ANNALES
POLONICI MATHEMATICI
XLVI (1985)

Neighbourhoods of meromorphic functions and Hadamard products

by JAN STANKIEWICZ (Rzeszów)

Franciszek Leja in memoriam

Abstract. Let $E = \{z\colon |z| > 1\}$ and let

$$F(z) = z(1+A_1 z^{-1}+A_2 z^{-2}+\ldots), \quad G(z) = z(1+B_1 z^{-1}+B_2 z^{-2}+\ldots),$$

be meromorphic in E. The convolution or Hadamard product $F*G$ we define as follows:

$$(F*G)(z) = z(1+A_1 B_1 z^{-1}+A_2 B_2 z^{-2}+\ldots).$$

In this paper some new definitions of the different classes of functions defined on E are given. For a fixed class Q a class Q' is construted such that

$$f \in Q \Leftrightarrow \forall_{H \in Q'} (F*H)(z) \neq 0.$$

Using these new definitions, we obtain some results on the neighbourhoods of functions. If we write

$$\mathcal{N}_\delta(F) = \{F\colon \sum_{k=1}^{\infty} k|A_k - B_k|\},$$

then some conditions on the function F are given such that $\mathcal{N}_\delta(F)$ is contained in a fixed class of functions.

Let $U = \{z\colon |z| < 1\}$ be the unit disc and let $E = \{z\colon |z| > 1\} = C\backslash\bar{U}$ be the exterior of the unit disc.

Let f, g be two functions holomorphic in the disc U and having the Taylor–Maclaurin expansions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k. \tag{1}$$

The convolution or Hadamard product $f*g$ of the functions f and g we define as follows

$$(f*g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k. \tag{2}$$

If f and g are holomorphic in U, then $f*g$ is holomorphic in U.

Now let F, G be two functions meromorphic in E and having there the following Laurent series expansions:

$$F(z) = z + A_1 + A_2 z^{-1} + \ldots = z\left(1 + \sum_{k=1}^{\infty} A_k z^{-k}\right), \tag{3}$$

$$G(z) = z + B_1 + B_2 z^{-1} + \ldots = z\left(1 + \sum_{k=1}^{\infty} B_k z^{-k}\right). \tag{4}$$

The convolution or Hadamard product $F*G$ of such functions we define as follows:

$$(F*G)(z) := z + A_1 B_1 + A_2 B_2 z^{-1} + \ldots = z\left(1 + \sum_{k=1}^{\infty} A_k B_k z^{-k}\right). \tag{5}$$

Using a concept of convolution, St. Ruscheweyh [3], [4] gave new definitions for some known classes of holomorphic functions and for some new classes. These new definitions are very useful in solving some extremal problems in those classes.

Let St denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{6}$$

which are holomorphic univalent in U and map the unit disc U onto domains starlike with respect to the origin.

If we put

$$St' = \left\{ h(z) = \frac{1}{1+it}\left[\frac{z}{(1-z)^2} - it\frac{z}{1-z}\right]:\ t \in \boldsymbol{R} \right\},$$

then the following theorem holds:

THEOREM A ([3], Ruscheweyh). *Let a function $f(z)$ of form* (6) *be holomorphic in U. Then*

$$f \in St \Leftrightarrow \forall z \in U_0\ \forall h \in St'\ (f*h)(z) \neq 0,$$

where $U_0 = \{z\colon 0 < |z| < 1\} = U \setminus \{0\}$.

A large number of results of this type for different classes of functions holomorphic in U were obtained by Ruscheweyh [3], Rahman and Stankiewicz [1], J. Stankiewicz and Z. Stankiewicz [7], Silverman, Silvia and Tellage [5].

Some results for meromorphic functions were obtained by Rahman and J. Stankiewicz [1], [6].

In this paper I would like to give different definitions of this type for the classes of functions meromorphic in E and some applications of those definitions leading to some results on the neighbourhoods of meromorphic functions.

We need the following notation:

$$M = \{F(z) = z(1+\sum_{k=1}^{\infty} A_k z^{-k}):\ F \text{ meromorphic and univalent in } E\},$$

$$\overset{0}{M} = \{F \in M:\ F(z) \neq 0 \text{ in } E\},$$

$$\overset{1}{M} = \{F \in M:\ F(z) = z(1+\sum_{k=2}^{\infty} A_k z^{-k})\} = \{F \in M:\ A_1 = 0\},$$

$$M_\alpha^* = \left\{F \in \overset{0}{M}:\ \operatorname{Re}\frac{zF'(z)}{F(z)} > \alpha \text{ for } z \in E\right\}, \quad \alpha \in \langle 0, 1),$$

$$\check{M}_\beta = \left\{F \in \overset{0}{M}:\ \operatorname{Re}\left[e^{i\beta}\frac{zF'(z)}{F(z)}\right] > 0 \text{ for } z \in E\right\}, \quad \beta \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$$

$$\check{M}_{\beta,\alpha} = \left\{F \in \overset{0}{M}:\ \operatorname{Re}\left[e^{i\beta}\frac{zF'(z)}{F(z)}\right] > \alpha\cos\beta \text{ for } z \in E\right\}, \quad \alpha \in \langle 0, 1),\ \beta \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi),$$

$$M(A, B) = \left\{F \in \overset{0}{M}:\ \frac{zF'(z)}{F(z)} \dashv \frac{z+A}{z-B},\ |A| \leqslant 1,\ |B| \leqslant 1,\ A+B \neq 0\right\}$$

and $F \dashv G$ if and only if there exists a function φ such that $|\varphi(z)| \geqslant |z|$ and $F(z) = G(\varphi(z))$ for $z \in E$.

More generally for a fixed univalent function $G(z)$, $G(\infty) = 1$ we put

$$M(G) = \left\{G \in \overset{0}{M}:\ \frac{zF'(z)}{F(z)} \dashv G(z)\right\}.$$

Let F_1 and F_2 be the functions defined below:

$$F_1(z) := z(1+\sum_{k=1}^{\infty}(1-k)z^{-k}) = z(1+\sum_{k=2}^{\infty}(1-k)z^{-k}) = z\left(1-\frac{1}{(z-1)^2}\right) = \frac{z^3-2z^2}{(z-1)^2}, \tag{7}$$

$$F_2(z) := z(1+\sum_{k=1}^{\infty} z^{-k}) = z(\sum_{k=0}^{\infty}(1/z)^k) = z\frac{1}{1-1/z} = \frac{z^2}{z-1}. \tag{8}$$

Then for every function $F \in M$ we have

$$F_2 * F = F * F_2 = F, \tag{9}$$

$$F_1 * F = F * F_1 = zF', \tag{10}$$

where F_1, F_2 are given by (7) and (8), respectively.

Using functions (7) and (8), we define the following families of functions:

$$M' = \left\{H(z) = \frac{F_2(xz)-F_2(yz)}{x-y} \colon |x| = |y| = 1,\ x \neq y\right\},$$

$$\overset{0}{M}{}' = \overset{1}{M}{}' = M',$$

$$M_\alpha^{*\prime} = \left\{H(z) = \frac{F_1-(\alpha+it)F_2}{1-(\alpha+it)} \colon t \in \boldsymbol{R}\right\},$$

$$\check{M}'_\beta = \left\{H(z) = \frac{F_1-ite^{-i\beta}F_2}{1-ite^{-i\beta}} \colon t \in \boldsymbol{R}\right\},$$

$$\check{M}'_{\beta,\alpha} = \left\{H(z) = \frac{F_1-(\alpha\cos\beta+it)e^{-i\beta}F_2}{1-(\alpha\cos\beta+it)e^{-i\beta}} \colon t \in \boldsymbol{R}\right\},$$

$$M(A, B)' = \left\{H(z) = \frac{F_1-[(e^{it}+A)/(e^{it}-B)]F_2}{1-(e^{it}+A)/(e^{it}-B)} \colon t \in \langle 0, 2\pi)\right\}.$$

For a fixed function $G(z)$, $G(\infty) = 1$, which is regular univalent for $|z| \geqslant 1$ we put

$$M(G)' = \left\{H(z) = \frac{F_1-G(e^{it})F_2}{1-G(e^{it})} \colon t \in \langle 0, 2\pi)\right\}.$$

Now we can give a new definition for some classes of functions.

THEOREM 1. *Let $F(z)$ of form* (1) *be holomorphic in E, and let Q be one of the classes M, M_α^*, $\check{M}_\beta$, $\check{M}_{\beta,\alpha}$, $M(A, B)$ and $M(G)$. Then $F \in Q$ if and only if*

$$\forall H \in Q'\ \forall z \in E\ (F * H)(z) \neq 0. \tag{11}$$

Proof. (a) $Q = M$. From (11) we have

$$F(xz) \neq F(yz) \tag{12}$$

for every x, y, z, $|x| = |y|$, $z \in E$. This gives univalency on every circle $|z| = r$ and hence in E. Reciprocally, if F is univalent in E, then (12) holds for every $z \in E$ and $|x| = |y| = 1$, $x \neq y$ and therefore

$$0 \neq F(xz) - F(yz) = (x-y)(F*H)(z),$$

and the proof is complete.

(b) $Q = M_\alpha^*$. From (11) we have

$$zF'(z) - (\alpha+it)F(z) \neq 0;$$

equivalently,

$$\frac{zF'(z)}{F(z)} \neq \alpha+it \tag{13}$$

for every $z \in E$ and every $t \in \boldsymbol{R}$.

Since $zF'(z)/F(z)|_{z=\infty} = 1$ and $\alpha+it$ covers the straight line $\operatorname{Re} w = \alpha$, we have by (13)

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > \alpha. \tag{14}$$

Therefore $F \in M_\alpha^*$.

Reciprocally, let $F \in M_\alpha^*$, that is, let F satisfy condition (14). For $H \in M_\alpha^{*\prime}$ we have the identity

$$(F*H)(z) = \frac{F(z)}{1-(\alpha+it)}\left[\frac{zF'(z)}{F(z)} - (\alpha+it)\right], \tag{15}$$

and therefore by (14) we have $(F*H)(z) \neq 0$.

(c) $Q = \check{M}_\beta$. From (11) we have

$$e^{i\beta}\frac{zF'(z)}{F(z)} \neq it, \qquad t \in \boldsymbol{R}. \tag{16}$$

Since

$$e^{i\beta}\frac{zF'(z)}{F(z)}\bigg|_{z=\infty} = e^{i\beta}, \qquad \operatorname{Re} e^{i\beta} > 0,$$

we have

$$\operatorname{Re}\left[e^{i\beta}\frac{zF'(z)}{F(z)}\right] > 0 \quad \text{for } z \in E \tag{17}$$

and $F \in \check{M}_\beta$.

Reciprocally, if $F \in \check{M}_\beta$, then (17) is satisfied. For $H \in M_\beta^{*\prime}$ we have the identity

$$(F*H)(z) = \frac{F(z)e^{-i\beta}}{1-ite^{-i\beta}}\left[\frac{e^{i\beta}zF'(z)}{F(z)} - it\right]. \tag{18}$$

By (17) and (18) we have

$$(F*H)(z) \neq 0,$$

and the proof is complete.

(d) $Q = \check{M}_{\beta,\alpha}$. From (11) we have

$$e^{i\beta}\frac{zF'(z)}{F(z)} \neq (\alpha\cos\beta + it) \quad \text{for } t \in \boldsymbol{R},\ z \in E. \tag{19}$$

Thus

$$\operatorname{Re}\left[e^{i\beta}\frac{zF'(z)}{F(z)}\right] > \alpha\cos\beta \quad \text{for } z \in E, \tag{20}$$

and therefore $F \in \check{M}_{\beta,\alpha}$.

If $F\in \check{M}_{\beta,\alpha}$, then by (20) and by the identity

$$(21)\quad (F*H)(z) = \frac{e^{-i\beta}F(z)}{1-(\alpha\cos\beta+it)}\left[\frac{e^{i\beta}zF'(z)}{F(z)}-(\alpha\cos\beta+it)\right],\qquad H\in \check{M}'_{\beta,\alpha},$$

we have

$$(F*H)(z)\neq 0 \qquad \text{for } H\in \check{M}'_{\beta,\alpha} \text{ and } z\in E.$$

(e) $Q = M(A, B)$. From (11) we have

$$\frac{zF'(z)}{F(z)}\neq\frac{e^{it}+A}{e^{it}-B},\qquad t\in\langle 0, 2\pi).$$

This means that $zF'(z)/F(z)$ do not take any boundary value of the function $(z+A)/(z-B)$ on E. Since $(z+A)/(z-B)$ is univalent in E, we have

$$\frac{zF'(z)}{F(z)} \dashv \frac{z+A}{z-B}$$

and therefore $F\in M(A, B)$.

If $H\in M(A, B)'$, then the following identity holds:

$$(22)\qquad (F*H)(z) = \frac{F(z)}{1-(e^{it}+A)/(e^{it}-B)}\left[\frac{zF'(z)}{F(z)}-\frac{e^{it}+A}{e^{it}-B}\right].$$

If $F\in M(A, B)$, then we have

$$(23)\qquad \frac{zF'(z)}{F(z)} \dashv \frac{z+A}{z-B},\qquad \frac{zF'(z)}{F(z)}\neq\frac{e^{it}+A}{e^{it}-B},\qquad t\in\langle 0, 2\pi).$$

By (22) and (23) we have

$$(F*H)(z)\neq 0 \qquad \text{for every } H\in M(A, B)',\ z\in E.$$

(f) $Q = M(G)$. From (11) we have

$$\frac{zF'(z)}{F(z)}\neq G(e^{it}) \qquad \text{for every } t\in\langle 0, 2\pi) \text{ and } z\in E.$$

Since G is regular univalent, $G(\infty) = 1 = \left.\frac{zF'(z)}{F(z)}\right|_{z=\infty}$, we immediately have $\frac{zF'(z)}{F(z)} \dashv G(z)$.

Reciprocally, let $F\in M(G)$; then

$$(24)\qquad \frac{zF'(z)}{F(z)} \dashv G(z).$$

For $H \in M(G')$ we have the identity

$$(F*H)(z) = \frac{F(z)}{1-G(e^{it})}\left[\frac{zF'(z)}{F(z)} - G(e^{it})\right]. \tag{25}$$

By (24) and (25) we have $(F*H)(z) \neq 0$ for every $H \in M(G)'$ and $z \in E$. This means that (11) holds.

The proof of Theorem 1 is complete.

For a function $F \in M$ and $\varepsilon \in C$, $n = 0, 1, 2, \ldots$, we define

$$F_{n,\varepsilon}(z) = \begin{cases} \dfrac{F(z)+\varepsilon z}{1+\varepsilon}, & n = 0, \\ F(z)+\varepsilon z^{1-n}, & n = 1, 2, \ldots \end{cases}$$

LEMMA 1. *If for every* ε, $|\varepsilon| < \delta$ *we have* $F_{n,\varepsilon} \in M_\alpha^*$, *then for every* $H \in M_\alpha^{*\prime}$

$$|(F*H)(z)| > \gamma_n \quad \text{for } z \in E,$$

where

$$\gamma_0 = \delta,$$

$$\gamma_1 = \begin{cases} \delta & \text{for } \alpha \geqslant \frac{1}{2}, \\ \delta\dfrac{\alpha}{1-\alpha} & \text{for } \alpha < \frac{1}{2}, \end{cases}$$

$$\gamma_n = \delta, \quad n \geqslant 2.$$

Proof. If $H(z) = z\left(1+\sum_{k=1}^{\infty} c_k z^{-k}\right) \in M_\alpha^{*\prime}$, then

$$c_k = c_k(t) = \frac{(1-k)-(\alpha+it)}{1-(\alpha+it)} = \frac{1-\alpha-k-it}{1-\alpha-it}$$

and

$$|c_k(t)|^2 = \frac{(1-\alpha-k)^2+t^2}{(1-\alpha)^2+t^2}.$$

Thus

$$1 \leqslant |c_k| \leqslant \frac{|k+\alpha-1|}{1-\alpha} < \frac{k}{1-\alpha} \quad \text{for } |k+\alpha-1| \geqslant 1-\alpha,$$

$$\frac{|k+\alpha-1|}{|1-\alpha|} \leqslant |c_k| \leqslant 1 \quad \text{for } |k+\alpha-1| < 1-\alpha.$$

This gives

$$1 \leqslant |c_k| \leqslant \frac{k+\alpha-1}{1-\alpha} < \frac{k}{1-\alpha} \quad \text{for } k = 2, 3, \ldots$$

and

$$1 \leqslant |c_1| \leqslant \frac{\alpha}{1-\alpha} < \frac{1}{1-\alpha}, \quad 1 > \alpha \geqslant \tfrac{1}{2},$$

$$\frac{\alpha}{1-\alpha} \leqslant |c_1| \leqslant 1 \leqslant \frac{1}{1-\alpha}, \quad 0 < \alpha < \tfrac{1}{2}.$$

Now, by Theorem 1, we have for $n \geqslant 2$

$$F_{n,\varepsilon} \in M_\alpha^* \Leftrightarrow (F_{n,\varepsilon} * H)(z) \neq 0 \Leftrightarrow (F*H)(z) + \varepsilon c_n z^{1-n} \neq 0$$
$$\Leftrightarrow z^{n-1}(F*H)(z) \neq -\varepsilon c_n.$$

Since this inequality holds for every ε, $|\varepsilon| < \delta$, it follows that $z^{n-1}(F*H)(z)$ do not take any value in the disc $|w| \leqslant \delta \min |c_n| = \delta$, that is,

$$|z^{n-1}(F*H)(z)| \geqslant \delta.$$

We have

$$\varphi(\xi) = [(1/\xi)^{n-1}(H*F)(1/\xi)]^{-1} = \xi^n + b_{n+1}\xi^{n+1} + \ldots, \quad \xi \in U.$$

By the Schwarz Lemma we have

$$|\varphi(\xi)| \leqslant \frac{1}{\delta}|\xi|^n, \quad \xi \in U,$$

or equivalently

$$|z^{n-1}(H*F)(z)| \geqslant \delta |z|^n, \quad z \in E,$$
$$|(F*H)(z)| \geqslant \delta |z| > \delta, \quad z \in E.$$

Now let $n = 0$:

$$F_{0,\varepsilon}(z) \in M_\alpha^* \Leftrightarrow (F_{0,\varepsilon} * H)(z) \neq 0 \Leftrightarrow \frac{1}{1+\varepsilon}[(F*H)(z) + \varepsilon z] \neq 0.$$

Thus

$$\frac{(F*H)(z)}{z} \neq -\varepsilon, \quad z \in E,$$

$$\left|\frac{(F*H)(z)}{z}\right| \geqslant \delta, \quad z \in E,$$

$$|(F*H)(z)| \geqslant \delta |z| > \delta \quad \text{for } z \in E.$$

For $n = 1$ we have

$$F_{1,\varepsilon}(z) \in M_\alpha^* \Leftrightarrow (F_{1,\varepsilon} * H)(z) \neq 0 \Leftrightarrow (F*H)(z) \neq -\varepsilon c_1$$
$$\text{for } H \in M_\alpha^{*\prime} \text{ and } z \in E.$$

This gives

$$|(F*H)(z)| \geqslant \delta \min |c_1| = \begin{cases} \delta & \text{for } \alpha \geqslant \frac{1}{2}, \\ \delta \dfrac{\alpha}{1-\alpha} & \text{for } \alpha < \frac{1}{2}. \end{cases}$$

In the class M we can introduce a pre-norm in the following way:

$$\|F\| = \sum_{k=1}^{\infty} k|A_k|.$$

Now we can introduce the distance between two functions and the neighbourhoods of functions.

For $F(z) = z\left(1+\sum_{k=1}^{\infty} A_k z^{-k}\right)$, $G(z) = z\left(1+\sum_{k=1}^{\infty} B_k z^{-k}\right)$ we define

$$d(F, G) = \|F-G\| = \sum_{k=1}^{\infty} k|A_k - B_k|$$

and

$$\mathcal{N}_\delta(F) = \{G \in M \colon d(F, G) < \delta\}.$$

THEOREM 2. *Let $n = 0, 1, 2, \ldots$ be fixed. If for every ε, $|\varepsilon| < \delta$, we have*

$$F_{n,\varepsilon}(z) \in M_\alpha^*,$$

then

$$\mathcal{N}_{\delta_n}(F) \subset M_\alpha^*,$$

where

$$\delta_n = \delta(1-\alpha) \quad \textit{for } n = 0, 2, 3, \ldots,$$

$$\delta_1 = \begin{cases} \delta(1-\alpha) & \textit{if } \frac{1}{2} \leqslant \alpha < 1, \\ \delta\alpha & \textit{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Proof. Let $G \in \mathcal{N}_{\delta_n}(F)$. For $H \in M_\alpha^{*\prime}$ we have

$$\begin{aligned} |(G*H)(z)| &= |((F+G-F)*H)(z)| \\ &= |(F*H)(z) + ((G-F)*H)(z)| \\ &\geqslant \big||(F*H)(z)| - |((G-F)*H)(z)|\big| \\ &\geqslant \gamma_n - \sum_{k=1}^{\infty} |A_k - B_k|\,|c_k|\,|z|^{1-k} \\ &> \gamma_n - \frac{1}{1-\alpha} d(F, G) = \gamma_n - \frac{\delta_n}{1-\alpha}. \end{aligned}$$

Then for every $H \in M_\alpha^{*\prime}$ and $z \in E$ we have

$$(G*H)(z) \neq 0.$$

This means that $G \in M_\alpha^*$ and Theorem 2 is proved.

Remark. In the special case $A_1 = 0$ we obtain Theorem 9 of paper [1].

LEMMA 2. *If for every* ε, $|\varepsilon| < 1$ *and a fixed* $n = 0, 2, 3, \ldots$ *we have*

$$F_{n,\varepsilon} \in \check{M}_\beta,$$

then for every $H \in M'_\beta$

$$|(F*H)(z)| > \gamma_n \quad \textit{for } z \in E,$$

where

$$\gamma_n = \delta \cdot \frac{|n - \sqrt{n^2 - 4(n-1)\cos^2\beta}|}{2\cos\beta}, \qquad n = 0, 2, 3, \ldots$$

Proof. We have $H(z) = z\left(1 + \sum_{k=1}^{\infty} c_k z^{-k}\right) \in \check{M}_\beta$ if and only if

$$c_k = c_k(t) = \frac{-k+1-ite^{-i\beta}}{1-ite^{-i\beta}} = 1 - \frac{k}{1-ite^{-i\beta}}.$$

It is easy to check that $c_k(t)$ covers a circle with centre $s_k = 1 - \frac{1}{2}ke^{-i\beta}/\cos\beta$ and radius $R_k = \frac{1}{2}k/\cos\beta$. Therefore

$$|c_k| \leqslant \frac{k + \sqrt{k^2 - 4(k-1)\cos^2\beta}}{2\cos\beta} \leqslant \frac{k}{\cos\beta},$$

$$|c_k| \geqslant \frac{k - \sqrt{k^2 - 4(k-1)\cos^2\beta}}{2\cos\beta} \geqslant \frac{k-1}{k}\cos\beta.$$

Thus for $n = 2, 3, \ldots$

$$F_{n,\varepsilon} \in \check{M}_\beta \Leftrightarrow (F_{n,\varepsilon}*H)(z) \neq 0 \Leftrightarrow (F*H)(z) + \varepsilon c_n z^{1-n} \neq 0,$$
$$z^{n-1}(F*H)(z) \neq -\varepsilon c_n.$$

Therefore

$$|z^{n-1}(F*H)(z)| \geqslant \delta \min |c_n(t)| = \delta \frac{n - \sqrt{n^2 - 4(n-1)\cos^2\beta}}{2\cos\beta}.$$

Using the Schwarz Lemma, we obtain

$$|(F*H)(z)| > \delta \frac{n - \sqrt{n^2 - 4(n-1)\cos^2\beta}}{2\cos\beta}.$$

For $n = 0$ we have as usual:

$$|(F*H)(z)| > \delta \quad (\gamma_0 = 1).$$

This completes the proof of Lemma 2.

THEOREM 3. *Let $n \in \{0, 2, 3, \ldots\}$ be fixed. If for every ε, $|\varepsilon| < \delta$ we have*

$$F_{n,\varepsilon} \in \check{M}_\beta,$$

then

$$\mathcal{N}_{\delta_n}(F) \subset \check{M}_\beta,$$

where

$$\delta_n = \gamma_n \cos\beta = \tfrac{1}{2}\delta\,|n - \sqrt{n^2 - 4(n-1)\cos^2\beta}|, \qquad n = 0, 2, 3, \ldots$$

$$(\delta_0 = \delta\cos\beta,\ \delta_n \geqslant \delta\frac{n-1}{n}\cos^2\beta \text{ for } n = 2, 3, \ldots).$$

Proof. Using Lemma 2 and the estimations on the coefficients c_k, we have for every $G \in \mathcal{N}_{\delta_n}(F)$ and $H \in \check{M}'_\beta$ the following relation:

$$\begin{aligned}|(G*H)(z)| &= |(F*H)(z) + ((G-F)*H)(z)| \\ &\geqslant |(F*H)(z) - \sum_{k=1}^{\infty} |B_k - A_k|\,|c_k z^{1-k}|| > \gamma_n - \Big(\sum_{k=1}^{\infty} k\,|B_k - A_k|\Big)\frac{1}{\cos\beta} \\ &\geqslant \gamma_n - d(F, G)\frac{1}{\cos\beta} > \gamma_n - \frac{\delta_n}{\cos\beta} = 0.\end{aligned}$$

Now the results follows from Theorem 1.

LEMMA 3. *Let $n \in \{0, 1, 2, \ldots\}$ be fixed. If for every ε, $|\varepsilon| < \delta$,*

$$F_{n,\varepsilon} \in \check{M}_{\beta,\alpha},$$

then for every $H \in \check{M}_{\beta,\alpha}$ we have

$$|(F*H)(z)| > \gamma_n \quad \text{for } z \in E,$$

where

$$\gamma_0 = \delta, \qquad \gamma_n = \delta\cdot\frac{n - \sqrt{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\beta}}{2(1-\alpha)\cos\beta} \geqslant \delta\frac{n+\alpha-1}{n}\cos\beta.$$

Proof. We have $H(z) = z\big(1 + \sum_{k=1}^{\infty} c_k z^{-k}\big) \in \check{M}'_{\beta,\alpha}$ if and only if

$$c_k = c_k(t) = \frac{1 - k - (\alpha\cos\beta + it)e^{-i\beta}}{1 - (\alpha\cos\beta + it)e^{-i\beta}} = 1 - \frac{ke^{i\beta}}{e^{i\beta} - \alpha\cos\beta - it}.$$

For real t the values $c_k(t)$ cover a circle with centre

$$s_k = 1-\frac{ke^{i\beta}}{2(1-\alpha)\cos\beta} = \frac{2(1-\alpha)\cos\beta - k\cos\beta - ik\sin\beta}{2(1-\alpha)\cos\beta}$$

and radius

$$R_k = \left|\frac{ke^{i\beta}}{2(1-\alpha)\cos\beta}\right| = \frac{k}{2(1-\alpha)\cos\beta}.$$

Thus

$$\frac{k+\alpha-1}{2k} \leqslant \frac{k-\sqrt{k^2-4(1-\alpha)(k+\alpha-1)\cos^2\beta}}{2(1-\alpha)\cos\beta} \leqslant |c_k| \leqslant \frac{k}{(1-\alpha)\cos\beta}.$$

For $n = 0$ we have as usual $|(F*H)(z)| > \delta$. For $n = 1, 2, \ldots$, using the estimations on $|c_k|$ in the class $\check{M}'_{\beta,\alpha}$, we have

$$F_{n,\varepsilon} \in \check{M}_{\beta,\alpha} \Leftrightarrow (F_{n,\varepsilon}*H)(z) \neq 0 \Leftrightarrow (F*H)(z) \neq -\varepsilon c_n z^{1-n} \Leftrightarrow \frac{(F*H)(z)}{z^{1-n}} \neq -\varepsilon c_n.$$

Thus

$$|z^{n-1}(F*H)(z)| \geqslant \delta \cdot \min_{t\in\langle 0,2\pi)} |c_n(t)|$$

$$= \delta\frac{n-\sqrt{n^2-4(1-\alpha)(n+\alpha-1)\cos^2\beta}}{2(1-\alpha)\cos\beta} \geqslant \delta\frac{n+\alpha-1}{n}\cos\beta.$$

THEOREM 4. *Let $n \in \{0, 1, 2, \ldots\}$ be fixed. If for every ε, $|\varepsilon| < \delta$, we have*

$$F_{n,\varepsilon} \in \check{M}_{\beta,\alpha},$$

then

$$\mathcal{N}_{\delta_n}(F) \subset \check{M}_{\beta,\alpha},$$

where

$$\delta_0 = \delta(1-\alpha)\cos\beta,$$

$$\delta_n = \gamma_n(1-\alpha)\cos\beta = \tfrac{1}{2}\delta\,|n-\sqrt{n^2-4(n+\alpha-1)(1-\alpha)\cos^2\beta}|$$

$$\geqslant \delta\frac{n+\alpha-1}{n}\cos\beta, \quad n = 1, 2, 3, \ldots$$

LEMMA 4. *Let*

$$H(z) = z\left(1+\sum_{k=1}^{\infty} c_k z^{-k}\right) \in M(A, B)'.$$

Then for $k = 1, 2,,,,$ *we have*

$$|c_k| \geqslant \frac{\big|k-|(\alpha-1)B-A|\big|}{|A+B|} \geqslant \frac{(k-1)(1-|B|)+1-|A|}{|A+B|} \tag{26}$$

$$\geqslant \frac{k(1-\max\{|A|, |B|\})}{|A+B|},$$

$$|c_k| \leqslant \frac{(k-1)(1+|B|)+1+|A|}{|A+B|} \tag{27}$$

$$\leqslant \frac{k(1+\max\{|A|, |B|\})}{|A+B|} \leqslant \frac{2k}{|A+B|}.$$

Proof. We have $H \in M(A, B)'$ if and only if its coefficients c_k are given by the formula

$$c_k = c_k(t) = \frac{-(k-1)-(e^{it}+A)/(e^{it}-B)}{1-(e^{it}+A)/(e^{it}-B)}$$

$$= \frac{ke^{it}-(k-1)B+A}{A+B} = \frac{A-(k-1)B}{|A+B|} + \frac{k}{A+B}e^{it}.$$

This means that $c_k(t)$ covers a circle with centre

$$s_k = \frac{A-(k-1)B}{A+B}$$

and radius

$$R_k = \frac{k}{|A+B|}.$$

Using the estimation $\big||s_k|-R_k\big| \leqslant |c_k| \leqslant |s_k|+R_k$, we obtain (26) and (27) after some calculations.

LEMMA 5. *Let* $n \in \{0, 1, 2, \ldots\}$ *be fixed. If for every* ε, $|\varepsilon| < \delta$, *we have*

$$F_{n,\varepsilon} \in M(A, B)$$

then, for every $H \in M(A, B)'$,

$$|(F*H)(z)| > \gamma_n \quad \text{for } z \in E,$$

where

$$\gamma_0 = \delta,$$

$$\gamma_n = \delta\frac{\big|n-|(n-1)B-A|\big|}{|A+B|} \geqslant \delta\frac{(n-1)(1-|B|)+(1-|A|)}{|A+B|}$$

$$\geqslant n\delta\frac{1-\max\{|A|, |B|\}}{|A+B|}.$$

Proof. If, for some F, we have $F_{n,\varepsilon} \in M(A, B)$, then by Theorem 1

$$F_{n,\varepsilon}(z) \in M(A, B) \Leftrightarrow (F_{n,\varepsilon} * H)(z) \neq 0 \Leftrightarrow (F * H)(z) + \varepsilon c_n z^{1-n} \neq 0$$
$$\Leftrightarrow z^{n-1}(F * H)(z) \neq -\varepsilon c_n.$$

This implies that

$$|z^{n-1}(F * H)(z)| \geqslant \delta \min_t |c_n(t)|.$$

Using the Schwarz Lemma, we have

$$|(F * H)(z)| > \delta \frac{\big|n - |(n-1)B - A|\big|}{|A+B|}$$
$$\geqslant \frac{\delta}{|A+B|}\big((n-1)(1-|B|) + 1 - |A|\big) \geqslant \frac{n\delta}{|A+B|}(1 - \max\{|A|, |B|\}).$$

For $n = 0$ we immediately have

$$|(F * H)(z)| > \delta.$$

THEOREM. 5. *Let* $n \in \{0, 1, 2, \ldots\}$ *be fixed. If for every* ε, $|\varepsilon| < \delta$, *we have*

$$F_{n,\varepsilon} \in M(A, B),$$

then

$$\mathcal{N}_{\delta_n}(F) \subset M(A, B),$$

where

$$\delta_0 = \delta \frac{|A+B|}{1 + \max\{|A|, |B|\}} \geqslant \tfrac{1}{2}\delta |A+B|,$$
$$\delta_n = \delta \frac{\big|n - |(n-1)B - A|\big|}{1 + \max\{|A|, |B|\}} \geqslant \delta \frac{(n-1)(1-|B|) + (1-|A|)}{1 + \max\{|A|, |B|\}}$$
$$\geqslant \delta n \frac{1 - \max\{|A|, |B|\}}{1 + \max\{|A|, |B|\}}.$$

Proof. Using Lemma 4 and Lemma 5, we obtain for every $G \in \mathcal{N}_{\delta_n}(F)$ the following relation:

$$|(G * H)(z)| = \big|(F * H)(z) - ((F - G) * H)(z)\big|$$
$$\geqslant \Big||(F * H)(z)| - |\sum_{k=1}^{\infty} (A_k - B_k) c_k z^{1-k}|\Big|$$
$$> \gamma_n - \frac{1 + \max\{|A|, |B|\}}{|A+B|} \sum_{k=1}^{\infty} k |A_k - B_k|$$
$$> \gamma_n - \delta_n \frac{1 + \max\{|A|, |B|\}}{|A+B|}.$$

Thus

$$|(G*H)(z)| > 0 \quad \text{and therefore} \quad (G*H)(z) \neq 0,$$

which by Theorem 1 implies $G \in M(A, B)$. The proof is complete.

In the general case, for the class $M(G)$, $G(\infty) = 1$, G univalent (convex) in E, we have

THEOREM 6. *If for a fixed $n \in \{0, 1, 2, \ldots\}$ and for every ε, $|\varepsilon| < \delta$, we have*

$$F_{n,\varepsilon} \in M(G),$$

then

$$\mathcal{N}_{\delta_n(G)}(F) \subset M(G),$$

where

$$\delta_0(G) = \delta \Big/ \sup_{k,t} \frac{|k-1+G(e^{it})|}{k\,|1-G(e^{it})|},$$

$$\delta_n = \delta \inf_t \frac{|n-1+G(e^{it})|}{|1-G(e^{it})|} \Big/ \sup_{k,t} \frac{|k-1+G(e^{it})|}{k\,|1-G(e^{it})|}.$$

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INSTYTUT MATEMATYKI i FIZYKI POLITECHNIKI RZESZOWSKIEJ
RZESZÓW

Reçu par la Rédaction le 20.03.1984