ANNALES
POLONICI MATHEMATICI
XLIII (1983)

On invariant measures for piecewise C^2-transformations of the n-dimensional cube

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Abstract. It is shown that a class of piecewise C^2-transformations of $[0, 1]^n$ have an absolutely continuous invariant measure.

1. Introduction. The purpose of this note is to show the existence of an absolutely continuous invariant measure for a transformation $\tau\colon [0, 1]^n \to [0, 1]^n$. Our theorem is a generalization of results of A. Rényi [9], A. O. Gel'fond [3], A. Lasota and J. A. Yorke [7] and W. Parry [8] to the n-dimensional case. In our paper the transformation is not of the form as in [9] or [12]. It is only piecewise continuous on the n-dimensional torus. The proof of our theorem is similar to the proof of the theorem given by Lasota and Yorke [7], but it is not a direct modification. For the proof of our theorem we must define the variation of a function of n variables, establish a lemma corresponding to Helly's theorem and prove an inequality concerning the behaviour of variation in a sequence of functions.

In Section 2 we recall certain basic definitions and state the main theorem. In Section 3 we prove necessary lemmas and in Section 4 we prove the theorem.

2. Existence theorem. Let $I^n = [0, 1]^n$. Denote by $L^1(I^n)$ the space of all integrable functions on I^n. The n-dimensional Lebesgue measure on I^n will be denoted by m_n, and we write $m_n(dx) = dx = dx_1 \ldots dx_n$.

We say that a measurable transformation $\tau\colon I^n \to I^n$ is *nonsingular* if $m_n(\tau^{-1}(A)) = 0$ whenever $m_n(A) = 0$.

For nonsingular $\tau\colon I^n \to I^n$ we define the *Frobenius–Perron operator* $P_\tau\colon L^1 \to L^1$ by the formula

$$\int_A P_\tau f\,dx = \int_{\tau^{-1}(A)} f\,dx,$$

which is valid for every measurable set $A \subset I^n$.

It is well known that the operator P_τ is linear and satisfies the following conditions:

(a) P_τ is positive: $f \geqslant 0 \Rightarrow P_\tau f \geqslant 0$;

(b) P_τ preserves integrals:

$$\int_{I^n} P_\tau f dx = \int_{I^n} f dx, \quad f \in L^1;$$

(c) $P_{\tau^k} = P_\tau^k$ (τ^k denotes the nth iterate of τ);

(d) $P_\tau f = f$ if and only if the measure $d\mu = f dx$ is invariant under τ, i.e. $\mu(\tau^{-1}(A)) = \mu(A)$ for any measurable A.

We shall not distinguish between functions $f\colon I^n \to R$ defined on I^n and functions $f\colon I^n \to R$ viewed as elements of the space L^1. Which is naturally the case, will become clear from the context.

Denote by $\prod_{i=1}^{n} A_i$ the Cartesian product of the sets A_i and denote by P_i the projection of R^n onto R^{n-1} given by

$$P_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Let $g\colon A \to R$ be a function of the n-dimensional interval $A = \prod_{i=1}^{n} [a_i, b_i]$ into R. Fixing i, we define a function $\bigvee_i^A g$ of the $n-1$ variables $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ by the formula

$$\bigvee_i^A g = \bigvee_i g = \sup \Big\{ \sum_{k=1}^{r} |g(x_1, \ldots, x_i^k, \ldots, x_n) - g(x_1, \ldots, x_i^{k-1}, \ldots, x_n)| :$$

$$a_i \leqslant x_i^0 < \ldots < x_i^r \leqslant b_i,\ r \in N \Big\}.$$

For $f\colon A \to R$ where $A = \prod_{i=1}^{n} [a_i, b_i]$ we define the variation $\overset{A}{\mathbf{V}} f$ as

$$\overset{A}{\mathbf{V}} f = \mathbf{V} f = \sup_i \overset{A}{\underset{i}{\mathbf{V}}} f$$

where

$$\overset{A}{\underset{i}{\mathbf{V}}} f = \overset{A}{\underset{i}{\mathbf{V}}} f$$

$$= \inf \Big\{ \int_{P_i(A)} \bigvee_i g \, dm_{n-1} : g = f \text{ almost everywhere, } \bigvee_i g \text{ measurable} \Big\}.$$

THEOREM 1. *Consider a partition*

$$\bigcup_{j=1}^{p} D_j = I^n, \quad D_j \cap D_k = \emptyset \quad \text{for} \quad j \neq k,$$

of I^n into sets D_j of the form $D_j = \prod_{i=1}^{n} D_{ij}$, $j = 1, 2, \ldots p$, where

$$D_{ij} = [a_{ij}, b_{ij}) \text{ if } b_{ij} < 1 \quad \text{and} \quad D_{ij} = [a_{ij}, b_{ij}] \text{ if } b_{ij} = 1.$$

Let $\tau\colon I^n \to I^n$ be the transformation given by the formula

$$\tau(x_1, \ldots, x_n) = (\varphi_{1j}(x_1), \ldots, \varphi_{nj}(x_n)), \quad (x_1, \ldots, x_n) \in D_j,$$

where $\varphi_{ij}\colon [a_{ij}, b_{ij}] \to [0, 1]$ are C^2-functions and

$$\inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \} > 1.$$

Then for any $f \in L^1$ the sequence

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$$

is convergent in norm to a function $f^* \in L^1$. The limit function has the following properties:

(1) $f \geqslant 0 \Rightarrow f^* \geqslant 0$.

(2) $\int_0^1 f^* dm = \int_0^1 f dm$.

(3) *$P_\tau f^* = f^*$ and consequently the measure $d\mu^* = f^* dm$ is invariant under τ.*

(4) *The function f^* is of bounded variation, moreover, there exists a constant c independent of the choice of initial f such that the variation of the limiting f^* satisfies the inequality*

$$\mathbf{V} f^* \leqslant c \, \|f\|.$$

3. Auxiliary lemmas. Now we state and prove some lemmas, which will be needed in the proof of Theorem 1.

The following lemma is proved in the standard way and therefore we omit the proof.

LEMMA 1. *If $f\colon A \to R$ is a function of the n-dimensional interval $A = \prod_{i=1}^{n} [a_i, b_i]$ into R and g is given by*

$$g = \int_{a_j}^{b_j} f dx_j,$$

then for $i \neq j$

$$\mathop{\mathbf{V}}_i g \leqslant \mathop{\mathbf{V}}_i f.$$

LEMMA 2. *Let S be a set of functions $f\colon I^n \to R$ such that*

(e) $f \geqslant 0$,

(f) $\mathbf{V} f \leqslant M$,

(g) $\|f\| \leqslant 1$.

Let f_i be such that

$$\int_{P_i(I^n)} \bigvee_i f_i \, dm_{n-1} \leqslant \mathop{\mathbf{V}}_i f + \varepsilon \qquad (\varepsilon > 0)$$

and $f_i = f$ almost everywhere. Then for $i = 1, 2, \ldots, n$

$$\lim_{k\to\infty} \{\sup_{f\in S} m_{n-1}(P_i(B_{f,k}))\} = 0 \tag{5}$$

where

$$B_{f,k} = \bigcup_{i=1}^{n} \{x \in I^n\colon f_i(x) \geqslant k\}. \tag{6}$$

Proof. Suppose that there are i and $a > 0$ such that for any k there exists $f^k \in S$ with

$$m_{n-1}(P_i(B_{f^k,k})) \geqslant a. \tag{7}$$

From (g) we have

$$\sup_{f\in S} m_n(B_{f,k}) \leqslant 1/k.$$

From this inequality it follows that there exists k_0 such that for $f \in S$

$$m_n(B_{f,k_0}) \leqslant a/2. \tag{8}$$

Write $H_t = [0, 1]^{i-1} \times \{t\} \times [0, 1]^{n-i}$, $t \in [0, 1]$. For $k > k_0$ it follows from (7) and (8) that there exists $t_k \in [0, 1]$ such that

$$m_{n-1}(H_{t_k} \cap B_{f^k,k_0}) \leqslant a/2.$$

Since $B_{f,p} \subset B_{f,q}$ for $p \geqslant q$, from the last inequality we obtain

$$\int_{P_i(I^n)} \bigvee_i f_i^k \, dm_{n-1} \geqslant (k-k_0)\, a/2 \qquad (k > k_0).$$

This contradiction ends the proof.

LEMMA 3. *If a set S of functions $f\colon I^n \to R$ satisfies the conditions of Lemma 2, then S is weakly relatively compact in L^1.*

Proof. We argue by induction on n the dimension of the cube. If $n = 1$, the lemma follows from Helly's theorem. Suppose the theorem is proved for $k < n$. Fix $\varepsilon > 0$ and define a set $\bar{S}$ by

$$\bar{S} = \{g\colon g = \int_0^1 f(x)\, dx_1, f \in S\}.$$

By the inductive hypothesis and Lemma 1 the set $\bar{S}$ is weakly relatively compact. Therefore, by the Pettis–Dunford theorem, there exists $\delta_1 > 0$ such that

(9) $$\int_F g\,dm_{n-1} < \varepsilon/2$$

whenever $m_{n-1}(F) < \delta_1$. Choose f_i as in Lemma 2 and choose k_1 (this is possible by Lemma 2) so large that

(10) $$m_{n-1}(P_1(B_{f,k_1})) < \delta_1 \quad \text{for} \quad f \in S.$$

Let $\delta = \varepsilon/2k_1$ and let $E \subset [0, 1]^n$ be such that $m_n(E) < \delta$. From (9) and (10) we have

$$\int_E f\,dm_n = \int_{E \cap (P_1(B_{f,k_1}) \times [0,1])} f\,dm_n + \int_{E \setminus (P_1(B_{f,k_1}) \times [0,1])} f\,dm_n$$
$$\leqslant \int_{P_1(B_{f,k_1})} \Big(\int_0^1 f\,dx_1\Big)\,dm_{n-1} + \frac{\varepsilon}{2k_1} k_1 \leqslant \varepsilon.$$

Therefore, by the Pettis–Dunford theorem, the set S is weakly relatively compact. Thus the inductive step is complete.

Let $f\colon \prod_{i=1}^{n} [a_i, b_i] \to R$ and let A be a subset of the interval $\prod_{i=1}^{n} [a_i, b_i]$. For this function and the set A we define a function $\bigvee_{i,A} f$ of the $n-1$ variables $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ by

$$\bigvee_{i,A} f = \sup\Big\{ \sum_{k=1}^{q} |f(x_1, \ldots, x_i^k, \ldots, x_n) - f(x_1, \ldots, x_i^{k-1}, \ldots, x_n)|:$$
$$a_i \leqslant x_i^0 < \ldots < x_i^q \leqslant b_i,\ (x_1, \ldots, x_i^k, \ldots, x_n) \notin A\Big\}.$$

The following lemma is easy to verify.

Lemma 4. *Let A be a subset of the interval $[a, b]$ and let a sequence of functions $f_n\colon [a, b] \to R$ converge to a function $f\colon [a, b] \to R$ pointwise on $[a, b]\setminus A$. Then*

$$\bigvee_{1,A} f \leqslant \liminf_{n\to\infty} \bigvee_{1,A} f_n \leqslant \liminf_{n\to\infty} \bigvee_{1} f_n$$

and there exists a function $\bar{f}\colon [a, b] \to R$ such that $\bar{f} = f$ almost everywhere on $[a, b]\setminus A$ and

$$\bigvee_1 \bar{f} = \liminf_{n\to\infty} \bigvee_1 f_n.$$

Lemma 5. *If a sequence of functions $f_n\colon [0, 1]^n \to R$ converges to a function $f\colon [0, 1]^n \to R$ in the norm of L^1, then*

(11) $$\mathbf{V} f \leqslant \limsup_{n\to\infty} \mathbf{V} f_n.$$

Proof. If $\limsup_{n\to\infty} \mathbf{V} f_n = \infty$, the theorem is obvious. Assume that $\limsup \mathbf{V} f_n < \infty$. Fix $\varepsilon > 0$ and let $\bar{f}_n$ be such that $\bar{f}_n = f_n$ almost everywhere and

$$\int_{P_i(I^n)} \bigvee_i \bar{f}_n \, dm_{n-1} \leqslant \mathbf{V}_i f_n + \varepsilon \quad (n = 1, 2, \ldots).$$

It is easy to verify that the sequence $\bar{f}_n$ is convergent in measure. Therefore, since $m_n([0, 1]^n) < \infty$, there exists a subsequence $\bar{f}_{n_j}$ which converges pointwise almost everywhere to a function $\bar{f} = f$ a.e. Denote by A the set of points for which the sequence $\bar{f}_{n_j}$ does not converge. From Lemma 4 it follows that there exists a function $\bar{\bar{f}}$ such that $\bar{\bar{f}} = f$ a.e. and

$$\bigvee_i \bar{\bar{f}} = \liminf_{n\to\infty} \bigvee_i \bar{f}_{n_j}.$$

From this, by Fatou's lemma we obtain

$$\int_{P_i(I^n)} \bigvee_i \bar{\bar{f}} \, dm_{n-1} \leqslant \liminf_{n\to\infty} \int_{P_i(I^n)} \bigvee_i \bar{f}_{n_j} \, dm_{n-1}$$

and consequently

$$\int_{P_i(I^n)} \bigvee_i \bar{\bar{f}} \, dm_{n-1} \leqslant \liminf_{n\to\infty} \mathbf{V} f_{n_j} + \varepsilon.$$

Since ε and i are arbitrary, this gives

$$\mathbf{V} f \leqslant \liminf \mathbf{V} f_{n_j} \leqslant \limsup \mathbf{V} f_{n_j},$$

which completes the proof.

Denote by $\mathscr{E}$ the set of functions of the form

$$g = \sum_{j=1}^{m} g_j \chi_{A_j}$$

where χ_{A_j} is the characteristic function of the set $A_j = \prod_{i=1}^{n} [\alpha_{ij}, \beta_{ij}] \subset [0, 1]^n$ (we do not assume that $\alpha_{ij} < \beta_{ij}$, the interval $[\alpha_{ij}, \beta_{ij}]$ can be degenerate) and $g_j \colon [0, 1]^n \to R$ is a C^1-function on A_j.

The following remarks are easy to verify.

Remark 1. *The set $\mathscr{E}$ is a dense subspace of the space L^1.*

Remark 2. *If $g \in \mathscr{E}$ then for any i and for any $A = [0,1]^{i-1} \times [0, x_i] \times \times [0, 1]^{n-i}$ $(x_i \in [0, 1])$ $\bigvee_i^A g$ is a measurable function of the variables $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.*

Remark 3. *Let $g \in \mathscr{E}$ and let $I^n = \bigcup_{j=1}^{r} B_j$, where $B_j = \prod_{i=1}^{n} [\gamma_{ij}, \sigma_{ij}]$ and $m_n(B_k \cap B_l) = 0$ for $k \neq l$. Then $\bigvee_i^{B_j} g$ are measurable functions and*

$$\int_{P_i(I^n)} \bigvee_i^{[0,1]^n} g\, dm_{n-1} = \sum_{j=1}^{r} \int_{P_i(B_j)} \bigvee_i^{B_j} g\, dm_{n-1}.$$

Remark 4. *For any $f \in \mathscr{E}$ there exists $\bar{f} \in \mathscr{E}$ such that*

$$\mathbf{V}_i f = \int_{P_i(I^n)} \bigvee_i \bar{f} dm_{n-1}.$$

Remark 5. *If $f \in \mathscr{E}$ then $\mathbf{V} f < \infty$.*

Let $f\colon \prod_{i=1}^{n} [a_i, b_i] \to R$ be a function of n variables and $g\colon [a_j, b_j] \to R$ be a function of one variable. Define a function $h(t; x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ of the variable t ($t \in [a_j, b_j]$) and parameters $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ by

$$h(t; x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n).$$

Denote by $\int_{a_i}^{b_i} g(x_j)\, d_j f(x_1, \ldots, x_n)$ the Riemann–Stieltjes integral of $g\colon [a_j, b_j] \to R$ with respect to the function $h(t; x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ of the variable t on $[a_j, b_j]$. This integral is a function of the variables $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

4. Proof of the existence theorem.

Proof of Theorem 1. Write $s = \inf_{i,j} \{ \inf_{[a_{ij}, b_{ij}]} |\varphi'_{ij}| \}$ and choose a number N such that $s^N > 2$. It is easy to see that the function $\Phi = \tau^n$ satisfies the assumptions of the theorem. Denote by $B_j = \prod_{i=1}^{n} [c_{ij}, d_{ij}]$ the corresponding partition for Φ.

Denoting by Φ_{ij} the corresponding C^2-functions we have

$$(12) \qquad |\Phi'_{ij}(x_i)| \geqslant s^N, \quad x_i \in [c_{ij}, d_{ij}], \quad i = 1, \ldots, n, \quad j = 1, \ldots, q.$$

Computing the Frobenius–Perron operator for Φ, we obtain

$$(13) \qquad P_\Phi f(x) = \sum_{j=1}^{q} f(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\, \sigma_{1j}(x_1) \ldots \sigma_{nj}(x_n)\, \chi_j(x)$$

where $\psi_{ij} = \Phi_{ij}^{-1}$, $\sigma_{ij}(x_i) = |\psi'_{ij}(x_i)|$ and χ_j is the characteristic function of the set $I_j = \prod_{i=1}^{n} \Phi_{ij}([c_{ij}, d_{ij}])$. From (1) it follows that

$$(14) \qquad \sigma_{ij}(x_i) \leqslant s^{-N}, \quad x_i \in \Phi_{ij}([c_{ij}, d_{ij}]), \quad i = 1, \ldots, n, \quad j = 1, \ldots, q.$$

By its very definition the operator P_Φ is a mapping from L^1 into L^1, but the last formula enables us to consider P_Φ as a map from the space of functions defined on I^n into itself.

Let $f\in\mathscr{E}$ be a nonnegative function and let a function $f_i\in\mathscr{E}$ be such that $f_i = f$ a.e. and $\int\limits_{P_i(I^n)} \bigvee\limits_i f_i\,dm_{n-1} = \mathbf{V}\limits_i f$. From (2) and (3) it follows that

$$\begin{aligned}(15)\quad \int\limits_{P_i(I^n)} \overset{I^n}{\bigvee_i} P_\Phi f_i\,dm_{n-1}
&\leqslant \sum_{j=1}^{q} \int\limits_{P_i(I_j)} \overset{I_j}{\bigvee_i} f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\,\sigma_{1j}(x_1)\ldots\sigma_{nj}(x_n)\,dm_{n-1}+\\
&\quad + s^{-N}\sum_{j=1}^{q}\int\limits_{P_i(I_j)} \big(|f_i(\psi_{1j}(x_1), \ldots, b_{ij}, \ldots, \psi_{nj}(x_n))|+\\
&\quad + |f_i(\psi_{1j}(x_1), \ldots, a_{ij}, \ldots, \psi_{nj}(x_n))|\big)\times\\
&\quad \times \sigma_{1j}(x_1)\ldots\sigma_{i-1,j}(x_{i-1})\sigma_{i+1,j}(x_{i+1})\ldots\sigma_{nj}(x_n)\,dm_{n-1}.\end{aligned}$$

In order to evaluate the first sum we write

$$\begin{aligned}&\int\limits_{P_i(I_j)} \overset{I_j}{\bigvee_i} f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\sigma_{1j}(x_1)\ldots\sigma_{nj}(x_n)\,dm_{n-1}\\
&= \int\limits_{P_i(I_j)} \Big(\int\limits_{c_{ij}}^{d_{ij}} \big|d_i\big(f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\sigma_{1j}(x_1)\ldots\sigma_{nj}(x_n)\big)\big|\Big)\,dm_{n-1}\\
&\leqslant \int\limits_{P_i(I_j)} \Big(\int\limits_{c_{ij}}^{d_{ij}} |f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))|\,\sigma_{1j}(x_1)\ldots|\sigma'_{ij}(x_i)|\ldots\sigma_{nj}(x_n)\,dx_i\Big)\,dm_{n-1}+\\
&\quad + \int\limits_{P_i(I_j)} \Big(\int\limits_{c_{ij}}^{d_{ij}} \sigma_{1j}(x_1)\ldots\sigma_{nj}(x_n)|d_i f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))|\Big)\,dm_{n-1}\\
&\leqslant K\int\limits_{I_j} f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\sigma_{1j}(x_1)\ldots\sigma_{nj}(x_n)\,dm_n+\\
&\quad + s^{-N}\int\limits_{P_i(I_j)} \overset{I_j}{\bigvee_i} f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\times\\
&\quad \times \sigma_{1j}(x_1)\ldots\sigma_{i-1,j}(x_{i-1})\sigma_{i+1,j}(x_{i+1})\ldots\sigma_{nj}(x_n)\,dm_{n-1}\end{aligned}$$

where $K = \dfrac{\max|\sigma'_{ij}|}{\min\sigma_{ij}}$. Changing the variables we obtain

$$\begin{aligned}(16)\quad \int\limits_{P_i(I_j)} \Big(\overset{I_j}{\bigvee_i} f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n))\sigma_{1j}(x_1)\ldots\sigma_{nj}(x_n)\Big)\,dm_{n-1}&\\
\leqslant K\int\limits_{I_j} f_i\,dm_n + s^{-N}\int\limits_{P_i(B_j)} \Big(\overset{B_j}{\bigvee_i} f_i\Big)\,dm_{n-1}&.\end{aligned}$$

In order to evaluate the second term in (15) we write

$$(17)\quad |f_i(\psi_{1j}(x_1), \ldots, b_{ij}, \ldots, \psi_{nj}(x_n))| + |f_i(\psi_{1j}(x_1), \ldots, a_{ij}, \ldots, \psi_{nj}(x_n))| \leqslant \bigvee_i^{I_j} f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n) + 2h_{ij}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))$$

where

$$h_{ij} = \inf\{f_i(\psi_{1j}(x_1), \ldots, \psi_{nj}(x_n)):\ x_i \in \Phi_{ij}([c_{ij}, d_{ij}])\}.$$

On the other hand, we have the obvious inequality

$$(18)\qquad h_{ij} \leqslant b^{-1} \int_{c_{ij}}^{d_{ij}} f_i(x_1, \ldots, x_n)\, dx_i$$

where $b = \min(|b_{ij} - a_{ij}|)$. From (17), (18) and Remark 3 it follows that

$$(19)\quad \sum_{j=1}^{q} \int_{P_i(I_j)} \big(|f_i(\psi_{1j}(x_1), \ldots, b_{ij}, \ldots, \psi_{nj}(x_n))| + |f_i(\psi_{1j}(x_1), \ldots, a_{ij}, \ldots, \psi_{nj}(x_n))|\big) \times \sigma_{1j}(x_1)\ldots \sigma_{i-1,j}(x_{i-1})\, \sigma_{i+1,j}(x_{i+1})\ldots \sigma_{nj}(x_n)\, dm_{n-1} \leqslant \int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, dm_{n-1} + 2b^{-1}\|f\|.$$

Applying (19) and (16) to (15) and applying Remark 3 once more, we obtain

$$\int_{P_i(I^n)} \bigvee_i^{I^n} P_\Phi f_i\, dm_{n-1} \leqslant \alpha\|f\| + \beta \int_{P_i(I^n)} \bigvee_i^{I^n} f_i\, dm_{n-1}$$

where $\alpha = (K + 2b^{-1})$ and $\beta = 2s^{-N} < 1$. Thus, for $f \in \mathscr{E}$ and $i = 1, 2, \ldots, n$, we obtain

$$\mathop{\mathbf{V}}_i^{I^n} P_\Phi f \leqslant \alpha\|f\| + \beta \mathop{\mathbf{V}}_i^{[0,1]^n} f$$

with $\alpha < \infty$ and $\beta < 1$.

Now, for the same function f, let us write $f_k = P_\tau^k f$. Since $P_\tau^N = P_\Phi$, we have

$$\mathop{\mathbf{V}}_i^{I^n} f_{Nk} \leqslant \alpha\|f_{N(k-1)}\| + \beta \mathop{\mathbf{V}}_i^{I^n} f_{N(k-1)} \leqslant \alpha\|f\| + \beta \mathop{\mathbf{V}}_i^{I^n} f_{N(k-1)}$$

and consequently

$$(20)\qquad \limsup_{k\to\infty} \mathop{\mathbf{V}}_i f_{Nk} \leqslant \alpha(1-\beta)^{-1}\|f\|.$$

From this inequality, the condition $\|f_k\| \leqslant \|f\|$ (which follows from (a) and (b)) and Lemma 3 it follows that the set $C = \{f_{Nk}\}_{k=0}^{\infty}$ is relatively weakly compact in L^1. Since $\{f_k\}_{k=0}^{\infty} \subset \bigcup_{k=0}^{N-1} P_\tau^k C$, the whole sequence $\{f_k\}_{k=0}^{\infty}$ is relatively compact. By Mazur's theorem the same is true for the sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f\right\}. \tag{21}$$

The subset of nonnegative functions of the space $\mathscr{E}$ is linearly dense in L^1. We have proved that for any such function f the sequence (21) is relatively weakly compact. Therefore, we are in a position to use the Kakutani–Yosida theorem (see [1], VIII.5.3) which says that for any $f \in L^1$ the sequence (21) converges strongly to a function f^*, which is invariant under P_τ. From (a) and (b) it follows that f^* satisfies (1) and (2). Therefore it remains only to prove (4).

Since the operator P_τ is given by a formula analogous to (13), it is easy to derive the inequality

$$\overset{[0,1]^n}{\underset{i}{\mathbf{V}}} P_\tau f \leqslant c_1 \overset{[0,1]^n}{\underset{i}{\mathbf{V}}} f + c_2\|f\|,$$

with some constants c_1 and c_2. Thus, the definition of variation and relation (20) imply the inequality

$$\limsup_{k\to\infty} \overset{I^n}{\mathbf{V}} P_\tau^k f \leqslant c\|f\|$$

(with a positive constant c), which is valid for any $f \in \mathscr{E}$. Consequently, for any such f we have also

$$\limsup_{k\to\infty} \mathbf{V}\left(\frac{1}{n}\sum_{k=1}^{n-1} P_\tau^k f\right) \leqslant c\|f\|.$$

Writing $Q = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} P_\tau^k$ and using Lemma 5 we have $\mathbf{V}\,Qf \leqslant c\|f\|$, for $f \in \mathscr{E}$. The operator Q is linear and contractive. We may therefore apply Lemma 5 once more to extend this inequality to the closure of the set $\mathscr{E}$, that is, to all of L^1. This finishes the proof.

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Reçu par la Rédaction le 20.09.1979