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Multiplicative linear functionals on some algebras of holomorphic functions with restricted growth

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Abstract. Let X be a Riemann domain over C^n and let $\delta\colon X \to (0, 1]$ be a weight function. Denote by $\mathcal{O}(X, \delta)$ the algebra of all functions f holomorphic on X such that, for some $k = k(f) \geqslant 0$, the function $\delta^k f$ is bounded. A characterization of the space of all (bounded) multiplicative linear functionals $\xi\colon \mathcal{O}(X, \delta) \to C$ will be presented.

1. Introduction. Let us fix an open set $X \subset C^n$. Let $\mathcal{O}(X)$ denote the algebra of all holomorphic functions on X and let $S(X)$ denote the spectrum of $\mathcal{O}(X)$, i.e., the space of all non-zero complex algebra-homomorphisms $\xi\colon \mathcal{O}(X) \to C$. It is well known that $S(X)$ may be represented as the space of all evaluations on the envelope of holomorphy of X (by the evaluation determined by a point x_0 we mean the functional $f \to f(x_0)$). In particular, every functional $\xi \in S(X)$ is bounded in the sense of the Mackey boundedness on $\mathcal{O}(X)$.

This classical result leads to the following general question. Given a subalgebra $A \subset \mathcal{O}(X)$ endowed with an algebra boundedness; what is a characterization of the space of all (bounded) complex homomorphisms $\xi\colon A \to C$?

From the point of view of the spectral theory (in the sense of [2]) the most interesting case is where A is an algebra $\mathcal{O}(X, \delta)$ of holomorphic δ-tempered functions defined as follows.

Let $\delta\colon X \to (0, +\infty)$ be bounded and lower semi-continuous. For $k \geqslant 0$, let $\mathcal{O}^{(k)}(X, \delta)$ denote the space of all $f \in \mathcal{O}(X)$ such that the function $\delta^k f$ is bounded. It is seen that

$$(1) \qquad \|f\|_K \leqslant (\min_K \delta)^{-k} \|\delta^k f\|_\infty, \qquad K \subset\subset X,\ f \in \mathcal{O}^{(k)}(X, \delta).$$

In particular, the space $\mathcal{O}^{(k)}(X, \delta)$ endowed with the norm $f \to \|\delta^k f\|_\infty$ is a Banach space. Put

$$\mathcal{O}(X, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(X, \delta).$$

One may easily check that $\mathcal{O}(X, \delta)$ is a complex algebra (with unit element).

Let $S(X, \delta)$ denote the space of all non-zero complex homomorphisms

$\xi\colon \mathcal{O}(X, \delta) \to C$ and let $S_b(X, \delta)$ be the space of those $\xi \in S(X, \delta)$ which are bounded, that is

for every $k \geq 0$, the restriction of ξ to $\mathcal{O}^{(k)}(X, \delta)$ is a bounded linear functional of the Banach space $\mathcal{O}^{(k)}(X, \delta)$ into C.

Note that, in view of (1), $S(X) \subset S_b(X, \delta)$.

Hence it is natural to look for conditions under which $S_b(X, \delta) = S(X)$ or, more generally, $S(X, \delta) = S(X)$ (the last equality means, in particular, that every functional $\xi \in S(X, \delta)$ is bounded).

It is clear that, after evident formal changes, the analogous problems may be posed in the case where X is an arbitrary complex analytic space.

In the question of applications of the spectral theory (cf. [2]), it is natural to reduce the class of admissible functions δ to so-called weight functions.

DEFINITION 1. A function $\delta\colon X \to (0, 1]$ is said to be a *weight function on* X ($\delta \in W(X)$) if

(i) $\delta \leq \delta_X = \min\{\varrho_X, (1+\|z\|^2)^{-1/2}\}$, where ϱ_X denotes the distance to the boundary of X taken with respect to the Euclidean norm $\|z\| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}$, $z = (z_1, \ldots, z_n) \in C^n$,

(ii) $|\delta(z') - \delta(z'')| \leq \|z' - z''\|$, $z', z'' \in X$.

The following result is known ([2], § 4.6).

THEOREM F. *Let X be a domain of holomorphy in C^n and let $\delta \in W(X)$. Then* $S_b(X, \delta) = S(X)$ (= the space of evaluations on X).

The proof of Theorem F is based on Waelbroeck's holomorphic functional calculus. Unfortunately, such a method of the proof cannot be adopted to more general cases.

The purpose of this paper is to find an analogue of Theorem F in the case where X is a Riemann domain over C^n. In particular, that will permit us to study the spectra $S(X, \delta)$, $S_b(X, \delta)$ for arbitrary open sets in C^n.

Now let X be a Riemann domain, countable at infinity spread over C^n and let $p = (p_1, \ldots, p_n)\colon X \to C^n$ denote its locally homeomorphic projection into C^n. Denote by $\varrho = \varrho_X$ "the distance function to the boundary of X", that is, for $x \in X$, $\varrho(x) =$ the supremum of all numbers $r > 0$ such that there exists an open neighbourhood $\hat{B}(x, r)$ of the point x mapped homeomorphically by p onto the Euclidean ball $B(p(x), r) \subset C^n$. Put $\hat{B}(x) = \hat{B}(x, \varrho(x))$.

DEFINITION 2. A function $\delta\colon X \to (0, 1]$ is said to be a *weight function on* X ($\delta \in W(X)$) if

$$\delta \leq \delta_X = \min\{\varrho_X, (1+\|p\|^2)^{-1/2}\}, \tag{2}$$

$$|\delta(x) - \delta(x')| \leq \|p(x) - p(x')\|, \quad x \in X,\ x' \in \hat{B}(x). \tag{3}$$

Note that, in view of (3)

(4) $$\delta(x') \geqslant \tfrac{1}{2}\delta(x), \qquad x \in X, \ x' \in \hat{B}(x, \tfrac{1}{2}\delta(x))$$

(by (2), the last "ball" is well-defined).

The notion of weight functions on Riemann domains was introduced by the author in [4].

One may easily prove that $\delta_X \in W(X)$. Observe that in the case $X \in \operatorname{top} C^n$, $p = \mathrm{id}_X$ the class of weight functions in the sense of Definition 2 is the same as that in Definition 1.

We shall study the spectra $S(X, \delta)$, $S_b(X, \delta)$ where δ is a fixed weight function on X. We always assume that $\mathcal{O}(X)$ separates points in X. At first we shall show that without loss of generality we may assume that (X, p) is a Stein domain (i.e., X considered as a complex n-dimensional analytic manifold is Stein) and $-\log\delta$ is plurisubharmonic.

Let $(\hat{X}, \hat{p})$ denote the envelope of holomorphy of (X, p) and let φ: $X \to \hat{X}$ be the natural embedding of X into $\hat{X}$. Define φ^*: $\mathcal{O}(\hat{X}) \to \mathcal{O}(X)$ and φ_*: $S(X) \to S(\hat{X})$ (note that $(\hat{X}, \hat{p})$ is a Stein domain so $S(\hat{X})$ = the space of evaluations on $\hat{X}$) by the formulae:

$$\varphi^*(f) = f \circ \varphi, \quad f \in \mathcal{O}(\hat{X}), \qquad \varphi_*(\xi) = \xi \circ \varphi^*, \qquad \xi \in S(X).$$

It is well known that φ^* is both algebraic and topological isomorphism, hence φ_* is a bijection of $S(X)$ onto $S(\hat{X})$.

THEOREM 1. *For every $\delta \in W(X)$ there exists $\hat{\delta} \in W(\hat{X})$ such that:*

(5) $$-\log\hat{\delta} \in \mathrm{PSH}(\hat{X}),$$

(6) $$\delta \leqslant \hat{\delta} \circ \varphi,$$

(7) *for every $k \geqslant 0$, $f \in \mathcal{O}^{(k)}(X, \delta)$: $\hat{f} = (\varphi^*)^{-1}(f) \in \mathcal{O}^{(k)}(\hat{X}, \hat{\delta})$ and $\|\hat{\delta}^k \hat{f}\|_\infty \leqslant \|\delta^k f\|_\infty$.*

The proof will be given in Section 3.

In view of (6), φ^* may be regarded as an algebra homomorphism of $\mathcal{O}(\hat{X}, \hat{\delta})$ into $\mathcal{O}(X, \delta)$ such that, for every $k \geqslant 0$, φ^* maps $\mathcal{O}^{(k)}(\hat{X}, \hat{\delta})$ into $\mathcal{O}^{(k)}(X, \delta)$ and, viewed as an operator between these spaces has the norm $\leqslant 1$. In consequence φ_* may be extended to a mapping of $S(X, \delta)$ into $S(\hat{X}, \hat{\delta})$ which maps $S_b(X, \delta)$ into $S_b(\hat{X}, \hat{\delta})$.

In view of (7), φ^* is an isomorphism of $\mathcal{O}(\hat{X}, \hat{\delta})$ onto $\mathcal{O}(X, \delta)$ such that for every $k \geqslant 0$, φ^* is an isometry of $\mathcal{O}^{(k)}(\hat{X}, \hat{\delta})$ onto $\mathcal{O}^{(k)}(X, \delta)$. In particular, φ_* is a bijection between $S(X, \delta)$ and $S(\hat{X}, \hat{\delta})$ which maps $S_b(X, \delta)$ onto $S_b(\hat{X}, \hat{\delta})$.

Thus we see that the equality $S(X, \delta) = S(X)$ (resp. $S_b(X, \delta) = S(X)$) is equivalent to $S(\hat{X}, \hat{\delta}) = S(\hat{X})$ (resp. $S_b(\hat{X}, \hat{\delta}) = S(\hat{X})$).

Note that φ is injective, because $\mathcal{O}(X)$ separates points. We shall show (see 2.3) that $\mathcal{O}^{(4n)}(\hat{X}, \hat{\delta})$ separates points in $\hat{X}$. Hence $\mathcal{O}^{(4n)}(X, \delta)$ separates points in X. Thus we have in fact one-to-one correspondence between the considered spectra.

Let $\mu = \mu_X$ denote the measure on X transported by p from the Lebesgue measure λ in C^n, that is, μ is the measure generated by the volume element $(2i)^{-n} d\bar{p}_1 \wedge \ldots \wedge d\bar{p}_n \wedge dp_1 \wedge \ldots \wedge dp_n$.

A Riemann domain (X, p) is said to be *finitely sheeted* if for every $x \in X$ the stalk $p^{-1}(p(x))$ is a finite set.

The main result of the paper is the following

THEOREM 2. *Let (X, p) be a Riemann–Stein domain over C^n and let $\delta \in W(X)$ be such that*

(ii) $-\log \delta \in \mathrm{PSH}(X)$,

(ii) *there exists $\alpha \geqslant 0$ such that $\delta^\alpha \in L^1(X, \mu)$.*

Then $S_b(X, \delta) = S(X)$ ($=$ the space of evaluations on X).

If, moreover, (X, p) is finitely sheeted, then $S(X, \delta) = S(X)$.

The proof will be presented in Section 4.

Observe that if $X \in \operatorname{top} C^n$, $p = \mathrm{id}_X$, then for any $\delta \in W(X)$:

$$\int_X \delta^{2(n+\varepsilon)} d\lambda \leqslant \int_{C^n} (1+\|z\|^2)^{-(n+\varepsilon)} d\lambda < +\infty, \qquad \varepsilon > 0.$$

Hence, by Theorems 1, 2 we get the following important

COROLLARY 1 (a generalization of Theorem F). *Let X be an open subset of C^n such that its envelope of holomorphy is univalent. Then, for every $\delta \in W(X)$, $S(X, \delta) = S(X)$.*

An application of Corollary 1 to the theory of holomorphic continuation with restricted growth will be given in Section 5.

2. Basic properties of the algebras $\mathcal{O}(X, \delta)$. In this section we present some auxiliary results which will be useful in the sequel.

2.1 ([4], Proposition 2). *Let (X, p) be a Riemann domain over C^n and let $\delta \in W(X)$. Then*

$$\left\|\delta^{k+1} \frac{\partial f}{\partial x_j}\right\|_\infty \leqslant \sqrt{n}\, 2^{k+1} \|\delta^k f\|_\infty, \quad f \in \mathcal{O}^{(k)}(X, \delta),\ j = 1, \ldots, n.$$

2.2 ([4], Theorem 1). *Let (X, p) be a Riemann–Stein domain over C^n and let $\delta \in W(X)$ be such that $-\log \delta \in \mathrm{PSH}(X)$. Then there exist a set of indices I and families $(n_i)_{i \in I} \subset N$, $(f_i)_{i \in I} \subset \mathcal{O}(X)$ such that*

$$-\log \delta = \sup_{i \in I} \left\{ \frac{1}{n_i} \log |f_i| \right\}.$$

2.3 ([4], Theorem 3). *Under the assumptions given in 2.2, for every* $s \geqslant 0$ *and* $x_0 \in X$ *there exists* $u \in \mathcal{O}(X)$ *such that*

(i) $u(x_0) = 1$,

(ii) $u(x) = 0$, $x \in p^{-1}(p(x_0))$, $x \neq x_0$ (the remark that the function u constructed in the proof of Theorem 3 in [4] satisfies (ii) is due to P. Pflug – see [8]),

(iii) $\|\delta^{s+4n} u\|_\infty \leqslant c(n, s)\delta^{s-2n}(x_0)$ ($c(n, s)$ *depends only on* n *and* s).

In particular, $\mathcal{O}^{(4n)}(X, \delta)$ *separates points in* X. *Hence (in view of Theorem 4 in* [4]*),* $\mathcal{O}(X, \delta)$ *is dense in* $\mathcal{O}(X)$ *in the topology of uniform convergence on compact subsets of* X.

THEOREM 3. *Under the assumptions of Theorem 2, for every* $f_0, f_1, \ldots \ldots, f_N \in \mathcal{O}(X, \delta)$, *if for some* $c > 0$, $\gamma \geqslant 0$

$$(|f_1|^2 + \ldots + |f_N|^2)^{1/2} \geqslant c\delta^\gamma |f_0|,$$

then there exist $g_1, \ldots, g_N \in \mathcal{O}(X, \delta)$ *such that*

$$g_1 f_1 + \ldots + g_N f_N = f_0^k,$$

where $k = \min\{2n+1, 2N-1\}$.

Theorem 3 is a generalization to the case of Riemann domains of the famous "Nullstellensatz" for holomorphic functions with restricted growth. In the case $X \in \operatorname{top} C^n$, $p = \mathrm{id}_X$ this result was proved in [3] (for $f_0 = 1$) and later, basing on the ideas given in [3], in [1] and independently in [7]. In the case of Riemann domains we follow, with formal changes only, the method of the proof given in [7] – all the required estimations for the δ-problem may be deduced from Theorem 2 in [4].

3. Proof of Theorem 1. Let $F_k = \{f \in \mathcal{O}(\hat{X}): \|\delta^k (f \circ \varphi)\|_\infty \leqslant 1\}$, $k \geqslant 0$. Put

$$\Phi = \sup_{k>0}\left\{\sup_{f \in F_k}\left\{\frac{1}{k}\log|f|\right\}\right\}.$$

Clearly, $\Phi \geqslant 0$ and $\Phi \circ \varphi \leqslant -\log\delta$. Since $(\varphi^*)^{-1}$ is continuous, for every compact $K \subset \hat{X}$ there exists a compact $L \subset X$ such that

$$\|f\|_K \leqslant \|f \circ \varphi\|_L, \quad f \in \mathcal{O}(\hat{X}).$$

Hence $\|f\|_K \leqslant (\min_L \delta)^{-k}$, $f \in F_k$, so Φ is locally bounded.

Let us put $\eta = e^{-\Phi^*}$, here Φ^* denotes the upper regularization of Φ. It is seen that $\eta\colon \hat{X} \to (0, 1]$, $-\log\eta \in \mathrm{PSH}(\hat{X})$ and $\delta \leqslant \eta \circ \varphi$. In consequence, in view of the definition of Φ, φ^* is an isometry of $\mathcal{O}^{(k)}(\hat{X}, \eta)$ onto $\mathcal{O}^{(k)}(X, \delta)$.

$\hat{X}$ is a Stein domain, so $-\log \delta_{\hat{X}} \in \mathrm{PSH}(\hat{X})$, and hence, by 2.2,

$$-\log \delta_{\hat{X}} = \sup_{i\in I} \left\{ \frac{1}{n_i} \log |f_i| \right\},$$

where $(n_i)_{i\in I} \subset N$, $(f_i)_{i\in I} \subset \mathcal{O}(\hat{X})$.

φ is injective; hence $\varphi(\hat{B}_X(x, r)) = \hat{B}_{\hat{X}}(\varphi(x), r)$. In particular this gives: $\varrho_X \leqslant \varrho_{\hat{X}} \circ \varphi$, and $\delta_X \leqslant \delta_{\hat{X}} \circ \varphi$. Hence $f_i \in F_{n_i}$, $i \in I$ and in consequence $\Phi \geqslant -\log \delta_{\hat{X}}$. Thus $\eta \leqslant \delta_{\hat{X}}$.

Let $\hat{\delta} = \tilde{\eta} = \inf\{\eta(y) + \|\hat{p}(x) - \hat{p}(y)\|: y \in \hat{B}_{\hat{X}}(x)\}$ (= the formal convolution of η) – cf. [5], Lemma 2. Obviously, $\hat{\delta} \leqslant \eta$. It is known that $\hat{\delta} \in W(X)$ ([5], Lemma 2) and $-\log \hat{\delta} \in \mathrm{PSH}(\hat{X})$ ([5], Theorem 3).

Thus $\hat{\delta}$ satisfies (5) and (7).

For the proof of (6), note that $\hat{\delta}(\varphi(x)) = \min\{A(x), B(x)\}$, where

$$A(x) = \inf\{\eta(\varphi(x')) + \|p(x) - p(x')\|: x' \in \hat{B}_X(x, \delta(x))\},$$

$$B(x) = \inf\{\eta(y) + \|\hat{p}(\varphi(x)) - \hat{p}(y)\|: y \in \hat{B}_{\hat{X}}(\varphi(x)) \setminus \hat{B}_{\hat{X}}(\varphi(x), \delta(x))\}.$$

Clearly $B(x) \geqslant \delta(x)$ and $A(x) \geqslant \inf\{\delta(x') + \|p(x) - p(x')\|: x' \in \hat{B}_X(x)\}$, so in view of (3), $A(x) \geqslant \delta(x)$. Finally $\hat{\delta}(\varphi(x)) \geqslant \delta(x)$, $x \in X$, which completes the proof of Theorem 1.

Remark. The function $\hat{\delta}$ constructed in the proof of Theorem 1 satisfies the following condition

$$\hat{\delta} = \inf\{\delta' \in W(\hat{X}): -\log \delta' \in \mathrm{PSH}(\hat{X}), \delta \leqslant \delta' \circ \varphi\}.$$

Proof. Let us fix $\delta' \in W(\hat{X})$ such that $-\log \delta' \in \mathrm{PSH}(\hat{X})$ and $\delta \leqslant \delta' \circ \varphi$. By 2.2, $-\log \delta' = \sup_{i\in I} \left\{ \frac{1}{n_i} \log |f_i| \right\}$. Since $\delta^* \leqslant \delta' \circ \varphi$ so $f_i \in F_{n_i}$. Hence $\eta \leqslant \delta'$ and therefore $\hat{\delta} = \tilde{\eta} \leqslant (\delta')^{\sim} = \delta'$ (cf. [5], Lemma 2).

The proof is finished.

4. Proof of Theorem 2. The theorem will be proved by seven lemmas.

Let us fix $\xi \in S(X, \delta)$ and let $I = \ker \xi = \{f \in \mathcal{O}(X, \delta): \xi(f) = 0\}$. For $f = (f_1, \ldots, f_N) \in I^N$ we shall write $\|f\| = (|f_1|^2 + \ldots + |f_N|^2)^{1/2}$.

LEMMA 1. *For every* $f \in I^N$, $\gamma \geqslant 0$: $\inf_X \{\delta^{-\gamma} \|f\|\} = 0$.

Proof. Suppose that $\inf_X \{\delta^{-\gamma} \|f\|\} > 0$. Then, by Theorem 3 (with $f_0 = 1$), the unit function 1 belongs to the ideal generated by $f_1, \ldots, f_N$ in $\mathcal{O}(X, \delta)$. This is impossible, because $\xi \not\equiv 0$.

LEMMA 2. $a = (\xi(p_1), \ldots, \xi(p_n)) \in p(X)$.

Proof. Since $p - a \in I^n$, so by Lemma 1 (with $\gamma = 1$) there exists $x \in X$ such that $\|p(x) - a\| < \delta(x)$. In particular, $a \in B(p(x), \delta(x)) \subset p(X)$.

Put $T = p^{-1}(a)$; note that T is a countable set.

LEMMA 3. *For every* $f \in I^N$: $\inf_T \|f\| = 0$.

Proof. Suppose that $\|f(x)\| \geqslant \varepsilon_0 > 0$, $x \in T$. Let $k \geqslant 0$, $A \geqslant 1$ be chosen such that $\|\delta^k f_j\|_\infty \leqslant A$, $j = 1, \ldots, N$ and let $\theta > 0$ be so small that $4^{k+1} n^2 A\theta < \frac{1}{2}\varepsilon_0$. Let $\varphi = \delta^{-(k+1)}(\|f\| + \|p-a\|)$. By Lemma 1, $\inf_X \varphi = 0$

Let $E = \{x \in X: \|p(x) - a\| < \theta\delta^{k+1}(x)\}$. Note that $\varphi \geqslant \theta$ on $X \setminus E$. Hence $\inf_E \|f\| = 0$.

Let us fix $y \in E$. By definition, there exists $x \in T \cap \hat{B}(y, \theta\delta^{k+1}(y))$. In view of 2.1 and (4), $\|f(x) - f(y)\| \leqslant 4^{k+1} n^2 A\theta < \frac{1}{2}\varepsilon_0$. Hence $\|f(y)\| > \frac{1}{2}\varepsilon_0$. Thus $\inf_E \|f\| \geqslant \frac{1}{2}\varepsilon_0$. We get the contradiction.

From Lemma 3 we immediately get

LEMMA 4. *For every* $f_1, f_2 \in \mathcal{O}(X, \delta)$ *if* $f_1 = f_2$ *on* T, *then* $\xi(f_1) = \xi(f_2)$.

Let us denote by $\mathscr{F}$ the set of all sequences $(u_x)_{x \in T} \subset \mathcal{O}(X, \delta)$ such that $u_x(y) = \delta_{xy}$, $x, y \in T$. Note that in view of 2.3, $\mathscr{F} \neq \emptyset$.

LEMMA 5. *The following disjunction holds true: either for every* $(u_x)_{x \in T} \in \mathscr{F}$: $\xi(u_x) = 0$, $x \in T$ *or there exists* $x_0 \in T$ *such that for every* $(u_x)_{x \in T} \in \mathscr{F}$: $\xi(u_x) = \delta_{xx_0}$, $x \in T$.

Proof. In view of Lemma 4 it is sufficient to verify those conditions for a fixed sequence $(u_x)_{x \in T} \in \mathscr{F}$.

Let us fix $N \in \mathbb{N}$ and $x_1, \ldots, x_N \in T$. Observe that $f = (u_{x_1} - \xi(u_{x_1}), \ldots, u_{x_N} - \xi(u_{x_N})) \in I^N$. Hence by Lemma 3

$$\min\{|1 - \xi(u_{x_1})| + |\xi(u_{x_2})| + \ldots + |\xi(u_{x_N})|, \ldots, |\xi(u_{x_1})| + \ldots + |\xi(u_{x_{N-1}})| + |1 - \xi(u_{x_N})|, |\xi(u_{x_1})| + \ldots + |\xi(u_{x_N})|\} = 0.$$

Now the thesis of Lemma 5 is clearly seen.

We pass to the proof of Theorem 2 for the case where (X, p) is finitely sheeted.

LEMMA 6. *If* T *is a finite set, then there exists* $x_0 \in T$ *such that for every* $f \in \mathcal{O}(X, \delta)$: $\xi(f) = f(x_0)$.

Proof. Let us fix $(u_x)_{x \in T} \in \mathscr{F}$. For $f \in \mathcal{O}(X, \delta)$ let

$$\tilde{f} = \sum_{x \in T} f(x) u_x.$$

Obviously $\tilde{f} \in \mathcal{O}(X, \delta)$ and $\tilde{f} = f$ on T. Hence by Lemma 4, $\xi(\tilde{f}) = \xi(f)$. Observe that $\xi(\tilde{f}) = \sum_{x \in T} f(x)\xi(u_x)$, so by Lemma 5, either for every $f \in \mathcal{O}(X, \delta)$: $\xi(f) = 0$ which is impossible, or there exists $x_0 \in T$ such that $\xi(f) = f(x_0)$, $f \in \mathcal{O}(X, \delta)$.

The proof is finished.

For the proof of the first part of Theorem 2, assume additionally that $\xi \in S_b(X, \delta)$. In view of the method of the proof of Lemma 6, it is clear that we only need to prove the following lemma.

LEMMA 7. *For every* $f \in \mathcal{O}^{(k)}(X, \delta)$ *there exists a sequence* $(u_x)_{x\in T} \in \mathcal{F}$ *such that* $u_x \in \mathcal{O}^{(k+8n+\alpha)}(X, \delta)$, $x \in T$ *and*

$$\sum_{x\in T} |f(x)| \, \|\delta^{k+8n+\alpha} u_x\|_\infty < +\infty .$$

Proof. Fix $f \in \mathcal{O}^{(k)}(X, \delta)$. By 2.3 there exists $(u_x)_{x\in T} \in \mathcal{F}$ such that

$$\|\delta^{k+8n+\alpha} u_x\|_\infty \leqslant c(n, k+4n+\alpha)\delta^{k+2n+\alpha}(x), \qquad x \in T.$$

Hence it is sufficient to show that $\sum_{x\in T} \delta^{2n+\alpha}(x) < +\infty$. Observe that in view of (4), $\delta^{2n+\alpha}(x) \leqslant \tau_n^{-1} 2^{2n+\alpha} \int_{\Delta_x} \delta^\alpha \, d\mu$, $x \in X$, where τ_n denotes the volume of the unit ball in C^n and $\Delta_x = \hat{B}(x, \frac{1}{2}\delta(x))$. Note that $\Delta_x \cap \Delta_y = \emptyset$, $x, y \in T$, $x \neq y$. Thus $\sum_{x\in T} \delta^{2n+\alpha}(x) \leqslant \mathrm{const} \cdot \int_X \delta^\alpha \, d\mu < +\infty$.

The proof is completed.

5. Holomorphic continuation of holomorphic functions with restricted growth. Let X be a connected domain of holomorphy in C^n and let $\delta \in W(X)$ be such that $-\log\delta \in \mathrm{PSH}(X)$. Fix a family $F \subset \mathcal{O}(X, \delta)$ and define

$$M = \bigcap_{f\in F} f^{-1}(0).$$

Let R denote the restriction operator $\mathcal{O}(X) \ni f \to f|_M \in \mathcal{O}(M)$.

In view of Corollary 1, by using the methods presented in [6], one may easily prove the following result.

THEOREM 4 (cf. [6], Theorem 3). (i) *If* $R(\mathcal{O}(X, \delta)) = \mathcal{O}(M, \delta)$, *then* $S(M, \delta) = S(M)$ (= *the space of evaluations on* M).

(ii) *For every algebra-homomorphism* $T\colon \mathcal{O}(M, \delta) \to \mathcal{O}(X, \delta)$ *with* $R \circ T = \mathrm{id}_{\mathcal{O}(M,\delta)}$ *there exists (uniquely determined) holomorphic retraction* $\pi\colon X \to M$ *such that, for some* $c > 0$, $\varkappa > 0$: $\delta^\varkappa \leqslant c\delta \circ \pi$ *and*

$$T(f) = f \circ \pi, \qquad f \in \mathcal{O}(M, \delta).$$

In particular, every such a homomorphism T *is bounded.*

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