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On a two-point boundary value problem for differential equations on the half-line

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Abstract. In the paper the existence of solutions of the second-order ordinary differential equations with a given value at 0 and vanishing at infinity is established by means of the degree theory of DC-mappings. The equations do not involve the first derivative.

Introduction. We deal with the *boundary value problem* (BVP): $x'' = f(x, t)$, $x(0) = \alpha$, $\lim_{t\to\infty} x(t) = 0$, where f is continuous and satisfies some other conditions. It is known that, for f strictly increasing and Lipschitz continuous on bounded sets, the problem has a unique solution (see [4]). But the existence of a solution needs weaker assumptions. In order to obtain the existence theorems, one should get a priori estimates for solutions, replace the BVP by an integral equation and apply the Leray–Schauder degree theory (or the topological transversality method due to Granas [2]). The last step is possible if the integral operator is compact. In the case of unbounded domain (the half-line), the operator is usually noncompact, so one needs a degree theory for a larger class of mappings. In the paper we use the theory of DC-mappings and their degree (see [6], [7]) since the Leray–Schauder theory can not be applied. For this purpose we have to get a priori bounds for approximate solutions, and the existence of such solutions is obtained after applying the homotopy arguments for the degree. An exact solution is found by a kind of compactness.

In the last section, a singular BVP of Emden–Fowler type: $\varphi(t)x'' = f(x, t)$, $x(0) = \alpha$, $\lim_{t\to\infty} x(t) = 0$, is considered.

The method we use in the paper is similar to and partially based on [3], where only compact problems are considered.

1. General setting. Let $f\colon \mathbf{R} \times \langle 0, \infty) \to \mathbf{R}$ be continuous and

$$x\big(f(x, t) - \mu^2 x\big) > 0 \quad \text{for } |x| > Ke^{-\mu t}. \tag{1}$$

Assume that

$$|f(x, t)| \leqslant C_1 |x|, \tag{2}$$

$$|f(x_1, t) - f(x_2, t)| \leqslant C_2 |x_1 - x_2| \tag{3}$$

for $x, x_1, x_2 \in \langle -\delta, \delta \rangle$, $t \geqslant t_0$. The constants $\mu, K, C_1, C_2, \delta, t_0$ are positive and fixed.

THEOREM 1. *Under these assumptions, the equation*

$$x'' = f(x, t) \tag{4}$$

has a solution satisfying the two-point boundary condition:

$$x(0) = \alpha \in \mathbf{R}, \quad \lim_{t \to \infty} x(t) = 0. \tag{5}$$

This solution tends to 0 *at infinity not slowlier than* $Me^{-\mu t}$ *for some positive* M.

The proof is contained in Sections 1–4.

Let us consider the family of BVP:

$$x'' = \lambda f(x, t) + (1-\lambda)\mu^2 x, \quad x(0) = \alpha, \ \lim_{t \to \infty} x(t) = 0, \tag{6}$$

where $\lambda \in \langle 0, 1 \rangle$. It is easy to see that the right-hand side satisfies (1)–(3). The first condition is not fulfilled for $\lambda = 0$ but we shall use it only for finding an a priori bound. When $\lambda = 0$, the solution is $\alpha e^{-\mu t}$ and it is bounded by a constant we shall obtain below.

Finding Green's function for the linear differential operator $Lx = x'' - \mu^2 x$, we can replace (6) by

$$x(t) = \alpha e^{-\mu t} - \frac{\lambda}{2\mu} \int_0^\infty (e^{-\mu|s-t|} - e^{-\mu(s+t)}) \big(f(x(s), s) - \mu^2 x(s)\big) ds. \tag{7}$$

Take $\nu \in (0, \mu)$ and denote by X the space of functions of the class C^2 on $\langle 0, \infty)$ such that

$$\lim_{t \to \infty} e^{\nu t} |x(t)| = 0, \quad \lim_{t \to \infty} e^{\nu t} |x''(t)| = 0. \tag{8}$$

Let

$$\|x\|_\nu = \sup_t e^{\nu t} |x(t)|,$$

and let

$$\|x\| = \max(\|x\|_\nu, \|x''\|_\nu)$$

be a norm in X. One can show that X is complete and separable.

Denote by A the integral operator in (7). Using (2), we see that if x satisfies the first part of (8), then, for sufficiently large t,

$$e^{\nu t}|Ax(t)| \leqslant (\mu^2+C_1)\Big[e^{(\nu-\mu)t}\int_0^t (e^{\mu s}-e^{-\mu s})|x(s)|\,ds+ \\ +(e^{(\nu+\mu)t}-e^{(\nu-\mu)t})\int_t^{\infty} e^{-\mu s}|x(s)|\,ds\Big].$$

By the l'Hospital theorem, both summands on the right-hand side tend to 0. Moreover, Ax is twice differentiable and

$$(Ax)''(t) = \mu^2(Ax)(t)-2\mu\big(f(x(t),\,t)-\mu^2 x(t)\big), \tag{9}$$

so Ax satisfies the second part of (8). Hence we can consider equation (7) within X.

LEMMA 1. *The operator $A\colon X\to X$ is continuous.*

Proof. It is sufficient to show the continuity of the Niemytzki operator $x\mapsto f(x(\cdot),\cdot)$ with respect to the norm $\|\cdot\|_\nu$. Let $x\in X$ and $t_1\geqslant t_0$ be such that $|x(t)|\leqslant \frac{1}{2}\delta$ for $t\geqslant t_1$. Obviously x is bounded, $|x(t)|\leqslant M_1$.

Take $\varepsilon>0$. There exists $\eta>0$ such that, for $x_1, x_2\in\langle -2M_1, 2M_1\rangle$, $|x_1-x_2|<\eta$ and $t\leqslant t_1$,

$$|f(x_1,t)-f(x_2,t)|\leqslant \varepsilon e^{-\nu t_1}.$$

Then, for any $y\in X$ such that $\|y-x\|_\nu\leqslant \min(\frac{1}{2}\delta,\,\eta,\,\varepsilon C_2^{-1})$,

$$e^{\nu t}\big|f(y(t),t)-f(x(t),t)\big|\leqslant\varepsilon.$$

The continuity of the linear integral operator is a consequence of standard calculations. Applying (9), we get the assertion. □

2. A priori bounds. The a priori bounds technique, introduced by Bernstein [1] and developed by many authors (see [3]), becomes complicated if we deal with approximate solutions. We shall see that this leads to a priori estimates for solutions of differential inclusions. The technical difficulties cause we restrict ourselves to equations not involving the first derivative.

Let us first notice that (1) implies

$$\begin{aligned} f(x,t) &\geqslant \nu^2 x+e^{-\nu t} \quad \text{for } x\geqslant \max\big(K,(\mu^2-\nu^2)^{-1}\big)e^{-\nu t},\\ f(x,t) &\leqslant \nu^2 x-e^{-\nu t} \quad \text{for } x\leqslant -\max\big(K,(\mu^2-\nu^2)^{-1}\big)e^{-\nu t}. \end{aligned} \tag{10}$$

Let $x_0(t)=\alpha e^{-\mu t}$. Suppose that

$$\Big\|x-x_0+\frac{\lambda}{2\mu}Ax\Big\|<\varepsilon \tag{11}$$

where $\varepsilon < (1+\mu^2)^{-1}$. Then

$$\left\|x - x_0 + \frac{\lambda}{2\mu} Ax\right\|_\nu < \varepsilon$$

and

$$\left\|x'' - \lambda f(x(\cdot),\cdot) - (1-\lambda)\mu^2 x + \mu^2\left(x - x_0 + \frac{\lambda}{2\mu} Ax\right)\right\|_\nu < \varepsilon.$$

Hence

$$\|x'' - g(x(\cdot),\cdot)\|_\nu < \varepsilon_1 < 1 \tag{12}$$

where $g(x, t) = \lambda f(x, t) + (1-\lambda)\mu^2 x$ and $\varepsilon_1 = (1+\mu^2)\varepsilon$. Obviously g satisfies (10). By (11), $|x(0)-\alpha| < \varepsilon$. An a priori bound for an ε-solution is given by

LEMMA 2. *If $x \in X$ satisfies* (12) *and* $|x(0)-\alpha| < \varepsilon$, *then*

$$\|x\|_\nu \leqslant \max\left(K, (\mu^2 - \nu^2)^{-1}, |\alpha| + \varepsilon\right) =: K_1.$$

Proof. Suppose on the contrary that $\|x\|_\nu > K_1$, and define $y(t) = e^{\nu t} x(t)$. Then y has a maximum $y(t_1) > K_1$ or a minimum $y(t_2) < -K_1$. We shall consider the first possibility.

Since $y'(t_1) = 0$ and $y''(t_1) \leqslant 0$, then, by (12) and (10),

$$\begin{aligned}0 \geqslant y''(t_1) &= \left(x''(t_1) - \nu^2 x(t_1)\right)e^{\nu t_1}\\ &\geqslant \left[g\left(x(t_1), t_1\right) - \varepsilon_1 e^{-\nu t_1} - \nu^2 x(t_1)\right]e^{\nu t_1} \geqslant 1 - \varepsilon_1 > 0.\end{aligned}$$

The case of the minimum leads to a contradiction in the same way. □

Having an a priori bound on x, we get, by (12), an estimate on $\|x''\|_\nu$. Therefore (11) implies $\|x\| \leqslant M$ for a certain M.

3. Information on DC-mappings. In this section, X stands for an arbitrary normed space. All results are proved in [6]. We shall quote only those which are used in the sequel.

Let $(X_n)_{n \in N}$ be a sequence of finite-dimensional linear subspaces of X such that $X_n \subset X_{n+1}$, $n \in N$, and $\operatorname{cl} \bigcup_{n \in N} X_n = X$. Such a sequence is called a *filtration* in X. Let V be an open bounded subset of X. Then $\operatorname{cl} \bigcup_{n \in N} (\operatorname{cl} V \cap X_n) = \operatorname{cl} V$.

We shall say that $A\colon \operatorname{cl} V \to X$ is a DC-mapping (with respect to the filtration (X_n)) if A is continuous and

$$\lim_{n\to\infty} \sup_{x \in \operatorname{cl} V \cap X_n} d(Ax, X_n) = 0$$

where $d(\cdot,\cdot)$ stands for the distance between a point and a set.

The class of DC-mappings contains the identity and the family of compact operators and it is a module over the ring of continuous bounded scalar functions. The last property makes the theory of DC-mappings better for applications. There are advantages but there are disadvantages, as well. DC-mappings are often not closed, so the theorems on solvability of equations involving such operators are of the form $\inf_x \|Ax-y\| = 0$.

APPROXIMATION LEMMA. *For any DC-mapping A: $\operatorname{cl} V \to X$ and $\varepsilon > 0$, there exists a continuous mapping A_ε: $\operatorname{cl} V \to X$ such that*

$$\|A_\varepsilon x - Ax\| < \varepsilon, \qquad x \in X,$$

and $A_\varepsilon(\operatorname{cl} V \cap X_n) \subset X_n$ for sufficiently large n.

The notion of the degree is based on this lemma. Let

$$\mathscr{G} = \prod_{n\in N} Z / \bigoplus_{n\in N} Z,$$

let A: $\operatorname{cl} V \to X$ be a DC-mapping and $y \in X \setminus \operatorname{cl} A(\partial V)$, where ∂V denotes the boundary of V. Take a mapping A_ε from the lemma for $\varepsilon = \frac{1}{2} d(y, \operatorname{cl} A(\partial V))$ and a point $y_\varepsilon \in X_{n_0}$ such that $\|y_\varepsilon - y\| < \frac{1}{2}\varepsilon$. We can assume that the second property of A from the lemma holds for $n \geqslant n_0$. Then $y_\varepsilon \in X_n \setminus A_\varepsilon(\partial(V \cap X_n))$ (the boundary in the topology of X_n) for $n \geqslant n_0$. We can define a sequence of Brouwer degrees

$$s_n = \deg(A_\varepsilon | \operatorname{cl} V \cap X_n, V \cap X_n, y_\varepsilon)$$

for such n. Putting $s_n = 0$ for $n < n_0$ and taking the equivalence class $[(s_n)_{n\in N}]$ in the quotient group $\mathscr{G}$, we get the definition of the degree of DC-mapping A on V at y:

$$\operatorname{Deg}(A, V, y) = [(s_n)_{n\in N}].$$

The definition is independent of the choice of y_ε and A_ε and the degree has many standard properties. We shall use:

(i) If H: $\langle 0, 1\rangle \times \operatorname{cl} V \to X$ is a DC-mapping (the filtration in $\langle 0, 1\rangle \times X$ is $(\langle 0, 1\rangle \times X_n)_{n\in N}$) and $y \in X \setminus \operatorname{cl} H(\langle 0, 1\rangle \times \partial V)$, then

$$\operatorname{Deg}(H(0, \cdot), V, y) = \operatorname{Deg}(H(1, \cdot), V, y).$$

(ii) If A: $\operatorname{cl} V \to X$ is a DC-mapping and the degree $\operatorname{Deg}(A, V, y)$ is defined and does not vanish, then

$$\inf_{x\in V} \|Ax - y\| = 0.$$

(iii) If A: $\operatorname{cl} V \to X$ is compact and $y \in X \setminus (I-A)(\partial V)$, then

$$\operatorname{Deg}(I - A, V, y) = [(s_n)],$$

where $s_n = \deg_{LS}(I-A, V, y)$ for $n \in N$, the Leray–Schauder degree of the compact field $I-A$.

4. Application of DC-mappings to the problem. In this section we shall see that there exists a filtration in X with respect to which the integral operator

$$Ax(t) = \int_0^\infty (e^{-\mu|t-s|} - e^{-\mu(t+s)})\big(f(x(s), s) - \mu^2 x(s)\big)\, ds$$

is a DC-mapping on the closed ball $\bar{B}(0, R)$. Let Y denote the space of all continuous functions $x\colon \langle 0, \infty) \to R$ such that

$$\|x\|_\mu = \sup_t e^{\mu t}|x(t)| < \infty .$$

Our considerations are based on the compactness of A as a mapping of any bounded set in Y into X.

LEMMA 3. *For any $R_1 > 0$, A transforms the closed ball $\bar{B}_\mu(0, R_1)$ in Y onto a precompact set in X.*

Proof. It is rather obvious that $D = A(\bar{B}_\mu(0, R_1))$ sits in X and is bounded. Moreover, standard calculations show that functions from D are equicontinuous on any bounded interval. Let $t_1 \geqslant t_0$ be such that $R_1 e^{\mu t} \leqslant \delta$ for $t \geqslant t_1$. Then, for such a t and $x \in \bar{B}_\mu(0, R_1)$,

$$e^{\nu t}|Ax(t)| \leqslant e^{(\nu-\mu)t}\Big[C + (C_1 + \mu^2)\delta \int_{t_1}^{t} (1 - e^{-2\mu s})\, ds\Big] +$$
$$+ (e^{(\nu+\mu)t} - e^{(\nu-\mu)t})(C_1 + \mu^2)\delta \int_t^\infty e^{-2\mu s}\, ds,$$

where C is a bound of the integral on $\langle 0, t_1\rangle$. It follows that

$$\lim_{t\to\infty} e^{\nu t} Ax(t) = 0$$

is uniform with respect to $x \in \bar{B}_\mu(0, R_1)$. By (9), the same is true for $(Ax)''$.

Combining the above with the Arzela–Ascoli Theorem, we get an ε-net of D for any positive ε. □

Now, note that $Y \cap X$ is dense in X. Hence ε-nets can be taken from $Y \cap X$, and

$$\mathrm{cl} \bigcup_{n\in N} \bar{B}(0, R) \cap \bar{B}_\mu(0, nR) = \bar{B}(0, R)$$

(the closure with respect to X). For $n \in N$, take a finite dimensional subspace

$X_n \subset Y \cap X$ such that

(13) $$\sup\{d(Ax, X_n) \colon x \in \bar{B}(0, R) \cap \bar{B}_\mu(0, nR)\} < n^{-1}.$$

Enlarging X_n, if necessary, we can assume that $X_n \subset X_{n+1}$ and $\mathrm{cl} \bigcup X_n = X$. This is possible, since X is separable. Therefore (X_n) is a filtration in X and, by (13), $A \colon \bar{B}(0, R) \to X$ is a DC-mapping.

Let $R > M$ where M is the a priori bound from Section 2, and let $H \colon \langle 0, 1 \rangle \times \bar{B}(0, R) \to X$ be given by the formula

$$H(\lambda, x) = x - x_0 + \frac{\lambda}{2\mu} Ax.$$

Then H is a DC-mapping and, due to Section 2,

$$\|H(\lambda, x)\| \geqslant \varepsilon_1 > 0$$

for $x \in \partial\bar{B}(0, R)$ and $\lambda \in \langle 0, 1 \rangle$. By (i), Section 3,

$$\mathrm{Deg}(H(0, \cdot), B(0, R), 0) = \mathrm{Deg}(H(1, \cdot), B(0, R), 0),$$

but $H(0, x) = x - x_0$ is a compact field with the Leray–Schauder degree 1 ($x_0 \in B(0, R)$), so, by (iii) and (ii), Section 3,

$$\inf_x \|H(1, x)\| = 0.$$

Therefore we get a sequence (x_n) of approximate solutions of (7) in X ($\lambda = 1$).

Repeating the arguments from the proof of Lemma 3, one can choose a subsequence (x_{n_k}) such that (Ax_{n_k}) is convergent with respect to the norm $\|\cdot\|_0$, i.e. uniformly convergent. It follows that (x_{n_k}) is uniformly convergent to a certain x and, therefore, x satisfies the integral equation (7) for $\lambda = 1$ which is equivalent that x is a solution of the BVP (4), (5).

We do not know, a priori, that $x \in Y$, but changing the proof of Lemma 2 slightly (it is simpler for exact solutions), one can obtain that $\|x\|_\mu \leqslant \max(K, |\alpha|)$.

5. Remarks. The present work is based on the comparison of the right-hand side of the differential equation with the simplest linear function $\mu^2 x$. This is impossible in many important cases, for instance the Thomas–Fermi equation. However, $\mu^2 x$ can be replaced by functions g such that solutions of $x'' = g(x)$ are known and simple.

Similarly, one can consider another BVP at 0 instead of the Dirichlet one. It is needed only to know Green's function. However, the described method works only for functions vanishing at infinity.

The solution of our BVP is unique if $f(\cdot, t)$ is increasing for any t (see

[3], [5], [8]). The uniqueness can be obtained in special cases for small α by the Contraction Principle or other methods.

6. Singular boundary value problems. Let us consider the equation

$$\varphi(t)x'' = f(x, t) \tag{14}$$

with boundary conditions (5). The assumptions on f are the same as above (1)–(3). Let φ: $\langle 0, \infty) \to \boldsymbol{R}$ be a continuous function such that

$$\varphi(0) = 0, \quad \varphi(t) > 0 \quad \text{for } t > 0, \tag{15}$$

$$\sup_t \varphi(t) = a < \infty, \quad \inf_{t \geqslant t_0} \varphi(t) = b > 0, \tag{16}$$

$$\varphi(t)^{-1} \quad \text{has an integrable singularity at } 0. \tag{17}$$

THEOREM 2. *Under the described assumptions, the BVP* (14), (5) *has a solution.*

Proof. Each of the problems

$$[(1-n^{-1})\varphi(t)+n^{-1}]x'' = f(x, t), \quad x(0) = \alpha, \ \lim_{t\to\infty} x(t) = 0 \tag{18}$$

for $n \geqslant 2$ has a solution. In fact, the functions

$$g_n(x, t) = f(x, t)/[(1-n^{-1})\varphi(t)+n^{-1}]$$

satisfy assumptions (1)–(3) with $\mu' = \mu/\max(1, \sqrt{a})$, $C_j' = C_j/\min(1, b)$ for $j = 1, 2$. Denote by x_n a solution of (18). We know that

$$\sup_t e^{\mu' t}|x_n(t)| \leqslant M,$$

where M does not depend on n. It follows that $\lim_{t\to\infty} x_n'(t) = 0$ or there is some t_n such that $x_n'(t_n) = 0$.

In the first case,

$$|x_n'(t)| = \Big|\int_t^\infty g_n(x_n(s), s)\,ds\Big| \leqslant L\max(1, b^{-1})\cdot\max(t_0-t, 0)+C_1' Me^{-\mu' t_0},$$

where $L = \sup\{|f(x, t)|: t \in \langle 0, t_0\rangle, |x| \leqslant M\}$. The second possibility is examined in the same way. Thus the functions x_n, $n \geqslant 2$, are uniformly bounded, equicontinuous and the limit $\lim_{t\to\infty} x_n(t) = 0$ is uniform. Repeating the arguments of the proof of Lemma 3, one can show that there is a subsequence (x_{n_k}) uniformly convergent to x.

On the other hand, each of x_{n_k} satisfies the integral equation of form (7) where μ is replaced by μ' and f by g_{n_k}. Passing to the limit $k \to \infty$ (it is admissible by the boundedness of $\varphi(t)^{-1}$ on $\langle t_0, \infty)$ and by the integrability of this function on $\langle 0, t_0 \rangle$), we get that x satisfies the integral equation equivalent to the BVP (14), (5). □

In general, the solution x is twice differentiable only for $t > 0$; on the closed half-line $\langle 0, \infty)$, x has the first derivative continuous.

The method we use in the proof works also for BVP on intervals. In this way, one can simplify the considerations from [3], III, §3.

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