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On the asymptotic behavior of solutions of the equation $y''+p(x)y = 0$

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Abstract. In this note we give sufficient conditions for each solution of $y''(x)+p(x)y(x) = 0$ approching to zero. Our result unifies a part of [5] and Theorem 1 of [4].

1. Introduction. We are concerned with the asymptotic behavior of solutions of the second order differential equation

$$y''(x)+p(x)y(x) = 0, \tag{1.1}$$

where $p(x) > 0$ for $x \geqslant a$ and $p'(x) \in C[a, \infty)$.

Recently, Stachurska [5] obtained the following result which is an improvement of [3], namely,

Theorem A. *If*

$$p(x) \in C^3(a, \infty), \quad p(x)\to\infty \quad \text{as } x\to\infty, \tag{1.2}$$

and

$$\limsup_{x\to\infty} \big(p^{-\frac{1}{4}}(x)\big) \int_a^x \big|\big(p^{-\frac{1}{4}}(t)\big)'''\big|\, dt < 1,$$

then, for each solution $y(x)$ *of* (1.1),

$$y(x)\to 0 \quad \text{as } x\to\infty. \tag{1.3}$$

In the present paper we will show that the same result may be obtained under more general assumptions than those of Theorem A and Theorem 1 of [4]. We also unify Theorem A and Theorem 1 of [4] into a single criterion for solutions of (1.1) satisfying (1.3). In this paper we say that a solution of (1.1) is oscillatory if it has no largest zero. The oscillation criteria which we use in this paper may be found in [1].

We will use the following notations throughout this paper:

$$u(x) = 2g'(x)p(x)+g(x)p'(x), \tag{N1}$$

$$u_+(x) = \max\big(u(x), 0\big), \tag{N2}$$

$$G(x) = -1/\Big(\int_b^x u(t)\, dt\Big), \tag{N3}$$

if $G(x)$ is well-defined, for some constant $b > 0$, where $g(x)$ satisfies

$$g(x) > 0 \quad \text{for } x > a \quad \text{and} \quad g(x) \in C^3[a, \infty). \tag{1.4}$$

2. Theorems. We will use an argument similar to Theorem 1 of [5] to prove the following theorems.

THEOREM 1. *Assume that there exists a function $g(x)$ which satisfies* (1.4), $g(x)p(x) \to \infty$ *as* $x \to \infty$, *and*

$$\limsup_{x\to\infty} \big(1/(g(x)p(x))\big) \int_a^x \big(|g'''(t)|/2 + u_+(t)\big)\,dt < 1.$$

Then, for each oscillatory solution $y(x)$ of (1.1), (1.3) *holds.*

Proof. Let $y(x)$ be a non-trivial oscillatory solution of (1.1). We shall first prove the boundedness of $y(x)$ for $a \leqslant x < \infty$. To show that $y(x)$ is bounded, it is sufficient to prove that the absolute values of $y(x)$ at its relative maximum and minimum points are bounded. Suppose that these values are unbounded. Then there exists a sequence $\{c_n\}$ such that

$$y'(c_n) = 0, \quad |y(c_n)| = \max\big(|y(x)|;\, x \in [a, c_n]\big) \quad (n = 1, 2, \ldots),$$
$$c_n \to \infty \quad \text{as } n \to \infty \quad \text{and} \quad |y(c_n)| \to \infty \quad \text{as } n \to \infty.$$

Define the function $w(x)$ by

$$w(x) = g(x)\big(y'(x)\big)^2 + \big(g''(x)/2 + g(x)p(x)\big)y^2(x) - g'(x)y(x)y'(x).$$

Then we obtain for $a \leqslant z \leqslant x$

$$w(x) = w(z) + \int_z^x \big(g'''(t)/2 + u(t)\big)y^2(t)\,dt. \tag{2.1}$$

From (2.1), for $x = c_n$ and $z = a$ $(n = 1, 2, \ldots)$, we obtain

$$w(c_n) = \big(g''(c_n)/2 + g(c_n)p(c_n)\big)y^2(c_n) = w(a) + \int_a^{c_n} \big(g'''(t)/2 + u(t)\big)y^2(t)\,dt.$$

Since

$$g''(x) = g''(a) + \int_a^x g'''(t)\,dt, \tag{2.2}$$

we have

$$\begin{aligned} g(c_n)p(c_n)y^2(c_n) &= w(a) + \tfrac{1}{2}\int_a^{c_n} g'''(t)\big(y^2(t) - y^2(c_n)\big)dt - \tfrac{1}{2}g''(a)y^2(c_n) + \int_a^{c_n} u(t)y^2(t)\,dt \\ &\leqslant |w(a)| + cy^2(c_n) + \tfrac{1}{2}\int_a^{c_n} |g'''(t)|\,|y^2(t) - y^2(c_n)|\,dt + \int_a^{c_n} u_+(t)y^2(t)\,dt \\ &\leqslant |w(a)| + cy^2(c_n) + \tfrac{1}{2}y^2(c_n)\int_a^{c_n} |g'''(t)|\,dt + y^2(c_n)\int_a^{c_n} u_+(t)\,dt, \end{aligned}$$

where $c = \frac{1}{2}|g''(a)|$. Hence we obtain the inequality

$$(2.3)\quad g(c_n)p(c_n)y^2(c_n)\times \times\Big(1-c/\big(g(c_n)p(c_n)\big)-\big(1/g(c_n)p(c_n)\big)\int_a^{c_n}\big(\tfrac{1}{2}|g'''(t)|+u_+(t)\big)dt\Big)\leqslant |w(a)|.$$

Since, by assumption, $g(c_n)p(c_n)y^2(c_n)\to\infty$ as $n\to\infty$, from (2.3) we obtain

$$\liminf_{x\to\infty}\Big(1-\big(1/(g(x)p(x))\int_a^x\big(\tfrac{1}{2}|g'''(t)|+u_+(t)\big)dt\big)\Big)\leqslant 0,$$

contrary to the assumption.

Thus we have

$$d = \limsup_{x\to\infty} y^2(x) < \infty.$$

To complete the proof of (1.3) it is sufficient to prove that $d = 0$. Suppose that $d > 0$. Then for every $\varepsilon > 0$ there exists a number M such that

$$(2.4)\qquad y^2(x) < d+\varepsilon \quad \text{for } x > M.$$

Let $x_1 < x_2 < \ldots$ be the successive relative maximum and minimum points of $y(x)$ and $M < x_1$. Then $y'(x_n) = 0$ and $x_n\to\infty$ as $n\to\infty$. From (2.1) for $x = x_n$ and $z = b$, where $M < b < x_1$ we have

$$(2.5)\quad \big(\tfrac{1}{2}g''(x_n)+g(x_n)p(x_n)\big)y^2(x_n) = w(a)+\tfrac{1}{2}\int_b^{x_n} g'''(t)y^2(t)\,dt+\int_b^{x_n} u(t)y^2(t)\,dt.$$

From (2.2) and (2.5) we obtain the inequality

$$g(x_n)p(x_n)y^2(x_n) \leqslant |w(b)|+\tfrac{1}{2}\int_b^{x_n}\big|g'''(t)\big(y^2(t)-y^2(x_n)\big)\big|\,dt+cy^2(x_n)+\int_b^{x_n} u_+(t)y^2(t)\,dt,$$

where $c = \frac{1}{2}|g''(b)|$. From (2.4) we obtain the further inequality

$$y^2(x_n) = \big(|w(b)|+c(d+\varepsilon)\big)/\big(g(x_n)p(x_n)\big)+ +\big((d+\varepsilon)/\big(2g(x_n)p(x_n)\big)\big)\int_a^{x_n}|g'''(t)|\,dt+\big((d+\varepsilon)/\big(g(x_n)p(x_n)\big)\big)\int_a^{x_n} u_+(t)\,dt.$$

Hence,

$$(2.6)\quad \limsup_{n\to\infty} y^2(x_n)\leqslant (d+\varepsilon)\limsup_{n\to\infty}\Big(\big(1/\big(g(x_n)p(x_n)\big)\big)\int_a^{x_n}\big(\tfrac{1}{2}|g'''(t)|+u_+(t)\big)dt\Big).$$

Write

$$d_1 = \limsup_{x\to\infty}\Big(\big(1/\big(g(x)p(x)\big)\big)\int_a^{x}\big(\tfrac{1}{2}|g'''(t)|+u_+(t)\big)dt\Big).$$

If $d_1 = 0$, then from (2.6) it follows that $d = 0$ and we have a contradiction. If $d_1 > 0$, then for $\varepsilon < d(1-d_1)/d_1$, we have $d_1(d+\varepsilon) < d$, contrary to (2.6). Therefore, (1.3) is satisfied.

If we now take $g(x) = p^{-\frac{1}{2}}(x)$ in Theorem 1 and use oscillation criteria of [1], we obtain the following corollary which improves Theorem A and is related to Theorem 9.5.1 of [2], p. 472.

Corollary 1.1. *Let $p(x)$ satisfy* (1.2). *If*

$$\limsup_{x\to\infty} p^{-\frac{1}{2}}(x)\int_a^x \left|\left(p^{-\frac{1}{2}}(t)\right)'''\right| dt < 2,$$

then every solution $y(x)$ of (1.1) *satisfies* (1.3).

If we take $g(x) = p^{-b}(x)$ for $0 < b < \frac{1}{2}$ in Theorem 1 and use the oscillation criteria of [1], we obtain the following result which improves Theorem 1 of [4] for $0 < b < \frac{1}{2}$.

Corollary 1.2. *Let $p(x)$ satisfy* (1.2) *and $p'(x) \geqslant 0$ for $x \geqslant a$. If*

$$\limsup_{x\to\infty} p^{b-1}(x)\int_a^x \left|\left(p^{-b}(t)\right)'''\right| dt < 2b/(1-b)$$

for some $0 < b < \frac{1}{2}$, then every solution $y(x)$ of (1.1) *satisfies* (1.3).

If we take $g(x) = p^{-b}(x)$ for $\frac{1}{2} < b < 1$ in Theorem 1, we obtain the following result which improves Theorem 1 of [4].

Corollary 1.3. *Let $p(x)$ satisfy* (1.2) *and $p'(x) \geqslant 0$ for $x > a$. If*

$$\limsup_{x\to\infty} p^{b-1}(x)\int_a^x \left|\left(p^{-b}(t)\right)'''\right| dt < 2$$

for some $\frac{1}{2} < b < 1$, then every solution $y(x)$ of (1.1) *satisfies* (1.3).

If we take $g(x) = p^{-1}(x)\left(\log p(x)\right)^b$, $b > 0$, in Theorem 1, we have the following result.

Corollary 1.4. *Let $p(x)$ satisfy* (1.2) *and $p'(x) \geqslant 0$ for $x \geqslant a$. If*

$$\limsup_{x\to\infty} \left(1/\left(\log p(x)\right)^b\right)\int_a^x \left|\left(p^{-1}(t)\left(\log p(t)\right)^b\right)'''\right| dt < 2,$$

then every oscillatory solution $y(x)$ of (1.1) *satisfies* (1.3).

We will now remove the condition $g(x)p(x)\to\infty$ as $x\to\infty$ in the above theorem. However, we require other more restrictive conditions than above on $p(x)$.

Theorem 2. *Let $p(x)$ satisfy* (1.2), *$g(x)$ satisfy* (1.4),

$$p'(x) \geqslant 0 \quad \text{and} \quad \left(g(x)p(x)\right)' \leqslant 0 \quad \text{for all } x \geqslant a, \tag{2.7}$$

$$\lim_{x\to\infty} G(x) = 0 \quad \text{and} \quad \limsup_{x\to\infty} G(x)\int_a^x |g'''(t)|\, dt = 0;$$

then every oscillatory solution $y(x)$ of (1.1) *satisfies* (1.3).

Proof. Let $y(x)$ be any non-trivial oscillatory solution of (1.1). Thus there exists the sequence $\{x_n\}$ such that $a < x_n$, $x_n \to \infty$ as $n \to \infty$, and $y'(x_n) = 0$, $n = 1, 2, \ldots$ Define

$$v(x) = y^2(x) + p^{-1}(x)\big(y'(x)\big)^2.$$

Since

$$dv(x)/dx = -p'(x)p^{-2}(x)\big(y'(x)\big)^2 \leqslant 0,$$

$\lim_{x\to\infty} v(x) = L$ exists. If $L = 0$, then (1.3) holds. Now assume that $L > 0$. Thus we have for $x \geqslant x_1$

$$y^2(x) \leqslant v(x) \leqslant v(x_1) = y^2(x_1) \quad \text{and} \quad L \leqslant v(x). \tag{2.8}$$

From (2.1) for $x = x_n$ and $z = x_1$ ($n = 2, 3, \ldots$), we obtain

$$\begin{aligned} w(x_n) &= \big(\tfrac{1}{2}g''(x_n) + g(x_n)p(x_n)\big)y^2(x_n) \\ &= w(x_1) + \int_{x_1}^{x_n} \big(\tfrac{1}{2}g'''(t) + u(t)\big)y^2(t)\,dt \end{aligned}$$

or

$$\begin{aligned} -\int_{x_1}^{x_n} u(t)y^2(t)\,dt &\leqslant w(x_1) + \tfrac{1}{2}\int_{x_1}^{x_n} g'''(t)y^2(t)\,dt - \tfrac{1}{2}g''(x_n)y^2(x_n) \\ &\leqslant w(x_1) + \tfrac{1}{2}\int_{x_1}^{x_n} g'''(t)\big(y^2(t) - y^2(x_n)\big)\,dt - \tfrac{1}{2}g''(x_1)y^2(x_n). \end{aligned}$$

Thus

$$\begin{aligned} -\int_{x_1}^{x_n} u(t)v(t)\,dt \leqslant w(x_1) &+ \tfrac{1}{2}\int_{x_1}^{x_n} g'''(t)\big(y^2(t) - y^2(x_n)\big)\,dt - \\ &- \tfrac{1}{2}g''(x_1)y^2(x_n) - \int_{x_1}^{x_n} u(t)p^{-1}(t)\big(y'(t)\big)^2\,dt. \end{aligned} \tag{2.9}$$

From (2.7), by integration by parts, we have

$$\begin{aligned} -\int_{x_1}^{x_n} u(t)p^{-1}(t)\big(y'(t)\big)^2\,dt &= -2\int_{x_1}^{x_n} g'(t)\big(y'(t)\big)^2\,dt - \int_{x_1}^{x_n} g(t)p'(t)p^{-1}(t)\big(y'(t)\big)^2\,dt \\ &\leqslant -2\int_{x_1}^{x_n} g'(t)\big(y'(t)\big)^2\,dt = 4\int_{x_1}^{x_n} g(t)y'(t)y''(t)\,dt \\ &= -4\int_{x_1}^{x_n} g(t)p(t)y(t)y'(t)\,dt \\ &= -2g(x_n)p(x_n)y^2(x_n) + 2g(x_1)p(x_1)y^2(x_1) + \\ &\quad + 2\int_{x_1}^{x_n} \big(g(t)p(t)\big)'y^2(t)\,dt \leqslant 2g(x_1)p(x_1)y^2(x_1); \end{aligned}$$

(2.9) now becomes

$$(2.10)\qquad -\int_{x_1}^{x_n} u(t)v(t)\,dt \leqslant w(x_1)+\tfrac{1}{2}\int_{x_1}^{x_n} g'''(t)\big(y^2(t)-y^2(x_n)\big)\,dt -$$
$$-\tfrac{1}{2}g''(x_1)y^2(x_n)+2g(x_1)p(x_1)y^2(x_1)$$
$$\leqslant c+\tfrac{1}{2}\int_{x_1}^{x_n} |g'''(t)|\big(|y^2(t)|+|y^2(x_n)|\big)\,dt+c_1y^2(x_n)$$
$$\leqslant c+y^2(x_1)\int_{x_1}^{x_n} |g'''(t)|\,dt+c_1y^2(x_1)$$

because of (2.8), where $c = |w(x_1)|+2g(x_1)p(x_1)y^2(x_1)$ and $c_1 = \frac{1}{2}|g''(x_1)|$. From (2.8) and (2.10) we obtain

$$-L\int_{x_1}^{x_n} u(t)\,dt \leqslant c_2+y^2(x_1)\int_{x_1}^{x_n} |g'''(t)|\,dt$$

or

$$L \leqslant G(x_n)\Big(c_2+y^2(x_1)\int_{x_1}^{x_n} |g'''(t)|\,dt\Big),$$

where $c_2 = c+c_1y^2(x_1)$ since from (2.7) we have $u(x)\leqslant 0$ for $x\geqslant a$. Let $n\to\infty$; then, by hypotheses we obtain $L = 0$ which contradicts the assumption $L>0$. This proves the theorem.

If we take $g(x) = p^{-1}(x)$ in Theorem 2 and use the oscillation criteria of [1], we obtain the following result which has been proved in Theorem 2 of [4].

COROLLARY 2.1. *Let $p(x)$ satisfy* (1.2) *and $p'(x)\geqslant 0$ for $a\leqslant x$. If*

$$\limsup_{x\to\infty}\big(1/\log p(x)\big)\int_a^x \big|\big(p^{-1}(t)\big)'''\big|\,dt = 0,$$

then every solution $y(x)$ of (1.1) *satisfies* (1.3).

If we take $g(x) = p^{-1}(x)\big(\log p(x)\big)^{-b}$, $0<b<1$, in Theorem 2 and use the oscillation criteria of [1], we obtain the following result.

COROLLARY 2.2. *Let $p(x)$ satisfy* (1.2) *and $p'(x)\geqslant 0$ for $a\leqslant x$. If*

$$\limsup_{x\to\infty}\big(\log p(x)\big)^{b-1}\int_a^x \Big|\Big(p^{-1}(t)\big(\log p(t)\big)^{-b}\Big)'''\Big|\,dt = 0,$$

then every solution $y(x)$ of (1.1) *satisfies* (1.3).

If we take $g(x) = \big(p(x)\log p(x)\big)^{-1}$ in Theorem 2 and use the oscillation criteria of [1], we obtain the following result.

COROLLARY 2.3. *Let $p(x)$ satisfy* (1.2) *and $p'(x) \geqslant 0$ for $a \leqslant x$. If*

$$\limsup_{x\to\infty}\big(\log(\log p(x))\big)^{-1}\int_a^x \Big|\Big(\big(p(t)\log p(t)\big)^{-1}\Big)'''\Big|\,dt = 0,$$

then every solution $y(x)$ of (1.1) *satisfies* (1.3).

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