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On balls and totally geodesic submanifolds

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0. Totally geodesic submanifolds of a Riemannian manifold M can be characterized as submanifolds N such that any point x of N admits an open neighbourhood $U \subset N$ such that $d_M | U \times U = d_N | U \times U$, where d_M and d_N denote the distance functions on M and N, respectively. In this note, we show that a compact submanifold N of M is totally geodesic if and only if there exists a positive number $\varepsilon > 0$ such that

$$B_N(x, r) = B_M(x, r) \cap N \tag{1}$$

for any x of N and r of $(0; \varepsilon)$, where $B_M(x, r)$ and $B_N(x, r)$ denote the centred at x open balls of radii r on M and N, respectively. (Recall, that the relation

$$B_N(x, r) \subset B_M(x, r) \cap N \tag{2}$$

holds for any submanifold N of M, any x and r.) If N is a submanifold of M and equality (1) holds for any x of N and r of $(0; \varepsilon)$, then we say that N is *ε-regular*. We denote by $\varepsilon(N)$ the smallest upper bound of the set of all ε such that N is ε-regular.

1. Theorem A. (a) *If a submanifold N of a Riemannian manifold M is ε-regular for some $\varepsilon > 0$, then it is totally geodesic.* (b) *If N is compact and totally geodesic, then it is ε-regular for some $\varepsilon > 0$.*

Proof. (a) Suppose that N is ε-regular. It is sufficient to show that if $x, y \in N$ and $d_N(x, y) < \varepsilon$, then $d_M(x, y) \geqslant d_N(x, y)$. In order to do this, let $a = d_N(x, y) < r < \varepsilon$. Then $y \in B_N(x, r) - B_N(x, a) = (B_M(x, r) \cap N) - (B_M(x, a) \cap N) = (B_M(x, r) - B_M(x, a)) \cap N$. It follows that $y \notin B_M(x, a)$, i.e. that $d_M(x, y) \geqslant a$.

(b) Suppose that N is compact and totally geodesic. For any x of N denote by $\varrho(x)$ the radius of injectivity of M at x ([2], § 5.2). The function $N \ni x \mapsto \varrho(x)$ is continuous. Therefore, the number $\delta = \min_N \varrho$ is positive. If $x \in N$, $v \in T_x N$, $|v| = 1$, $|t| < \delta$ and $y = \exp(tv)$, then $d_N(x, y) = d_M(x, y) = |t|$. In fact, if $a = d_N(x, y)$, then $a \leqslant L(c) = |t| < \delta$, where $c\colon \langle 0, 1 \rangle \to N$ is a regular curve given by $c(s) = \exp(stv)$, and $y = \exp(aw)$ for some unit vector w of $T_x N$. Since $|aw| < \varrho(x)$, $|tv| < \varrho(x)$, and $\exp(aw) = \exp(tv)$, we have $aw = tv$ and $a = |t|$. The similar argumentation shows that $d_M(x, y) = |t|$.

For any x of N put

$$\varepsilon_x = d_M(x, N - B_N(x, \tfrac{1}{2}\delta)).$$

Let us take points x and y of N such that $d_N(x, y) < \frac{1}{2}\delta$. There exists a point z_0 of the set $N - B_N(x, \frac{1}{2}\delta)$ such that $\varepsilon_x = d_M(x, z_0)$. Clearly,

$$d_N(y, z_0) \geqslant d_N(x, z_0) - d_N(x, y) \geqslant \tfrac{1}{2}\delta - d_N(x, y).$$

If $d_N(y, z_0) \geqslant \frac{1}{2}\delta$, then $z_0 \in N - B_N(y, \frac{1}{2}\delta)$ and

(3) $$\varepsilon_y \leqslant d_M(y, z_0) \leqslant d_M(y, x) + d_M(x, z_0) \leqslant \varepsilon_x + d_N(x, y).$$

If $d_N(y, z_0) < \frac{1}{2}\delta$, then there exists an unit vector v of $T_y N$ such that $z_0 = \exp(tv)$, $t = d_N(y, z_0)$. Put $z = \exp((t + d_N(x, y))v)$. Since $t + d_N(x, y) < \delta \leqslant \varrho(y)$, we have $d_N(y, z) = t + d_N(x, y)$ and $d_N(z, z_0) = d_N(x, y)$. Therefore,

(4) $$\varepsilon_y \leqslant d_M(y, z) \leqslant d_M(y, x) + d_M(x, z_0) + d_M(z_0, z) \leqslant \varepsilon_x + 2d_N(x, y).$$

From (3) and (4) it follows that if $x, y \in N$ and $d_N(x, y) < \frac{1}{2}\delta$, then

$$|\varepsilon_x - \varepsilon_y| \leqslant 2d_N(x, y).$$

We conclude that the function $N \ni x \mapsto \varepsilon_x$ is continuous.

Put $\varepsilon = \min\{\varepsilon_x; x \in N\}$. We claim that N is ε-regular. In fact, if $x \in N$, $r < \varepsilon$, and $y \in N \cap B_M(x, r)$, then $d_N(x, y) < \frac{1}{2}\delta$ (otherwise $y \in N - B_N(x, \frac{1}{2}\delta)$ and $d_M(x, y) \geqslant \varepsilon_x \geqslant \varepsilon > r$) and $d_N(x, y) = d_M(x, y) < r$.

This ends the proof.

In other words, Theorem A says that (a) if $\varepsilon(N) > 0$, then N is totally geodesic and (b) if a submanifold N is compact and totally geodesic, then $\varepsilon(N) > 0$.

Examples. If S^k (resp. $P^k R$) is considered as a totally geodesic submanifold of S^m (resp., of $P^m R$), $k < m$, then $\varepsilon(S^k) = \varepsilon(P^k R) = +\infty$. If N is the totally geodesic submanifold of the torus $T = R^2/Z^2$ obtained by the projection of the line $L \subset R^2$ given by the equation

$$px_1 + qx_2 + c = 0,$$

then

(a) $\varepsilon(N) = 0$ when the number p/q is irrational,

(b) $\varepsilon(N) = +\infty$ when $p, q \in Z$ and $p^2 + q^2 = 1$,

(c) $\varepsilon(N) = 1/\sqrt{p^2 + q^2}$ when $p, q \in Z$, $(p, q) = 1$, and $p^2 + q^2 > 1$.

If N_i $(i = 1, 2)$ are totally geodesic submanifolds of Riemannian manifolds M_i, then $N_1 \times N_2$ is a totally geodesic submanifold of $M_1 \times M_2$ and $\varepsilon(N_1 \times N_2) = \min(\varepsilon(N_1), \varepsilon(N_2))$.

2. THEOREM B. *Let G be a group of isometries acting freely and properly discontinously on a complete Riemannian manifold M. If N is a complete G-invariant connected submanifold embedded in M and $\tilde{N} = \pi(N)$, where π: $M \to \tilde{M} = M/G$ is the projection, then*

$$\min\left(\tfrac{1}{2}d, \varepsilon(\tilde{N})\right) \leqslant \varepsilon(N) \leqslant \varepsilon(\tilde{N}), \tag{5}$$

where $d = \inf\{d_M(x, gx);\ x \in N,\ g \in G,$ and $g \neq e\}$.

Proof. At first, we shall establish the equality

$$d_{\tilde{M}}\left(\pi(x), \pi(y)\right) = \inf_{g \in G} d_M(x, yg), \qquad x, y \in M. \tag{6}$$

Let us take points x and y of M and put $\tilde{x} = \pi(x)$, $\tilde{y} = \pi(y)$. Since M and $\tilde{M}$ are complete and π: $M \to \tilde{M}$ is a covering, there exist a curve $\tilde{c}$: $\langle 0, 1\rangle \to \tilde{M}$ and its lift c: $\langle 0, 1\rangle \to M$ such that $\tilde{c}(0) = \tilde{x}$, $\tilde{c}(1) = \tilde{y}$, $c(0) = x$, and $L(\tilde{c}) = d_{\tilde{M}}(\tilde{x}, \tilde{y})$. It is clear that $L(c) = L(\tilde{c})$ and $c(1) = yg$ for some g of G. Therefore,

$$d_{\tilde{M}}(\tilde{x}, \tilde{y}) = L(\tilde{c}) = L(c) \geqslant d_M(x, yg) \geqslant \inf_{g \in G} d_M(x, yg). \tag{7}$$

On the other hand, if $g \in G$, then there exists a curve c: $\langle 0, 1\rangle \to M$ such that $c(0) = x$, $c(1) = yg$, and $L(c) = d_M(x, yg)$. Then

$$d_M(x, yg) = L(c) = L(\pi \circ c) \geqslant d_{\tilde{M}}(\tilde{x}, \tilde{y}). \tag{8}$$

Inequalities (7) and (8) yield (6).

From (6), it follows immediately that

$$B_{\tilde{M}}\left(\pi(x), r\right) = \pi\left(B_M(x, r)\right) \tag{9}$$

for any x of M and $r > 0$.

From the assumptions of the Theorem, it follows that $G_N = \{g|N;\ g \in G\}$ is a group of isometries of N acting freely and properly discontinously on N in such manner that $\tilde{N} = N/G_N$. Therefore, we can prove analogously to (6) and (9) that

$$d_{\tilde{N}}\left(\pi(x), \pi(y)\right) = \inf_{g \in G} d_N(x, yg) \tag{10}$$

and

$$B_{\tilde{N}}\left(\pi(x), r\right) = \pi\left(B_N(x, r)\right) \tag{11}$$

for any $x \in N$, $y \in N$, and $r > 0$.

Comparing (9) and (11) we can conclude that if $x \in N$ and $r < \varepsilon(N)$, then $B_{\tilde{N}}(\pi(x), r) = B_{\tilde{M}}(\pi(x), r) \cap \tilde{N}$. This yields the inequality $\varepsilon(N) \leqslant \varepsilon(\tilde{N})$.

Using equalities (6) and (10), we can show that

$$\pi^{-1}\left(B_{\tilde{M}}\left(\pi(x), r\right)\right) = \bigcup_{g \in G} B_M(xg, r)$$

and

$$\pi^{-1}\big(B_{\tilde{N}}(\pi(x), r)\big) = \bigcup_{g\in G} B_N(xg, r)$$

for any x and r. Therefore, if $r < \varepsilon(\tilde{N})$ and $x \in N$, then

$$\begin{aligned}\bigcup_{g\in G} B_N(xg, r) &= \pi^{-1}\big(B_{\tilde{M}}(\pi(x), r)\big) \cap \pi^{-1}(\tilde{N}) \\ &= \bigcup_{g\in G} \big(B_M(xg, r) \cap N\big).\end{aligned}$$

If, in addition, $r < \frac{1}{2}d$, then $B_M(x, r) \cap B_M(xg, r) = \emptyset$ for any $g \in G$, $g \neq e$. This implies equality (1) for any x of N and $r < \min\big(\frac{1}{2}d, \varepsilon(\tilde{N})\big)$. Consequently, we have the inequality $\varepsilon(N) \geqslant \min\big(\frac{1}{2}d, \varepsilon(\tilde{N})\big)$ which completes the proof.

Let us note that if M is compact, then the number d in (5) is positive. Simple examples (geodesic lines on the cylinder and on the torus) show that inequalities (5) need not be satisfied if N is not G-invariant and that equalities $\varepsilon(N) = \varepsilon(\tilde{N})$ and $\varepsilon(N) = \frac{1}{2}d$ appear occasionally.

3. Let us recall that a submersion $f\colon M \to B$, where M and B are Riemannian manifolds, is called Riemannian [7] if and only if $|df(v)| = |v|$ for any vector v of TM orthogonal to $\ker df$.

THEOREM C. *If $f\colon M \to B$ is a Riemannian submersion with totally geodesic fibres and M is complete, then $\varepsilon(N) = +\infty$ for any fibre N of f.*

Proof. For any smooth curve $\gamma\colon \langle 0; 1\rangle \to B$, $\gamma(0) = x$, $\gamma(1) = y$, let us define a mapping $F_\gamma\colon f^{-1}(x) \to f^{-1}(y)$ as follows. If $z \in f^{-1}(x)$, then there exists a curve $\gamma_z\colon \langle 0; 1\rangle \to M$ such that $\gamma_z(0) = z$, $f \circ \gamma_z = \gamma$ and $\dot{\gamma}_z(t) \perp \ker df\big(\gamma_z(t)\big)$ for any t of $\langle 0; 1\rangle$. γ_z is uniquely determined by these conditions and is called the *horizontal lift of* γ. Put

$$F_\gamma(z) = \gamma_z(1).$$

The mappings F_γ are diffeomorphisms and, according to [3], a necessary and sufficient condition for F_γ to be isometries is that the fibres of f be totally geodesic.

Let us take an arbitrary curve $c\colon \langle 0; 1\rangle \to M$ and define a new curve $C\colon \langle 0; 1\rangle \to M$ putting

$$C(t) = F_{\gamma_t}^{-1}\big(c(t)\big),$$

where $\gamma_t\colon \langle 0; 1\rangle \to B$ is given by $\gamma_t(s) = f\big(c(st)\big)$. It is evident that C lies on the fibre $f^{-1}\big(f(c(0))\big)$. The vector $dF_{\gamma_t}\big(\dot{C}(t)\big)$ is equal to the vertical component of $\dot{c}(t)$. Therefore,

$$|\dot{C}(t)| = \big|dF_{\gamma_t}\big(\dot{C}(t)\big)\big| \leqslant |\dot{c}(t)|, \qquad t \in \langle 0; 1\rangle,$$

and

$$L(C) \leqslant L(c).$$

The above argumentation shows that for any curve on M joining two points of a fibre N of a submersion f we are able to find a curve on N which joins the same points and is shorter than the given one. Therefore, if N is a fibre of f and $x, y \in N$, then

$$d_N(x, y) \leqslant d_M(x, y).$$

Our Theorem follows immediately from this inequality.

4. Assume that N is a submanifold of a Riemannian manifold M, the Ricci curvature Ric_M of M is bounded by a positive number k from below, and the diameter $d(N)$ of N is greater than $\pi\sqrt{m-1}/\sqrt{k}$, where $m = \dim M$. Let us take points x and y of N such that $d_N(x, y) > \pi\sqrt{m-1}/\sqrt{k}$. If $c\colon \langle 0; 1\rangle \to M$ is a minimal geodesic on M joining x to y, then according to the well-known Myers theorem [6], $L(c) \leqslant \pi\sqrt{m-1}/\sqrt{k}$. Consequently, $a = d_N(x, y) - d_M(x, y) > 0$ and

$$y \in B_M(x, d_N(x, y)-b) \cap N - B_N(x, d_N(x, y)-b)$$

for any b of $(0; a)$. In this manner, we established the following:

PROPOSITION D. *If N is a submanifold of a complete m-dimensional Riemannian manifold M and the Ricci curvature of M is bounded by a number $k > 0$ from below, then either $d(N) \leqslant \pi\sqrt{m-1}/\sqrt{k}$ or $\varepsilon(N) \leqslant \pi\sqrt{m-1}/\sqrt{k}$.*

Replacing in the above argumentation the classical Myers theorem by its generalization due to Galloway [1] we can generalize Proposition D as follows:

PROPOSITION D′. *Assume that M is a complete m-dimensional Riemannian manifold and that there exist constants $k > 0$ and $c \geqslant 0$, and a differentiable function $h\colon M \to R$ such that $|h| \leqslant c$ and*

$$\mathrm{Ric}_M(v, v) \geqslant k + v(h)$$

for any unit vector v of TM. Then the inequality

$$\min(d(N), \varepsilon(N)) \leqslant \frac{\pi}{k}\left(c + \sqrt{c^2 + k(m-1)}\right)$$

holds for any submanifold N of M.

Proposition D′ and Theorem C imply the following:

COROLLARY. *Under the hypotheses of Proposition D′, any fibre N of a Riemannian submersion $f\colon M \to B$ with totally geodesic fibres satisfies the inequality*

$$d(N) \leqslant \frac{\pi}{k}\left(c + \sqrt{c^2 + k(m-1)}\right). \tag{12}$$

Examples. In the case of the Riemannian submersion $f\colon P^{2n+1}C \to P^nQ$ estimation (12) (with $c = 0$ and $k = \frac{1}{2}(n+1)$) is not informative: The fibres of f are isometric to S^2 and have diameter equal to π while the right-hand side of (12) equals $\pi\sqrt{(4n-2)/(n+1)}$ and tends to 2π as $n \to \infty$. The situation is different in the case of the orthogonal group $O(n)$ equipped with the standard biinvariant Riemannian metric. If $n < m$, then $O(n)$ is a closed subgroup of $O(m)$ and the projection $O(m) \to O(m)/O(n)$ is a Riemannian submersion with totally geodesic fibres isometric to $O(n)$. The Ricci curvature of $O(m)$ is constant and equals $\frac{1}{2}(m-1)(m-2)$. From (12), it follows that

$$d(O(n)) \leqslant \pi\sqrt{m/(m-2)}$$

for any $m > n$. Passing with m to the infinity, we get

$$d(O(n)) \leqslant \pi.$$

On the other hand, $d(O(n)) \geqslant d(O(2)) = \pi$. It follows that $d(O(n)) = \pi$ for any $n \geqslant 2$.

5. Let F be a foliation of a Riemannian manifold M. If all the leaves of F are compact minimal submanifolds of M, then F is stable, i.e. the quotient M/F is Hausdorff [8]. If X is an arbitrary subset of M saturated by compact minimal leaves, then the quotient X/F need not be Hausdorff even if all the leaves of F are totally geodesic. For example, if $M = R \times S^1 \times S^1$ (endowed with the standard Riemannian metric), F is the 1-dimensional foliation of M defined by the vector field

$$Z = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

where (t, x, y) are standard coordinates on M, and $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \times S^1 \times S^1$, then X is saturated by closed geodesics but X/F is not Hausdorff. In [9], we proved the following:

Proposition E. *If $X \subset M$ is a set saturated by compact totally geodesic leaves of a foliation F of a Riemannian manifold M and the function*

$$X \ni x \mapsto \varepsilon(L_x), \tag{13}$$

where L_x denotes the leaf of F passing through x, is locally bounded by positive numbers from below (i.e., for any x of X there exist a neighbourhood U of x and a number $a > 0$ such that $\varepsilon(L_y) \geqslant a$ for any y of $U \cap X$), then X/F is Hausdorff.

The converse is not true. For example, if F is the standard foliation of the Möbius strip M by closed geodesics (Figure 1), then M/F is Hausdorff but the function (13) is not bounded from below by any positive number in any neighbourhood of the "central leaf" L_0. In fact, if $L \in F$ and $d(L, L_0) = a > 0$ is sufficiently small, then $\varepsilon(L) = 2a$.

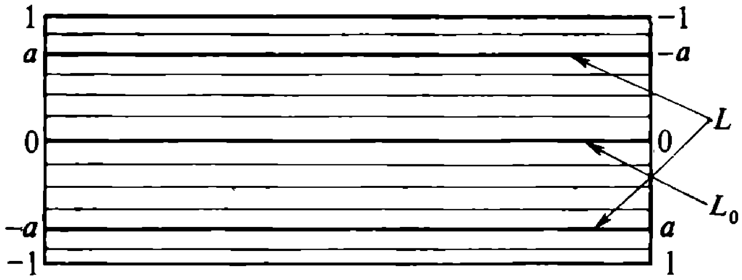

Fig. 1

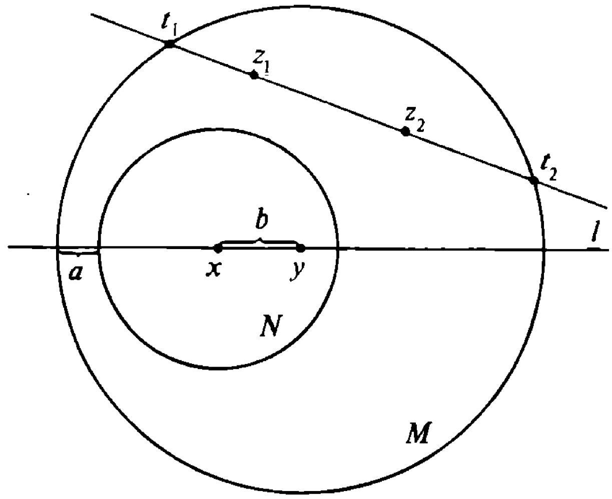

Fig. 2

6. A manifold M with a projective structure can be equipped with a projective invariant pseudometric δ_M (see [4], [5], [10]). If δ_M is a metric, then M is said to be hyperbolic. Let N be a submanifold of a hyperbolic manifold M. If N carries the projective structure induced from M, then N is hyperbolic and

$$\delta_N(x, y) \geqslant \delta_M(x, y)$$

for all x and y of N. It follows that the balls $B_N(x, r) = \{y \in N;\ \delta_N(x, y) < r\}$ and $B_M(x, r) = \{y \in M;\ \delta_M(x, y) < r\}$ satisfy condition (2) for any x of N and $r > 0$. One can expect that equality (1) holds for sufficiently small r in this case. The following example shows that this is not true even if M and N are domains on the plane.

Example. Let us consider the situation described in Figure 2. M is a convex bounded domain (a disc of the radius 2) on the plane. Therefore, M is hyperbolic and the metric δ_M is given by

$$\delta_M(z_1, z_2) = \left|\log \frac{(z_1 - t_2)(z_2 - t_1)}{(z_1 - t_1)(z_2 - t_2)}\right|,$$

where t_1 and t_2 are the points of intersection of the boundary of M with the line passing through z_1 and z_2. Exactly the same can be said about the domain N (a disc of the radius 1). Therefore,

$$\delta_M(x, y) = \log\frac{(3-a)(1+a+b)}{(1+a)(3-a-b)}$$

and

$$\delta_N(x, y) = \log\frac{1+b}{1-b}.$$

It is easy to see that for any $r > 0$ there exists a point y of $l \cap N$ such that

$$\delta_N(x, y) > r > \delta_M(x, y).$$

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