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Further results on the univalent functions with the monotonic modulus property

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Abstract. We give some analytic and geometric characterizations of univalent functions with the monotonic modulus property. We show that their logarithms are convex in the direction of the imaginary axis.

1. Introduction. Let S be the set of all functions f that are analytic and univalent in $D = \{z \in \mathbb{C}: |z| < 1\}$, and have the normalization $f(0) = 0$ and $f'(0) = 1$. Given $\beta \in [0, 2\pi)$ and $\alpha \in [\beta, \beta + 2\pi)$, we denote by $S(\alpha, \beta)$ the set of all functions f in S such that the modulus $|f(e^{i\theta})|$ is nonincreasing in (β, α), and nondecreasing in $(\alpha, \beta + 2\pi)$. The classes $S(\alpha, \beta)$ were first introduced in [1]. In this paper, we show that functions f in $S(\alpha, \beta)$ are closely related to close to convex functions that are convex in the direction of the imaginary axis. Particularly, we prove that $\log(f(z)/z)$ is convex in the direction of the imaginary axis for each $f \in S(\alpha, \beta)$. This result, in turn, implies that, among other functions, $\log(f(z)/z)$ and $f(z)/z$ are univalent for each $f \in S(\alpha, \beta)$. Moreover, we show that $\log(f(z)/z)$ and $f(z)/z$ can be embedded in explicit Löwner chains.

2. Analytic and geometric characterizations of $S(\alpha, \beta)$. Since

$$\lambda(z) = -i(e^{i\alpha/2} - e^{-i\alpha/2} z)/(e^{i\beta/2} - e^{-i\beta/2} z)$$

maps the unit disc D onto the right half plane, $w = \log \lambda(z)$ maps the unit disc D onto the strip $-\pi/2 < \operatorname{Im} w < \pi/2$. Define

$$J_\tau = \{z \in D: \operatorname{Arg}\lambda(z) = \tau\} \quad \text{and} \quad I_x = \{z \in D: f \in S(\alpha, \beta),\ |f(z)/z| = e^x\}.$$

Let l_x be the length of the image of I_x under the function $\log(f(z)/z)$.

THEOREM 1. *The following are equivalent*:

(i) $f \in S(\alpha, \beta)$.

(ii) f *is univalent and* $\log(f(z)/z)$ *is convex in the direction of the imaginary axis.*

(iii) *f is univalent and the function*

$$H(z,\zeta)=\frac{\log(f(z)/z)-\log(f(\zeta)/\zeta)}{\log\lambda(z)-\log\lambda(\zeta)} \tag{2.1}$$

has nonnegative real part in $D\times D$.

(iv) $\log|f(z)/z|$ and $\operatorname{Arg}\lambda(z)$ *are monotonic functions on* J_τ *and* I_x *respectively, and*

$$0\leqslant \operatorname{Arg} f(e^{i\varrho})-\operatorname{Arg} f(e^{i\theta})\leqslant \varrho-\theta-l_x<\varrho-\theta, \tag{2.2}$$

for every $\theta\in(\beta,\alpha)$ *and* $\varrho\in(\alpha,\beta+2\pi)$ *satisfying* $|f(e^{i\theta})|=|f(e^{i\varrho})|=e^x$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $f\in S(\alpha,\beta)$. It follows from Lemma 1 in [1] that $\phi(z)$ defined by

$$\phi(z)=\frac{d\log(f(z)/z)}{dz}\bigg/\frac{d\log\lambda(z)}{dz} \tag{2.3}$$

has nonnegative real part in D. Hence, $\log(f(z)/z)$ is close to convex. If we define $z\colon D\to D$ by $\lambda(z(w))=(1+w)/(1-w)$, then

$$\phi(z(w))=\frac{1-w^2}{2}\frac{d}{dw}\log\frac{f(z(w))}{z(w)}.$$

This shows that $\log(f(z)/z)$ is convex in the direction of the imaginary axis (see, for example, Hengartner and Schober [3]).

(ii) $\Rightarrow$ (iii). Define $F(w)$ on $\{w\colon -\pi/2<\operatorname{Im} w<\pi/2\}$ by $\log(f(z)/z)=F(\log\lambda(z))$. Then $F'(w)=\phi(z)$ has nonnegative real part by (ii). Since

$$H(z,\zeta)=\int_0^1 F'\{(1-t)\log\lambda(\zeta)+t\log\lambda(z)\}\,dt,$$

(iii) follows.

(iii) $\Rightarrow$ (iv). Let $z,\zeta\in J_\tau$ be such that $|\lambda(z)|>|\lambda(\zeta)|$. Since $\operatorname{Re} H(z,\zeta)\geqslant 0$, we obtain $\log|f(z)/z|\geqslant\log|f(\zeta)/\zeta|$. This shows that $\log|f(z)/z|$ is monotonic on J_τ. A similar argument establishes the monotonicity of $\operatorname{Arg}\lambda(z)$ on I_x. To prove (2.2), we first observe that $l_x>0$ and

$$l_x\leqslant\varrho-\theta-\{\operatorname{Arg} f(e^{i\varrho})-\operatorname{Arg} f(e^{i\theta})\}$$

since $\log(f(z)/z)$ is univalent for each $f\not\equiv z$. On the other hand $\{\operatorname{Arg} f(e^{i\varrho})-\operatorname{Arg} f(e^{i\theta})\}$ is nonnegative and is less than 2π by the univalency of f. This shows that (iii) implies (iv).

(iv) $\Rightarrow$ (i). Since $\varrho-\theta\leqslant 2\pi$, the inequalities (2.2) imply that f is univalent. The monotonicity of $\log|f(z)/z|$ on J_τ shows that f has the monotonic modulus property. Therefore $f\in(\alpha,\beta)$. ∎

COROLLARY 1. $\log(f(z)/z)$ *and* $f(z)/z$ *are univalent for each* $f\in S(\alpha,\beta)$ *which is different from the identity function.*

Proof. $\log(f(z)/z)$ is univalent because $\operatorname{Re}H(z,\zeta)\geqslant 0$ on $D\times D$. The inequalities (2.2) imply that $0<l_x<2\pi$. Therefore e^w is univalent over the image of $\log(f(z)/z)$. Hence, $f(z)/z$ is univalent. ∎

Hengartner and Schober [2] were the first to show the univalency on D of $\log(f(z)/z)$ and $f(z)/z$ for the extreme points of S. The corollary above generalizes this result considerably (see also Kirwan and Pell [5]).

Part (iv) of Theorem 1 is remarkable in that the monotonicity of $\log|f(z)/z|$ on $J_{\pi/2}$ and $J_{-\pi/2}$ implies the monotonicity of $\log|f(z)/z|$ on J_τ for every $\tau\in(-\pi/2,\pi/2)$. From this we obtain the following:

COROLLARY 2. *Define* $w_{\tau_1,\tau_2}\colon D\to D$ *by*

$$\lambda\{w_{\tau_1,\tau_2}(z)\}=e^{i(\tau_1+\tau_2)/2}\lambda(z)^{(\tau_1-\tau_2)/\pi}$$

where $\lambda(z)$ *as in Theorem* 1. *Then for all* τ_1,τ_2 *satisfying* $-\pi/2<\tau_2<\tau_1<\pi/2$ *the function*

$$g_{\tau_1,\tau_2}(z)=zf(w_{\tau_1,\tau_2}(z))w_{\tau_1,\tau_2}(0)/w_{\tau_1,\tau_2}(z)f(w_{\tau_1,\tau_2}(0))$$

belongs to $S(\alpha,\beta)$ *whenever it is univalent.*

3. Löwner chains. Let p, q, t be positive constants, let r be any real constant and let λ and ϕ be defined as in Section 2. For a given $f\in S(\alpha,\beta)$, we define

$$\begin{aligned}F(z,t)&=p\log(f(z)/z)+(q+ir)\log(\lambda(z)/\lambda(0))+tz\lambda'(z)/\lambda(z),\\ H(z,t)&=\exp\{F(z,t)\}\quad\text{and}\quad G(z,t)=zH(z,t).\end{aligned}\tag{3.1}$$

Then $\psi(z)$ defined by

$$\psi(z)=F'(z,t)\Big/\frac{d\log\lambda(z)}{dz}=p\phi(z)+q+ir+t\left\{1+z\frac{(\log\lambda)''}{(\log\lambda)'}\right\}$$

has nonnegative real part on D since $\log\lambda$ is convex, and ϕ has nonnegative real part on D. This shows that $F(z,t)$ is close to convex (actually convex in the direction of the imaginary axis) in D. Note also that $\psi(z)=zF'/\dot F$ where $\dot F=dF/dt$. Thus, $F(z,t)$ is a Löwner chain except for the easily adjusted normalization (see, for example, Pommerenke [6], Th. 6.2). From these and similar considerations we obtain:

THEOREM 2. *Let* $f\in S(\alpha,\beta)$ *and define* $F(z,t)$, $H(z,t)$ *and* $G(z,t)$ *as above. Then*

(i) $F(z,t)$ *and* $H(z,t)$ *are Löwner chains except the usual normalizations.*

(ii) $F(z,t)$ *and* $H(z,t)$ *are univalent in* D *for each set of parameters* p, q, r *and* t.

(iii) $G(z,t)\in S(\alpha,\beta)$ *whenever it is univalent.*

We remark that Corollary 1 to Theorem 1 also follows from this theorem. We also note that $G(z,t)$ converges to $f(z)$ as $p\to 1$ and $q, r, t\to 0$. Therefore,

$G(z, t)$ can be compared with $f(z)$. This suggests the possibility of using variational methods in some extremal problems over $S(\alpha, \beta)$. A similar remark applies to the function $g_{\tau_1,\tau_2}(z)$ defined in Corollary 2 of Theorem 1.

References

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