G_{i_a}(x_a, t_a, v_1^{(1)}H, \ldots, v_n^{(1)}H) - G_{i_a}^{(2)}(x_a, t_a, v_1^{(1)}H, \ldots, v_n^{(1)}H) \leqslant 0 .$$

Let q $(1 \leqslant q \leqslant n)$ denote the number of those functions among $v_1, \ldots, v_n$ which assume a positive value at point (x_a, t_a). Without loss of generality it may be assumed that these are the functions $v_1, \ldots, v_q$ (thereby $i_a \leqslant q$). According to the (W)-condition we get

$$(3.7) \quad G_{i_a}^{(2)}(x_a, t_a, v_1^{(1)}H, \ldots, v_q^{(1)}H, v_{q+1}^{(1)}H, \ldots, v_n^{(1)}H) - \\ - G_{i_a}^{(2)}(x_a, t_a, v_1^{(1)}H, \ldots, v_q^{(1)}H, v_{q+1}^{(2)}H, \ldots, v_n^{(2)}H) \leqslant 0 .$$

By virtue of condition $(\bar{\mathcal{L}})$ we derive

$$
\begin{aligned}
(3.8)\qquad & v_{i_a}\frac{dH}{dl_{i_a}} + G^{(2)}_{i_a}(x_a, t_a, v_1^{(1)}H, \ldots, v_q^{(1)}H, v_{q+1}^{(2)}H, \ldots, v_n^{(2)}H) - \\
& - G^{(2)}_{i_a}(x_a, t_a, v_1^{(2)}H, \ldots, v_q^{(2)}H, v_{q+1}^{(2)}H, \ldots, v_n^{(2)}H) \\
& \leqslant v_{i_a}\frac{dH}{dl_{i_a}} + L\sum_{i=1}^{q}|v_i|H \leqslant v_{i_a}\left(\frac{dH}{dl_{i_a}} + nLH\right).
\end{aligned}
$$

But in view of (1.1), (1.2), (1.4), (2.5) and (2.18) we have, as in the proof of theorem I,

$$\frac{dH}{dl_{i_a}} + nLH \leqslant -H < 0\,.$$

Now adding (3.8), (3.7), (3.6) we obtain, by (3.5), $dv_{i_a}/dl_{i_a} > v_{i_a} > 0$ which contradicts (2.16). Thus we have proved that $\bar{A}_a \leqslant 0$ in each of the cases 1)-3). In case 4), by (2.14) and (2.1), the following relation is satisfied

$$\bar{A}_a \leqslant \frac{M\exp(K_0R_a^2)}{\exp\left\{\dfrac{K(R_a-p)^2}{1-\mu t_a} + \nu t_a\right\}} \to 0 \quad \text{as} \quad a\to\infty\,.$$

Thereby this relation holds for the cases previously considered. On the other hand the sequence $\{\bar{A}_a\}$ is non-decreasing. Hence we conclude that for every a, $\bar{A}_a \leqslant 0$, which completes the proof.

The corollary inserted in [2] concerning the weakening of the $(\mathcal{L})$-condition is also true here relating to theorems I and II.

§ 4. Using the notation of § 2 let us put

$$(4.1)\qquad \bar{R} = \max\big(R_0, p(K)\big)\,, \quad h_1 = \frac{1-\gamma}{\mu(K+k_0)}\,, \quad 0<\gamma<1\,, \quad k_0>0\,.$$

Lemma. *If* $K > \dfrac{1}{2}\sqrt{\dfrac{nL_2+1}{AL_0}}$ *and* $k_0 > 0$, *then the inequality*

$$\frac{H(x,t;K)}{H(x,t;K+k_0)} \leqslant \exp\{-k_0[\Gamma(x)-p(K+k_0)]^2\}$$

holds for $|x| > \bar{R}$, $(x,t)\in D^{h_1}$ (see (1.3)).

Proof. By (2.5) we have $p(K+k_0) < p(K)$. Hence and from (2.2) we deduce that $[\Gamma(x)-p(K)]^2 \leqslant [\Gamma(x)-p(K+k_0)]^2$ for $|x| > \bar{R}$. By virtue of (2.7) $\nu(K+k_0) > \nu(K)$. Owing to (2.6), for $K > \frac{1}{2}\sqrt{\frac{nL_2+1}{AL_0}}$, we obtain $\mu(K+k_0) > \mu(K)$. Consequently

$$\frac{H(x,t;K)}{H(x,t;K+k_0)}$$
$$\leqslant \exp\left\{\frac{K[\Gamma-p(K)]^2}{1-\mu(K)t} - \frac{K[\Gamma-p(K+k_0)]^2}{1-\mu(K+k_0)t} - \frac{k_0[\Gamma-p(K+k_0)]^2}{1-\mu(K+k_0)t} + \right.$$
$$\left. +[\nu(K)-\nu(K+k_0)]t\right\}$$
$$\leqslant \exp\left\{-\frac{k_0[\Gamma-p(K+k_0)]^2}{1-\mu(K+k_0)t}\right\} \leqslant \exp\{-k_0[\Gamma-p(K+k_0)]^2\}.$$

Now let $\Phi_i(x,t)$ $(i = 1, \dots, n)$ be arbitrary continuous functions of class E_2 in $\overline{D^{h_1}}$ and let $G_s(x,t,y_1,\dots,y_n)$ be given functions defined for $(x,t)\in\sigma^{h_1}$, $-\infty < y_i < +\infty$, and fulfilling the $(\overline{\mathfrak{L}})$-condition.

Take an increasing sequence $\{R_a\}$, $R_a > \bar{R}$, $R_a \to \infty$ as $a \to \infty$.

THEOREM III. *If*

1° *for arbitrary* $\Phi_i(x,t)$ *continuous in* $\overline{D^{h_1}}$ *and for every* a *there exists a parabolic solution* $u_i^a(x,t)$ $(i = 1, \dots, n)$ *of* (0.1), *regular in* $D_a^{h_1}$ *and fulfilling the conditions*

$$(4.2)\qquad u_i^a(x,t) = \Phi_i(x,t) \quad (i=1,\dots,n) \quad \text{on the set } S_a^0 + C_a^{h_1},$$

$$(4.3)\qquad \frac{du_s^a}{dl_s} + G_s(x,t,u_1^a,\dots,u_n^a) = 0 \quad (s=1,\dots,n) \quad \text{for} \quad (x,t)\in\sigma^{h_1},$$

2° *the functions* F_s *and* G_s $(s=1,\dots,n)$ *satisfy the conditions* $(\mathfrak{L})$ *and* $(\overline{\mathfrak{L}})$ *respectively,*

3° *the functions* $F_s(x,t,0,0,0)$ *belong to* E_2 *in* D^{h_1} *while* $g_s(x,t) \stackrel{\text{df}}{=} G_s(x,t,0,\dots,0)$ *are bounded for* $(x,t)\in\sigma^{h_1}$,

4° $\varphi_i(x)$ *are given continuous functions of class* E_2 *in* S;

then the problem (F) (for $h = h_1$, see § 1) *possesses at least one solution and this solution belongs to* E_2 *in* D^{h_1}.

Proof. First, we shall prove that the sequences $\{u_i^a(x,t)\}$ $(i=1,\dots,n)$, $a\to\infty$ (see assumption 1°), are almost uniformly convergent in D^{h_1}. For this purpose let us choose continuous functions $\Phi_i(x,t)$ of class E_2 in D^{h_1} so that $\Phi_i(x,0) = \varphi_i(x)$ for $x\in S$ $(i=1,\dots,n)$.

It may be assumed that the functions $\varphi_i(x)$, $\Phi_i(x,t)$ and $F_i(x,t,0,0,0)$ $(i=1,\dots,n)$ belong to the class $E_2(M,K_0)$ with the same constants M and K_0.

We introduce the substitution

$$(4.4)\qquad u_i^a = v_i^a H(x,t;K) \quad (i=1,\dots,n),$$

where $H(x, t; K)$ is given by (2.4) with

$$K = \max\left(K_0, \frac{1}{2}\sqrt{\frac{nL_3+1}{AL_0}}\right). \tag{4.5}$$

For an arbitrary fixed a let i_a and (x_a, t_a) be an integer and a point respectively for which

$$\bar{\bar{A}}_a \stackrel{\text{df}}{=} \max_{(i)} \max_{(x,t)\in \overline{D^{h_1}_a}} |v^a_i(x, t)| = |v^a_{i_a}(x_a, t_a)|$$

and consider the following cases:

1) $(x_a, t_a) \in D^{h_1}$. Making use of the property of function (2.4) one can prove, as in the proof of theorem III included in [2], that for this case

$$|v^a_{i_a}(x_a, t_a)| \leqslant M, \tag{4.6}$$

M being a constant appearing in the definition of class $E_2(M, K_0)$.

2) $(x_a, t_a) \in S^0_a + C^{h_1}_a$. Now from (4.2) and (4.4) we derive (4.6);

3) $(x_a, t_a) \in \sigma^{h_1}$. Suppose $v^a_{i_a}(x_a, t_a) > 0$. By (4.3) and (4.4) the function $v^a_{i_a}$ satisfies the boundary condition which may be written in the form

$$-\frac{dv^a_{i_a}}{dl_{i_a}} H = v^a_{i_a}\frac{dH}{dl_{i_a}} + G_{i_a}(x_a, t_a, v^a_1 H, \ldots, v^a_n H) - $$
$$- G_{i_a}(x_a, t_a, 0, 0, \ldots, 0) + G_{i_a}(x_a, t_a, 0, 0, \ldots, 0).$$

According to the $(\bar{\mathcal{L}})$-condition we have

$$-\frac{dv^a_{i_a}}{dl_{i_a}} \leqslant v^a_{i_a}\frac{dH}{dl_{i_a}} + L\sum_{i=1}^{n} |v^a_i| H + g_{i_a}(x_a, t_a)$$

and hence, by (1.1) and (2.18) we get

$$-\frac{dv_{i_a}}{dl_{i_a}} \leqslant v^a_{i_a}\left[\frac{-2Kp}{1-\mu t}|\operatorname{grad}\Gamma|\cos(n_0, l_{i_a}) + nL\right] + \frac{g_{i_a}(x_a, t_a)}{H}.$$

By our assumption $g_{i_a}(x, t)$ is bounded. We may assume that $g_{i_a}(x_a, t_a)/H \leqslant M$. Furthermore, relations (1.2), (1.3) and (2.5) yield

$$-\frac{dv^a_{i_a}}{dl_{i_a}} \leqslant v^a_{i_a}(-2Kp\Gamma_0\gamma_0 + nL) + M \leqslant -v^a_{i_a} + M.$$

On the other hand $dv^a_{i_a}/dl_{i_a} \leqslant 0$. Hence we deduce that $v^a_{i_a} \leqslant M$. If $v^a_{i_a}(x_a, t_a) < 0$, then, repeating the above reasoning for the function $\bar{v}^a_{i_a} = -v^a_{i_a}$, we should have $v^a_{i_a}(x_a, t_a) \geqslant -M$, or in each of the cases 1)-3) $\bar{\bar{A}}_a \leqslant M$ for every a. Therefore for a natural number $\beta > a$ we get $\bar{\bar{A}}_\beta \leqslant M$ too. Putting

$$u^{a\beta}_i = u^a_i - u^\beta_i, \quad v^{a\beta}_i = v^a_i - v^\beta_i \quad (i = 1, \ldots, n) \tag{4.7}$$

we have

$$(4.8)\qquad u_i^{\alpha\beta} = v_i^{\alpha\beta} H(x, t; K) \quad \text{and} \quad |v_i^{\alpha\beta}| \leqslant 2M \quad \text{for} \quad (x, t) \in \overline{D_\alpha^{h_1}}.$$

Let $\overset{*}{v}{}_i^{\alpha}$, $\overset{*}{v}{}_i^{\beta}$ be the functions defined by the relations

$$(4.9)\qquad u_i^{\alpha} = \overset{*}{v}{}_i^{\alpha} H(x, t; K + k_0), \quad u_i^{\beta} = \overset{*}{v}{}_i^{\beta} H(x, t; K + k_0) \quad (i = 1, \dots, n),$$

k_0 being a positive constant. Denoting $\overset{*}{v}{}_i^{\alpha\beta} = \overset{*}{v}{}_i^{\alpha} - \overset{*}{v}{}_i^{\beta}$ we obtain

$$(4.10)\qquad u_i^{\alpha\beta} = \overset{*}{v}{}_i^{\alpha\beta} H(x, t; K + k_0) \quad (i = 1, \dots, n).$$

Let us denote

$$A_{\alpha\beta} = \max_{(i)} \max_{(x,t) \in \overline{D_\alpha^{h_1}}} |\overset{*}{v}{}_i^{\alpha\beta}(x, t)|.$$

Evidently, there exist an index $i_{\alpha\beta}$ and a point $(x_{\alpha\beta}, t_{\alpha\beta}) \in \overline{D_\alpha^{h_1}}$ for which $A_{\alpha\beta} = |\overset{*}{v}{}_{i_{\alpha\beta}}^{\alpha\beta}(x_{\alpha\beta}, t_{\alpha\beta})|$. If this point belongs to $D_\alpha^{h_1}$, then likewise as in the proof of theorem III in [2] it can be shown that $A_{\alpha\beta} = 0$. Further, if $(x_{\alpha\beta}, t_{\alpha\beta}) \in \sigma^{h_1}$, then using the same reasoning as in the proof of theorem I of this paper, one can show that $A_{\alpha\beta} = 0$ too. Finally, let us consider the case: $(x_{\alpha\beta}, t_{\alpha\beta}) \in S_\alpha^0 + C_\alpha^{h_1}$. Owing to (4.8), (4.10) and the lemma we have

$$|\overset{*}{v}{}_{i_{\alpha\beta}}^{\alpha\beta}(x_{\alpha\beta}, t_{\alpha\beta})| \leqslant \frac{2MH(x_{\alpha\beta}, t_{\alpha\beta}; K)}{H(x_{\alpha\beta}, t_{\alpha\beta}; K + k_0)} \leqslant 2M \exp\{-k_0[R_\alpha - p(K + k_0)]^2\}.$$

According to the above considerations we conclude that the inequalities

$$(4.11)\qquad |\overset{*}{v}{}_i^{\alpha\beta}(x, t)| \leqslant 2M \exp\{-k_0[R_\alpha - p(K + k_0)]^2\} \quad (i = 1, \dots, n)$$

hold for $(x, t) \in \overline{D_\alpha^{h_1}}$. Now let α_0 be an arbitrary integer. Put

$$N_0 = \max_{\overline{D_{\alpha_0}^{h_1}}} H(x, t; K + k_0).$$

From (4.10) and (4.11) we derive the inequalities

$$|u_i^{\alpha\beta}(x, t)| \leqslant 2MN_0 \exp\{-k_0[R_\alpha - p(K + k_0)]^2\} \quad \text{for} \quad \alpha > \alpha_0,\ (x, t) \in \overline{D_{\alpha_0}^{h_1}}.$$

Hence we deduce that for every $\varepsilon > 0$ there exists $\alpha_1(\varepsilon)$, $\alpha_1 > \alpha_0$, such that for $\beta > \alpha > \alpha_1$ the inequalities $|u_i^{\alpha\beta}(x, t)| < \varepsilon$ $(i = 1, \dots, n)$ hold for $(x, t) \in \overline{D_{\alpha_0}^{h_1}}$. It means that the sequences $\{u_1^{\alpha}(x, t)\}, \dots, \{u_n^{\alpha}(x, t)\}$ almost uniformly converge in $\overline{D^{h_1}}$.

It is easy to see that the limit functions of these sequences belong to class E_2 in D^{h_1}.

Similarly as in the proof of theorem III of [2] it can be shown that these functions constitute a solution of the (F)-problem for the domain D^{h_1}.

Remark I. By theorem I it is the unique solution of the considered problem.

Remark II. If the domain S (see § 1) is the half-space

$$S\{x_1 > 0, -\infty < x_i < +\infty, (i = 1, \ldots, n)\},$$

then the theorems similar to I, II and III can be proved when instead of $H(x, t; K)$ defined by (2.4) the function

$$H_1(x, t; K) = \exp\left\{\frac{K\left[(x_1 - p_1(K))^2 + \sum_{i=2}^{m} x_i^2\right]}{1 - \mu_1(K)t} + \nu_1(K)t\right\}$$

is applied (see [3]).

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