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Difference inequalities and error estimates for the Runge–Kutta method

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Dedicated to the memory of Jacek Szarski

Abstract. The paper deals with the convergence and error estimates for the Runge–Kutta method of the fourth order. The method of proofs does not change essentially for the Runge–Kutta method of the third and second order.

1. This paper contains a theorem on the convergence and error estimates for the Runge–Kutta method of the fourth order, cf. Theorem 3.

The proofs are based on the method of difference inequalities, cf. Remark 2, Section 6.

2. We shall assume that the right-hand member of the equation

$$y' = f(x, y), \tag{2.1}$$

satisfies the classical conditions of the existence and uniqueness theorem:

(i) $f(x, y)$ is a continuous function of its arguments and satisfies the Lipschitz condition in the set Q:

$$Q:\ |x-\xi| \leqslant k, \quad |y-\eta| \leqslant k, \tag{2.2}$$

$$|f(x, y)-f(x, \bar{y})| \leqslant \mathscr{L}\,|y-\bar{y}|, \tag{2.3}$$

for $(x, y) \in Q$, $(x, \bar{y}) \in Q$, $0 < \mathscr{L} = \text{const}$.

We assume that

$$|f(x, y)| \leqslant M \quad \text{for } (x, y) \in Q,\ 0 < M = \text{const}. \tag{2.4}$$

Let us denote by q, P, T the sets

$$\begin{aligned} &q:\ |x-\xi| \leqslant h_1, \quad |y-\eta| \leqslant h_1, \\ &P:\ |x-\xi| \leqslant h_1, \quad |y-\eta| \leqslant k, \\ &T:\ x_0 \leqslant x \leqslant \xi + h_1, \quad |y-y_0| \leqslant M\cdot(x-x_0) \end{aligned} \tag{2.5}$$

where (x_0, y_0) is an arbitrary point in q, and

$$h_1 = \frac{k}{2M+1}. \tag{2.6}$$

If the point $(x_0, y_0) \in q$ is given, then there exists a unique integral curve $y = \varphi(x)$, $y_0 = \varphi(x_0)$, in the interval $|x-\xi| \leqslant h_1$.

For an application of the Runge–Kutta method the class C^5 of the functions φ and f will be required. This additional assumption will be explicitly mentioned at appropriate places.

3. We shall use the classical notation for the Runge–Kutta method:

$$\begin{aligned} k_1^{(n)} &= h\cdot f(x_n, \varphi_n), \\ k_2^{(n)} &= h\cdot f(x_n+a_2 h,\ \varphi_n+\beta_{21}\cdot k_1^{(n)}), \\ k_3^{(n)} &= h\cdot f\Big(x_n+a_3 h,\ \varphi_n+\sum_{s=1}^{2}\beta_{3s}\cdot k_s^{(n)}\Big), \\ k_4^{(n)} &= h\cdot f\Big(x_n+a_4 h,\ \varphi_n+\sum_{s=1}^{3}\beta_{4s}\cdot k_s^{(n)}\Big), \end{aligned} \tag{3.1}$$

where x_n denote the nodal points and the numbers a_i, β_{is} ($i = 2, 3, 4$; $s = 1, 2, 3$; $i > s$) are independent on the choice of the function $f(x, y)$.

Let us also denote by ω_j ($j = 1, 2, 3, 4$) numbers independent on the choice of the function $f(x, y)$.

Let us assume for a moment that the quantities (3.1) are defined, $\varphi(x)$ and $f(x, y)$ are of class C^5 and let us introduce the following condition W and condition U:

CONDITION W. We have the relations

$$\varphi_{n+1}-\varphi_n = \sum_{l=0}^{4} a_l h^l + \varepsilon_1(x_n, h), \tag{3.2}$$

$$\sum_{j=1}^{4} \omega_j k_j^{(n)} = \sum_{l=0}^{4} a_l h^l + \varepsilon_2(x_n, h), \tag{3.3}$$

that is, the first five terms on the right-hand sides of (3.2) and (3.3) coincide, and

$$\varepsilon_1(x_n, h) = O(h^5), \qquad \varepsilon_2(x_n, h) = O(h^5), \tag{3.4}$$

holds for every function $f(x, y)$ of class C^5 in the set Q.

CONDITION U. The numbers $\omega_j, a_i, \beta_{is}$ ($j = 1, 2, 3, 4$; $i = 1, 2, 3$;

$i > s$) ($s = 1, 2, 3$) satisfy the system

$$
\begin{gathered}
\alpha_2 = \beta_{21}, \quad \alpha_3 = \beta_{31} + \beta_{32}, \quad \alpha_4 = \beta_{41} + \beta_{42} + \beta_{43}, \\
\omega_1 + \omega_2 + \omega_3 + \omega_4 = 1, \quad \omega_2 \alpha_2 + \omega_3 \alpha_3 + \omega_4 \alpha_4 = \tfrac{1}{2}, \\
\omega_2 \alpha_2^2 + \omega_3 \alpha_3^2 + \omega_4 \alpha_4^2 = \tfrac{1}{3}, \quad \omega_2 \alpha_2^3 + \omega_3 \alpha_3^3 + \omega_4 \alpha_4^3 = \tfrac{1}{4}, \\
\omega_3 \beta_{32} \alpha_2 + \omega_4 \cdot (\beta_{42} \alpha_2 + \beta_{43} \alpha_3) = \tfrac{1}{6}, \\
\omega_3 \beta_{32} \alpha_3 \alpha_2 + \omega_4 \cdot (\beta_{42} \alpha_4 \alpha_2 + \beta_{43} \alpha_4 \alpha_3) = \tfrac{1}{8}, \\
\omega_3 \beta_{32} \alpha_2^2 + \omega_4 \cdot (\beta_{42} \alpha_2^2 + \beta_{43} \alpha_3^2) = \tfrac{1}{12}, \quad \omega_4 \beta_{43} \beta_{32} \alpha_2 = \tfrac{1}{24}.
\end{gathered} \tag{3.5}
$$

We have the following theorem:

THEOREM 1. *Let us assume that $\varphi(x)$ and $f(x, y)$ are of class C^5 in the interval $x_0 \leqslant x \leqslant x_0 + h_1$ and in the set Q, respectively, and that the quantities* (3.1) *are defined.*

Under these assumptions

$$\textit{Condition } \mathrm{W} \Leftrightarrow \textit{Condition } \mathrm{U}. \tag{3.6}$$

The proof can be found, for example, in Berezin, Zhidkov [1].

4. The difference equation for the Runge–Kutta method can be found in the following way:

From relation (3.3) it follows that

$$\sum_{l=0}^{4} a_l h^l = \sum_{j=1}^{4} \omega_j k_j^{(n)} - \varepsilon_2(x_n, h); \tag{4.1}$$

hence (3.2) can be written in the form:

$$\varphi_{n+1} - \varphi_n = \sum_{j=1}^{4} \omega_j k_j^{(n)} + \varepsilon_1(x_n, h) - \varepsilon_2(x_n, h). \tag{4.2}$$

Thus the solution $\varphi(x)$ satisfies equality (4.2). If on the right-hand side of (4.2) the member $\varepsilon_1(x_n, h) - \varepsilon_2(x_n, h)$ is dropped, then we obtain a difference equation for the unknown discrete function $y_n = y(x_n)$:

$$y_{n+1} - y_n = \sum_{j=1}^{4} \omega_j \cdot k_{jy}^{(n)}, \tag{4.3}$$

where

$$
\begin{gathered}
k_{1y}^{(n)} = h \cdot f(x_n, y_n), \quad k_{2y}^{(n)} = h \cdot f(x_n + \alpha_2 h, y_n + \beta_{21} k_{1y}^{(n)}), \\
k_{3y}^{(n)} = h \cdot f\Big(x_n + \alpha_3 h, y_n + \sum_{s=1}^{2} \beta_{3s} k_{sy}^{(n)}\Big), \\
k_{4y}^{(n)} = h \cdot f\Big(x_n + \alpha_4 h, y_n + \sum_{s=1}^{3} \beta_{4s} k_{sy}^{(n)}\Big).
\end{gathered} \tag{4.4}
$$

We shall impose the same initial condition

$$y(x_0) = y_0, \tag{4.5}$$

for the solution $y(x)$ $(x = x_n)$ $(n = 0, 1, \ldots)$ of the difference equation (4.3), (4.4) as for the solution $\varphi(x)$ of the differential equation (2.1).

5. The location of the solution y_n of the difference equation is connected with the determination of the set where the quantities (4.4) and (3.1) can be defined.

Let us write

$$\gamma = \max(\gamma_1, \gamma_2, \gamma_3), \tag{5.1}$$

where

$$\gamma_1 = M\cdot\sum_{j=1}^{4}|\omega_j|, \quad \gamma_2 = M\cdot\sum_{s=1}^{2}|\beta_{3s}|, \quad \gamma_3 = M\cdot\sum_{s=1}^{3}|\beta_{4s}|. \tag{5.2}$$

We have $\gamma \geqslant M$, since $\sum_{j=1}^{4}\omega_j = 1$ and $1 \leqslant \sum_{j=1}^{4}|\omega_j|$; therefore $M \leqslant M\cdot\sum_{j=1}^{4}|\omega_j| \leqslant \gamma_1 \leqslant \gamma$.

Let us denote by (ξ_1, η_1) the point of intersection of the straight-line $y = y_0+\gamma(x-x_0)$ $(x \geqslant x_0)$ with the boundary ∂P of the rectangle P and by (ξ_2, η_2) the point of intersection of the line $y = y_0-\gamma(x-x_0)$ $(x \geqslant x_0)$ with ∂P.

Let us denote

$$\beta = \min(\xi+h_1, \xi_1, \xi_2), \tag{5.3}$$

$$T^{(4)}\colon\ x_0 \leqslant x \leqslant \beta, \quad |y-y_0| \leqslant \gamma\cdot(x-x_0). \tag{5.4}$$

Thus the triangle $T^{(4)}$ is contained in the rectangle P.

We introduce the nodal points

$$\begin{aligned} &x_0 < x_1 < x_2 < \ldots < x_N = \beta, \quad \beta - x_0 = N\cdot h, \\ &x_{n+1}-x_n = h \quad (n = 0, 1, \ldots, N-1) \end{aligned} \tag{5.5}$$

in the interval $x_0 \leqslant x \leqslant \beta$. The mesh size h will be regarded as a constant. With a variable mesh size h_n the main idea of the paper does not change essentially.

6. We now prove

THEOREM 2. *Let us suppose that $\varphi(x)$ and $f(x, y)$ are of class C^5 in the interval $x_0 \leqslant x \leqslant x_0+h_1$ and the set Q, respectively, and $0 < a_i \leqslant 1$ $(i = 2, 3, 4)$.*

Under these assumptions:

(i) *the solution y_n of the difference equation* (4.3), (4.4) *satisfying the initial condition* (4.5), *is defined for $n = 0, 1, \ldots, N$;*

(ii) *the points* (x_n, y_n) $(n = 0, 1, \ldots, N)$ *are in the triangle* $T^{(4)}$, *cf.* (5.4);

(iii) *the quantities* (4.4) *are defined for* $n = 0, 1, \ldots, N$.

Proof. We proceed by induction.

(a) The point (x_0, y_0) belongs to the triangle $T^{(4)}$.

(b) Let us assume that for a fixed natural number p $(0 \leqslant p \leqslant N-1)$ the value y_p is defined and

$$(x_p, y_p) \in T^{(4)}. \tag{6.1}$$

We shall prove that the next value y_{p+1} is defined and

$$(x_{p+1}, y_{p+1}) \in T^{(4)}. \tag{6.2}$$

For this purpose we shall verify that

$$(x_p + a_2 h,\ y_p + \beta_{21} k_{1y}^{(p)}) \in T^{(4)}, \tag{6.3}$$

$$\Big(x_p + a_3 h, y_p + \sum_{s=1}^{2} \beta_{3s} k_{sy}^{(p)}\Big) \in T^{(4)}, \tag{6.4}$$

$$\Big(x_p + a_4 h, y_p + \sum_{s=1}^{3} \beta_{4s} k_{sy}^{(p)}\Big) \in T^{(4)}. \tag{6.5}$$

First we prove (6.3). From the induction assumption (6.1) and definition (5.1) of the number γ it follows that

$$|\beta_{21} k_{1y}^{(p)}| \leqslant |\beta_{21}| \cdot h \cdot |f(x_p, y_p)| \leqslant a_2 h M \leqslant h M \leqslant h\gamma, \tag{6.6}$$

cf. (3.5), which means that relation (6.3) holds. Thus the value $f(x, y$ at the point (6.3) and the value $k_{2y}^{(p)}$ are defined.

Secondly, we prove (6.4). From the induction assumption (6.1) and definition (5.1) of the number γ we get

$$\Big|\sum_{s=1}^{2} \beta_{3s} k_{sy}^{(p)}\Big| \leqslant \sum_{s=1}^{2} |\beta_{3s}| \cdot hM \leqslant h\gamma, \tag{6.7}$$

cf. (3.5), which means that relation (6.4) holds. Therefore the value $f(x, y)$ at the point (6.4) and the value $k_{3y}^{(p)}$ are defined.

Finally, we verify (6.5). From (6.1) and (5.1) it follows that

$$\Big|\sum_{s=1}^{3} \beta_{4s} k_{sy}^{(p)}\Big| \leqslant \sum_{s=1}^{3} |\beta_{4y}| \cdot hM \leqslant h\gamma, \tag{6.8}$$

cf. (3.5); hence we obtain (6.5). This means that the value $f(x, y)$ at the point (6.5) and the value $k_{4y}^{(p)}$ are defined.

Now we write the difference equation (4.3) for $n = p$, and we see that the value y_{p+1} is defined.

In addition

$$(6.9)\qquad |y_{p+1}-y_p| = \Big|\sum_{j=1}^{4} \omega_j k_{jy}^{(p)}\Big| \leqslant \sum_{j=1}^{4} |\omega_j|\cdot hM \leqslant h\gamma,$$

cf. (5.2); hence relation (6.2) holds. Thus the solution y_n of the difference equation (4.3), (4.4) satisfying the initial condition (4.5) is defined for $n = 0, 1, \ldots, N$ and the points (x_n, y_n) $(n = 0, 1, \ldots, N)$ are in the triangle $T^{(4)}$.

This completes the proof of Theorem 2.

Remark 1. In a similar way we can prove that

$$(6.10)\qquad (x_n+a_2 h, \varphi_n+\beta_{21} k_1^{(n)}) \in T^{(4)},$$

$$(6.11)\qquad \Big(x_n+a_3 h, \varphi_n+\sum_{s=1}^{2} \beta_{3s} k_s^{(n)}\Big) \in T^{(4)},$$

$$(6.12)\qquad \Big(x_n+a_4 h, \varphi_n+\sum_{s=1}^{3} \beta_{4s} k_s^{(n)}\Big) \in T^{(4)},$$

for $x_0 \leqslant x_n \leqslant \beta$ $(n = 0, 1, \ldots, N-1)$, $0 < a_i \leqslant 1$ $(i = 2, 3, 4)$, and for $\varphi(x)$ and $f(x, y)$ of class C^5. This means that the quantities (3.1) are defined.

Remark 2. In the next section we shall use the following well-known theorem on difference inequalities:

If R_n $(n = 0, 1, 2, \ldots)$ denotes a discrete function satisfying the difference inequality

$$(6.13)\qquad R_n^{\sim} \leqslant \Lambda\cdot R_n+\mu \quad (\Lambda, \mu - \text{const}), \qquad R_0 = 0,$$

where $R_n^{\sim} = h^{-1}\cdot(R_{n+1}-R_n)$, then the estimate

$$(6.14)\qquad R_n \leqslant \frac{\mu}{\Lambda}\cdot(e^{nh\Lambda}-1) \quad (n = 0, 1, 2, \ldots),$$

holds for $n = 0, 1, \ldots$

7. The convergence of the Runge–Kutta method of the fourth order. Let us write

$$(7.1)\qquad \mu(h) = \max_{x_n} |\varepsilon(x_n, h)|,$$

where

$$(7.2)\qquad \varepsilon(x_n, h) = h^{-1}\cdot[\varepsilon_1(x_n, h)-\varepsilon_2(x_n, h)].$$

From definition (7.2) and (3.4) it follows that

$$(7.3)\qquad \mu(h) = O(h^4).$$

Let us denote, additionally,

$$r_n = \varphi_n - y_n, \quad r_n^{\sim} = h^{-1}\cdot(r_{n+1}-r_n). \tag{7.4}$$

THEOREM 3. *Suppose that $\varphi(x)$ and $f(x, y)$ are of class C^5 in the interval $x_0 \leqslant x \leqslant \xi + h_1$ and in the set Q, respectively* (cf. Section 2).

Further, assume that $0 < a_i \leqslant 1$ $(i = 2, 3, 4)$ and denote by L, M the constants occurring in Section 2, cf. (2.3) *and* (2.4).

Under these assumptions:

(i) *the Runge–Kutta method is convergent*

$$r_n \to 0, \quad \text{as } h \to 0, \quad x_0 \leqslant x_n \leqslant \beta \quad (n = 0, 1, \ldots, N), \tag{7.5}$$

(ii) *we have the error estimate*

$$|r_n| \leqslant \frac{\mu(h)}{\Lambda}\cdot(e^{nh\Lambda}-1) \quad (n = 0, 1, \ldots, N), \tag{7.6}$$

where

$$\Lambda = \mathscr{L}\cdot\Big\{\sum_{j=1}^{4}|\omega_j| + \mathscr{L}h\cdot\sum_{i=2}^{4}\sum_{s=1}^{i-1}|\omega_i\beta_{is}| + \tag{7.7}$$
$$+\mathscr{L}^2h^2\sum_{k=3}^{4}\sum_{i=2}^{k-1}\sum_{s=1}^{i-1}|\omega_k\beta_{ki}\beta_{is}| + \mathscr{L}^3h^3|\omega_4\beta_{43}\beta_{32}\beta_{21}|\Big\}.$$

Proof. The difference quotient $r_n^{\sim}$, cf. (7.4), can be written in the form

$$r_n^{\sim} = \sum_{j=1}^{4}\omega_j\cdot h^{-1}\cdot(k_j^{(n)}-k_{jy}^{(n)}) + \varepsilon(x_n, h), \tag{7.8}$$

because of formula (4.2) and (4.3).

From the definition of the quantities $k_j^{(n)}$ and $k_{jy}^{(n)}$, cf. (3.1), (4.4), it follows that

$$h^{-1}\cdot(k_1^{(n)}-k_{1y}^{(n)}) = f_y(\sim_1)\cdot r_n, \tag{7.9}$$

where $(\sim_1)$ denotes a suitable point in the triangle $T^{(4)}$, cf. (5.4) and the second part of Theorem 2.

In a similar way we obtain from (3.1) and (4.4) the formulas

$$\frac{1}{h}\cdot(k_2^{(n)}-k_{2y}^{(n)}) = f_y(\sim_2)\cdot\{r_n+\beta_{21}(k_1^{(n)}-k_{1y}^{(n)})\}, \tag{7.10}$$

$$\frac{1}{h}\cdot(k_3^{(n)}-k_{3y}^{(n)}) = f_y(\sim_3)\cdot\Big\{r_n+\sum_{s=1}^{2}\beta_{3s}\cdot(k_s^{(n)}-k_{sy}^{(n)})\Big\}, \tag{7.11}$$

$$\frac{1}{h}\cdot(k_4^{(n)}-k_{4y}^{(n)}) = f_y(\sim_4)\cdot\Big\{r_n+\sum_{s=1}^{3}\beta_{4s}\cdot(k_s^{(n)}-k_{sy}^{(n)})\Big\}, \tag{7.12}$$

where $(\sim_i)$ $(i = 2, 3, 4)$ denote suitable points in the triangle $T^{(4)}$, cf. Remark 1, formulas (6.3)–(6.5) and (6.10)–(6.12).

If we insert into (7.10) expression (7.9), we get

$$\frac{1}{h}\cdot(k_2^{(n)}-k_{2y}^{(n)}) = f_y(\sim_2)\cdot r_n + f_y(\sim_2)\cdot f_y(\sim_1)\cdot\beta_{21}\cdot hr_n. \tag{7.13}$$

Now we can insert into (7.11) expressions (7.9) and (7.10). This yields

$$\begin{aligned}\frac{1}{h}\cdot(k_3^{(n)}-k_{3y}^{(n)}) = {} & f_y(\sim_3)\cdot r_n + \\ & + f_y(\sim_3)\cdot f_y(\sim_1)\cdot\beta_{31}\cdot hr_n + f_y(\sim_3)\cdot f_y(\sim_2)\cdot\beta_{32}\cdot hr_n + \\ & + f_y(\sim_3)\cdot f_y(\sim_2)\cdot f_y(\sim_1)\cdot\beta_{32}\beta_{21}h^2 r_n.\end{aligned} \tag{7.14}$$

Finally, we insert into (7.12) expressions (7.9), (7.13) and (7.14) and we obtain

$$\begin{aligned}\frac{1}{h}\cdot(k_4^{(n)}-k_{4y}^{(n)}) = {} & f_y(\sim_4)\cdot r_n + f_y(\sim_4)\cdot f_y(\sim_1)\cdot\beta_{41}\cdot hr_n + \\ & + f_y(\sim_4)\cdot f_y(\sim_2)\cdot\beta_{42}\cdot hr_n + f_y(\sim_4)\cdot f_y(\sim_3)\cdot\beta_{43}\cdot hr_n + \\ & + f_y(\sim_4)\cdot f_y(\sim_2)\cdot f_y(\sim_1)\cdot\beta_{42}\beta_{21}\cdot h^2 r_n + \\ & + f_y(\sim_4)\cdot f_y(\sim_3)\cdot f_y(\sim_1)\cdot\beta_{43}\beta_{31}\cdot h^2 r_n + \\ & + f_y(\sim_4)\cdot f_y(\sim_3)\cdot f_y(\sim_2)\cdot\beta_{43}\beta_{32}\cdot h^2 r_n + \\ & + f_y(\sim_4)\cdot f_y(\sim_3)\cdot f_y(\sim_2)\cdot f_y(\sim_1)\cdot\beta_{43}\beta_{32}\beta_{21}\cdot h^3 r_n.\end{aligned} \tag{7.15}$$

Let us again consider formula (7.8). Taking absolute values on both sides of (7.8) we get the inequality

$$|\tilde{r_n}| \leqslant \sum_{j=1}^{4} |\omega_j|\cdot\left|\frac{1}{h}\cdot(k_j^{(n)}-k_{jy}^{(n)})\right| + \mu(h). \tag{7.16}$$

Now we can insert into (7.16) the calculated expressions (7.9), (7.13), (7.14) and (7.15), which permits us to write (7.16) in the form

$$\begin{aligned}|\tilde{r_n}| \leqslant |r_n|\cdot\mathscr{L}\cdot\Big\{ & \sum_{j=1}^{4}|\omega_j| + \mathscr{L}h\sum_{i=2}^{4}\sum_{s=1}^{i-1}|\omega_i\beta_{is}| + \\ & + \mathscr{L}^2h^2\sum_{k=3}^{4}\sum_{i=2}^{k-1}\sum_{s=1}^{i-1}|\omega_k\beta_{ki}\beta_{is}| + \mathscr{L}^3h^3\cdot|\omega_4\beta_{43}\beta_{32}\beta_{21}|\Big\} + \mu(h).\end{aligned} \tag{7.17}$$

To simplify notation we use the symbol (7.7), so that (7.17) becomes

$$|\tilde{r_n}| \leqslant \Lambda\cdot|r_n| + \mu(h). \tag{7.18}$$

Write $R_n = |r_n|$. Then we have $|r_n^{\sim}| \geqslant R_n^{\sim}$ and we get from (7.18) the desired difference inequality and the initial condition for R_n:

(7.19) $$R_n^{\sim} \leqslant \Lambda \cdot R_n + \mu(h), \quad R_0 = 0.$$

Inequality (7.19) and the theorem on difference inequalities, cf. Remark 2, permit us to write the estimate

(7.20) $$R_n \leqslant \frac{\mu(h)}{\Lambda} \cdot (e^{nh\Lambda} - 1) \quad (n = 0, 1, \ldots, N).$$

But $nh \leqslant Nh = d$, where $d = \text{const}$ denotes the length of the interval: $d = \beta - x_0$. Thus, inequality (7.20) and the condition $0 < \mu(h) \to 0$, as $h \to 0$, imply that

(7.21) $$0 \leqslant R_n \to 0, \quad \text{as } h \to 0 \quad (n = 0, 1, \ldots, N).$$

Therefore we have

(7.22) $$r_n \to 0, \quad \text{as } h \to 0, \quad x_0 \leqslant x_n \leqslant \beta \quad (n = 0, 1, \ldots, N).$$

This completes the proof of convergence.

The error estimate follows from (7.20) and has the form

(7.23) $$|r_n| \leqslant \frac{\mu(h)}{\Lambda} \cdot (e^{nh\Lambda} - 1) \quad (n = 0, 1, \ldots, N),$$

for $x_0 \leqslant x \leqslant \beta$.

This completes the proof of Theorem 3.

8. Remark 3. Let us restrict our considerations to the values a_2, a_3, $0 < a_i \leqslant 1$ $(i = 2, 3)$ such that ω_j and β_{is} are non-negative numbers:

(8.1) $$\omega_j \geqslant 0, \quad \beta_{is} \geqslant 0$$

for $j = 1, 2, 3, 4$; $i = 2, 3, 4$; $s = 1, 2, 3$; $i > s$ ($a_4 = 1$ and the set a_2, a_3 is not empty, cf. for example Berezin, Zhidkov [1]).

Then we have

(8.2) $$\Lambda = \mathscr{L} \cdot \Big\{ \sum_{j=1}^{4} \omega_j + \mathscr{L} \cdot h \sum_{i=2}^{4} \sum_{s=1}^{i-1} \omega_i \beta_{is} + \mathscr{L}^2 h^2 \cdot \sum_{k=3}^{4} \sum_{i=2}^{k-1} \sum_{s=1}^{i-1} \omega_k \beta_{ki} \beta_{is} + \mathscr{L}^3 h^3 \cdot \omega_4 \beta_{43} \beta_{32} \beta_{21} \Big\}$$

by the definition of Λ, cf. (7.7) and (8.1).

From (8.2) and (3.5) it follows that for non-negative ω_j, β_{is}, cf. (8.1), the corresponding formula for the quantity Λ has an interesting form

(8.3) $$\Lambda = \mathscr{L} \cdot \Big(1 + \frac{1}{2!} \mathscr{L} h + \frac{1}{3!} \mathscr{L}^2 h^2 + \frac{1}{4!} \mathscr{L}^3 h^3\Big).$$

References

[1] I. Berezin, N. Zhidkov, *Numerical methods*, vol. I and II, Moscow 1962.
[2] Z. Mikołajska, *Une remarque sur l'équation des erreurs dans la solution approximative des équations différentielles*, Ann. Polon. Math. 26 (1972), p. 341–344.

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