ANNALES
POLONICI MATHEMATICI
XIX (1967)

On the absolute harmonic summability of a series related to a Legendre series

by G. S. PANDEY (Ujjain, India)

1. Let $f(x)$ be Lebesgue integrable in $\langle -1, +1\rangle$. The Legendre series corresponding to this function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) , \tag{1.1}$$

where

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{+1} f(x) P_n(x)\, dx , \tag{1.2}$$

and $P_n(x)$ is defined by the following expansion:

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) .$$

2. Let $\{S_n\}$ be the sequence of partial sums of a given infinite series $\sum a_n$. Let the sequence $\{t_n\}$ be defined by

$$t_n = \frac{(n+1)^{-1}S_0 + n^{-1}S_1 + \ldots + 1\cdot S_n}{p_n},$$

where

$$p_n = 1+\tfrac{1}{2}+ \ldots + \frac{1}{n+1} .$$

The series $\sum a_n$ is said to be *absolutely harmonic summable*, if the series

$$\sum_n |t_n - t_{n-1}| \tag{2.1}$$

is convergent. It is known that this method of summability is absolutely regular and implies absolute Cesàro summability of every positive order (see [2]).

In a recent paper Varshney ([8]) has applied the method of absolute harmonic summability to the series

$$\sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{\log(n+1)} \tag{2.2}$$

where a_n and b_n are the Fourier coefficients of a function $f(x)$, which is periodic with period 2π and integrable (L) over $(-\pi, \pi)$.

He proved the following theorem:

THEOREM A. *If*

$$\varphi(t) \equiv \tfrac{1}{2}\{f(x+t)+f(x-t)\}$$

is of bounded variation in $(0, \pi)$ *then the series* (2.2) *is absolutely summable by harmonic means.*

This theorem of Varshney is analogous to that of Mohanty ([3]) on the absolute Riesz summability of the series (2.2).

In the present paper the author intends to apply the method of absolute harmonic summability to the series

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{\log(n+1)}, \tag{2.3}$$

where a_n is given by relation (1.2).

We establish the following new result:

THEOREM. *If* $f(x)$ *is of bounded variation in* $\langle -1, +1\rangle$, *then the series* (2.3) *is absolutely summable by harmonic means at an internal point* x *of the interval* $(-1, +1)$.

3. The proof of the theorem will be based on the following lemmas:

LEMMA 1 ([7], p. 440). *Uniformly for* $0 < t < \pi$

$$\left|\sum_{m}^{n} \frac{\sin \nu t}{\nu}\right| \leqslant K, \tag{3.1}$$

where m *and* n *are any positive integers.*

LEMMA 2 ([1]). *If* $0 < t < \pi$, *then*

$$\left|\sum_{k=0}^{m} \frac{\cos(k+1)t}{k+1}\right| = O\left(1+\log\frac{1}{t}\right). \tag{3.2}$$

LEMMA 3. *If* $0 < t < \pi$, *then for all positive integers* m *and* m'

$$\sum_{k=m}^{m'} \frac{\sin(n-k)t}{k+1} = O\left(1+\log\frac{1}{t}\right). \tag{3.3}$$

The proof follows from Lemmas 1 and 2.

LEMMA 4 ([2]). *If $\{p_n\}$ is a non-negative and non-increasing sequence, then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$, and for any n and a,*

$$\left|\sum_{a}^{b} p_k e^{i(n-k)t}\right| \leqslant KP_T \,, \tag{3.4}$$

where $P_T = P_{[1/t]}$, $T = [1/t]$; $P_n = p_0+p_1+p_2+\ldots+p_n$, and K is a fixed constant.

LEMMA 5 ([8]). *For $p_n = 1+\frac{1}{2}+\ldots+\frac{1}{n+1}$, we have*

$$\sum_{k=0}^{[n/2]-2} \left|\Delta\left[\frac{p_n(n+1)-p_k(k+1)}{(n-k)\log(n-k+1)}\right]\right| = O(1)\,, \tag{3.5}$$

$$\sum_{k=[n/2]}^{n-2} \left|\Delta\left(\frac{p_k}{k+1}\cdot\frac{1}{\log(n-k+1)}\right)\right| = O\left(\frac{p_n}{n}\right). \tag{3.6}$$

LEMMA 6 ([4]), p. 78). *For $0 \leqslant \theta \leqslant \pi$*

$$|P_{n+1}(\cos\theta)-P_{n-1}(\cos\theta)| \leqslant M_1\sqrt{\frac{\sin\theta}{n}}\,, \tag{3.7}$$

where M_1 is a constant.

LEMMA 7 ([5], p. 208 and [6], p. 196; for $\alpha=\beta=0$).

$$P_n(\cos\theta) = \begin{cases} n^{-1/2}k(\theta)\cos\{(n+\frac{1}{2})\theta-\pi/4\}+O(n^{-3/2}), \\ \qquad\qquad \textit{for } \varepsilon \leqslant \theta \leqslant \pi-\varepsilon,\ 0<\varepsilon<\pi/2;\ k=\sqrt{2/\pi\sin\theta}\,; \\ n^{-1/2}k(\theta)[\cos\{(n+\frac{1}{2})\theta-\pi/4\}+(n\sin\theta)^{-1}O(1)], \\ \qquad\qquad \textit{for } c/n \leqslant \theta \leqslant \pi-c/n\,. \end{cases}$$

Here c is a fixed number.

4. Proof of the theorem. From the definition, we have

$$\begin{aligned} t_n-t_{n-1} &= \sum_{k=0}^{n-1}\left(\frac{p_k}{p_n}-\frac{p_{k-1}}{p_{n-1}}\right)\frac{a_{n-k}P_{n-k}(x)}{\log(n-k+1)} \\ &= \frac{1}{p_np_{n-1}}\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{a_{n-k}P_{n-k}(x)}{\log(n-k+1)} \\ &= \frac{1}{p_np_{n-1}}\int_{-1}^{+1} f(y)\left[\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{(n-k+\frac{1}{2})}{\log(n-k+1)}P_{n-k}(x)P_{n-k}(y)\right]dy \\ &= \frac{1}{2p_np_{n-1}}\int_{-1}^{+1} f(y)\left[\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{P_{n-k}(x)}{\log(n-k+1)}\times\right. \\ &\qquad\qquad \left.\times\left(\frac{d}{dy}\{P_{n-k+1}(y)-P_{n-k-1}(y)\}\right)\right]dy\,. \end{aligned}$$

Thus, in order to prove the theorem, we have to show that

$$(4.1)\qquad \sum_n \frac{1}{2p_n p_{n-1}}\left|\int_{-1}^{+1} f(y)\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{P_{n-k}(x)}{\log(n-k+1)}\times\right.$$
$$\left.\times\left[\frac{d}{dy}\{P_{n-k+1}(y)-P_{n-k-1}(y)\}\right]dy\right|<\infty .$$

On integration by parts the expression on the left hand side becomes

$$\sum_n \frac{1}{2p_n p_{n-1}}\left|\left[f(y)\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{P_{n-k}(x)}{\log(n-k+1)}\{P_{n-k+1}(y)-P_{n-k-1}(y)\}\right]_{-1}^{+1}-\right.$$
$$\left.-\int_{-1}^{+1}\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{P_{n-k}(x)}{\log(n-k+1)}\{P_{n-k+1}(y)-P_{n-k-1}(y)\}\,df(y)\right|$$
$$=\sum_n \frac{1}{2p_n p_{n-1}}\left|\int_{-1}^{+1}\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\times\right.$$
$$\left.\times\frac{P_{n-k}(x)}{\log(n-k+1)}\{P_{n-k+1}(y)-P_{n-k-1}(y)\}\,df(y)\right| .$$

Substituting $x=\cos\theta$ and $y=\cos\varphi$ the above expression becomes

$$(4.2)\qquad \sum_n \frac{1}{2p_n p_{n-1}}\left|\int_0^{\pi}\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)\frac{P_{n-k}(\cos\theta)}{\log(n-k+1)}\times\right.$$
$$\left.\times\{P_{n-k+1}(\cos\varphi)-P_{n-k-1}(\cos\varphi)\}\,df(\cos\varphi)\right|$$
$$=\sum_n \frac{1}{2p_n p_{n-1}}\left|\int_0^{c/\sqrt{n}}+\int_{c/\sqrt{n}}^{\theta-c/np_n^2}+\int_{\theta-c/np_n^2}^{\theta+c/np_n^2}+\int_{\theta+c/np_n^2}^{\pi-c/\sqrt{n}}+\int_{\pi-c/\sqrt{n}}^{\pi}\right|$$
$$=J_1+J_2+J_3+J_4+J_5\,,$$

say, where c is a fixed number, $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$, $\varepsilon>0$ but fixed.

Now,

$$(4.3)\qquad J_1\leqslant\sum_n\frac{1}{p_n p_{n-1}}\left|\int_0^{c/\sqrt{n}}\sum_{k=0}^{n-1}\frac{p_n}{k+1}\cdot\frac{\sqrt{\sin\varphi}}{(n-k)\log(n-k+1)}\,|df(\cos\varphi)|\right|$$
$$=\sum_n\frac{1}{np_{n-1}}\cdot\frac{1}{n^{1/4}}\sum_{k=0}^{n-1}\left(\frac{1}{n-k}+\frac{1}{k+1}\right)\int_0^{c/\sqrt{n}}|df(\cos)\varphi|$$
$$=O(1)\int_0^{\pi}|df(\cos\varphi)|=O(1)\,,$$

by the hypothesis.

We now discuss J_3.

Denoting $\nu = [1/\varphi]$, we have

$$(4.4)\quad J_3 \leqslant \sum_{1}^{\nu} \frac{1}{2p_n p_{n-1}} \left| \int_{\theta - c/np_n^2}^{\theta + c/np_n^2} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) \frac{P_{n-k}(\cos\theta)}{\log(n-k+1)} \times \right.$$

$$\left. \times \{P_{n-k+1}(\cos\varphi) - P_{n-k-1}(\cos\varphi)\}\, df(\cos\varphi) \right|$$

$$+ \sum_{\nu+1}^{\infty} \frac{1}{2p_n p_{n-1}} \left| \int_{\theta - c/np_n^2}^{\theta + c/np_n^2} \sum_{k=0}^{[n/2]-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) \frac{P_{n-k}(\cos\theta)}{\log(n-k+1)} \times \right.$$

$$\left. \times \{P_{n-k+1}(\cos\varphi) - P_{n-k-1}(\cos\varphi)\}\, df(\cos\varphi) \right|$$

$$+ \sum_{\nu+1}^{\infty} \frac{1}{2p_n p_{n-1}} \left| \int_{\theta - c/np_n^2}^{\theta + c/np_n^2} \sum_{k=[n/2]}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) \frac{P_{n-k}(\cos\theta)}{\log(n-k+1)} \times \right.$$

$$\left. \times \{P_{n-k+1}(\cos\varphi) - P_{n-k-1}(\cos\varphi)\}\, df(\cos\varphi) \right|$$

$$= J_{3,1} + J_{3,2} + J_{3,3}\,,$$

say.

$$(4.5)\quad J_{3,1} \leqslant \sum_{1}^{\nu} \frac{1}{2p_n p_{n-1}} \left| \int_{\theta - c/np_n^2}^{\theta + c/np_n^2} \sum_{k=0}^{n-1} \frac{p_n}{k+1} \cdot \frac{\sqrt{\sin\varphi}}{(n-k)\log(n-k+1)} \,|df(\cos\varphi)| \right|$$

$$\leqslant A\sqrt{\varphi} \sum_{1}^{\nu} \frac{1}{p_n p_{n-1}} \sum_{k=0}^{n-1} \frac{p_n}{(n-k)(k+1)} = A\sqrt{\varphi} \sum_{1}^{\nu} \frac{1}{n+1} = O(1),$$

A being a fixed constant.

Using the first part of Lemma 7, we have, after some simplification

$$J_{3,2} = \sum_{\nu+1}^{\infty} \frac{1}{2p_n p_{n-1}} \left| \int_{\theta - c/np_n^2}^{\theta + c/np_n^2} \sum_{k=0}^{[n/2]-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) \frac{k(\theta)k(\varphi)}{(n-k)\log(n-k+1)} \times \right.$$

$$\left. \times \cos\left\{ (n-k+\tfrac{1}{2})\theta - \frac{\pi}{4} \right\} \sin\left\{ (n-k+\tfrac{1}{2})\varphi - \frac{\pi}{4} \right\} \sin\tfrac{1}{2}\varphi\, df(\cos\varphi) \right| + O(1)$$

$$= \sum_{\nu+1}^{\infty} \frac{M}{(n+1)p_n p_{n-1}} \left| \int_{\theta - c/np_n^2}^{\theta + c/np_n^2} \sum_{k=0}^{[n/2]-1} \frac{p_n(n+1) - p_k(k+1)}{(n-k)(k+1)} \cdot \frac{k(\theta)k(\varphi)\sin\frac{1}{2}\varphi}{\log(n-k+1)} \times \right.$$

$$\left. \times \left[\sin\left\{ (n-k+\tfrac{1}{2})(\theta+\varphi) - \frac{\pi}{2} \right\} - \sin\{(n-k+\tfrac{1}{2})(\theta-\varphi)\} \right] df(\cos\varphi) \right| + O(1)$$

$$\leqslant J_{3,2,1} + J_{3,2,2} + O(1)\,,$$

say, where M is a constant not necessarily the same at each occurrence.

By Abel's transformation and using Lemmas 2, 3 and 5, we obtain

$$J_{3,2,1} = O\left[\sum_{\nu+1}^{\infty} \frac{M}{(n+1)p_n p_{n-1}} \left| \int_{\theta-c/np_n^2}^{\theta+c/np_n^2} \frac{p_n(n+1)-p_{[n/2]-1}([n/2])}{([n/2]+1)\log([n/2]+2)} \times \right.\right.$$

$$\left. \times \left(\log \frac{1}{\theta+\varphi}\right) |df(\cos\varphi)| \right| +$$

$$+ O\left[\sum_{\nu+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}} \left(\log \frac{1}{\theta+\varphi}\right) \sum_{k=0}^{[n/2]-2} \left| \Delta \frac{p_n(n+1)-p_k(k+1)}{(n-k)\log(n-k+1)} \right|\right.$$

$$= O(1) \sum_{\nu+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}}$$

$$= O(1)\,.$$

$$J_{3,2,2} \leqslant \sum_{\nu+1}^{\infty} \frac{M}{p_n p_{n-1}} \left| \int_{\theta-c/np_n^2}^{\theta+c/np_n^2} \sum_{k=0}^{n-1} \frac{p_n|\theta-\varphi|}{(k+1)\log(n-k+1)} |df(\cos\varphi)| \right|$$

$$= O\left[\sum_{\nu+1}^{\infty} \frac{M}{np_n^2} \int_{\theta-c/np_n^2}^{\theta+c/np_n^2} |df(\cos\varphi)|\right]$$

$$= O(1)\,.$$

Thus, we have

$$J_{3,2} = O(1)\,. \tag{4.6}$$

Proceeding in the same way, using Lemma 5 (3.6) and the identity

$$\frac{1}{(n+1)(n-k)} = \frac{1}{(k+1)(n-k)} - \frac{1}{(n+1)(k+1)},$$

it can be easily seen that

$$J_{3,3} = O(1)\,. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we get

$$J_3 = O(1)\,. \tag{4.8}$$

Finally we consider J_2.

$$J_2 = \sum_{n} \frac{1}{2p_n p_{n-1}} \left| \int_{c/\sqrt{n}}^{\theta-c/np_n^2} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{P_{n-k}(\cos\theta)}{\log(n-k+1)} \times \right.$$

$$\left. \times \{P_{n-k+1}(\cos\varphi) - P_{n-k-1}(\cos\varphi)\}\, df(\cos\varphi) \right|.$$

By using the first part of Lemma 7 and simplifying, we obtain

$$J_2 = \sum_n \frac{1}{2p_n p_{n-1}} \left| \int_{c/\sqrt{n}}^{\theta - c/np_n^2} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{k(\theta)k(\varphi)}{(n-k)\log(n-k+1)} \times \right.$$

$$\left. \times \cos\left\{(n-k+\tfrac{1}{2})\theta - \frac{\pi}{4}\right\} \sin\left\{(n-k+\tfrac{1}{2})\varphi - \frac{\pi}{4}\right\} \sin\tfrac{1}{2}\varphi \, df(\cos\varphi) \right| + O(1)$$

$$= \sum_n \frac{1}{2p_n p_{n-1}} \left| \int_{c/\sqrt{n}}^{\theta - c/np_n^2} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{k(\theta)k(\varphi)}{(n-k)\log(n-k+1)} \times \right.$$

$$\left. \times \left[\sin\left\{(n-k+\tfrac{1}{2})(\theta+\varphi) - \frac{\pi}{4}\right\} - \sin\{(n-k+\tfrac{1}{2})(\theta-\varphi)\}\right] \sin\tfrac{1}{2}\varphi \, df(\cos\varphi) \right| +$$

$$+ O\left[\sum_n \frac{1}{p_n p_{n-1}} \int_{c/\sqrt{n}}^{\theta - c/np_n^2} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{1}{(n-k)^2} \cdot \frac{1}{\log(n-k+1)} |df(\cos\varphi)|\right] +$$

$$+ O(1)$$

$$\leqslant J_{2,1} + J_{2,2} + J_{2,3} + O(1),$$

say. Writing $\theta - \varphi = t$ and $T = [1/t]$, we have

$$J_{2,2} \leqslant \sum_{1}^{T} \frac{1}{2p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta - c/\sqrt{n}} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{k(\theta)k(\theta - t)}{\log(n-k+1)} \times \right.$$

$$\left. \times \frac{\sin(n-k+\frac{1}{2})t}{n-k} \sin\tfrac{1}{2}(\theta - t) \, df\{\cos(\theta - t)\} \right| +$$

$$+ \sum_{T+1}^{\infty} \frac{1}{2p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta - c/\sqrt{n}} \sum_{k=0}^{[n/2]-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{k(\theta)k(\theta - t)}{\log(n-k+1)} \times \right.$$

$$\left. \times \frac{\sin(n-k+\frac{1}{2})t}{n-k} \sin\tfrac{1}{2}(\theta - t) \, df\{\cos(\theta - t)\} \right| +$$

$$+ \sum_{T+1}^{\infty} \frac{1}{2p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta - c/\sqrt{n}} \sum_{k=[n/2]}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{k(\theta)k(\theta - t)}{\log(n-k+1)} \right.$$

$$\left. \times \frac{\sin(n-k+\frac{1}{2})t}{n-k} \sin\tfrac{1}{2}(\theta - t) \, df\{\cos(\theta - t)\} \right|$$

$$= J_{2,2,1} + J_{2,2,2} + J_{2,2,3},$$

say. Now, since $|\sin(n-k)t| \leqslant (n-k)t$ and $p_n(n+1) \geqslant p_k(k+1)$ for $k \leqslant n$, we have

$$(4.9) \quad J_{2,2,1} \leqslant \sum_{1}^{T} \frac{1}{p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta-c/\sqrt{n}} \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) \frac{(n-k+\frac{1}{2})t}{\log(n-k+1)} \times \right.$$
$$\left. \times \frac{1}{(n-k)} |df\{\cos(\theta-t)\}| \right|$$
$$= O(t) \sum_{1}^{T} \{p_n p_{n-1}\}^{-1} \sum_{k=0}^{n-1} \frac{p_n}{k+1}$$
$$= O(t) \sum_{1}^{T} \frac{p_n p_{n-1}}{p_n p_{n-1}}$$
$$= O(1) .$$

Using Abel's transformation and Lemma 5, we have

$$J_{2,2,2} = O\left[\sum_{T+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}} \left(\log \frac{r}{t}\right) \frac{p_n(n+1) - p_{[n/2]-1}([n/2])}{([n/2]+1)\log([n/2]+2)} \times \right.$$
$$\left. \times \int_{c/np_n^2}^{\theta-c/\sqrt{n}} |df\{\cos(\theta-t)\}| \right] +$$
$$+ O\left[\sum_{T+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}} \left(\log \frac{r}{t}\right) \times \right.$$
$$\left. \times \sum_{k=0}^{[n/2]-2} \left| \Delta \frac{p_n(n+1) - p_k(k+1)}{(n-k)\log(n-k+1)} \right| \int_{c/np_n^2}^{\theta-c/\sqrt{n}} |f\{\cos(\theta-t)\}| \right]$$
$$= O\left(\log \frac{r}{t}\right) \sum_{T+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}}$$
$$= O\left(\frac{\log(r/t)}{p_T}\right),$$

where r is a fixed number $> \pi$. Hence

$$(4.10) \qquad J_{2,2,2} = O(1) .$$

By the identity

$$\frac{1}{(n+1)(n-k)} = \frac{1}{(k+1)(n-k)} - \frac{1}{(n+1)(k+1)},$$

we have

$$J_{2,2,3} \leqslant \sum_{T+1}^{\infty} \frac{M}{2p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta-c/\sqrt{n}} \sum_{k=[n/2]}^{n-1} \frac{p_n-p_k}{k+1} \cdot \frac{\sin(n-k+\frac{1}{2})t}{(n-k)\log(n-k+1)} \times \right.$$
$$\left. \times \sqrt{\sin\tfrac{1}{2}(\theta-t)}\, df\{\cos(\theta-t)\} \right| +$$
$$+ \sum_{T+1}^{\infty} \frac{M}{2(n+1()p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta-c\sqrt{n}} \sum_{k=[n/2]}^{n-1} \frac{p_k}{k+1} \cdot \frac{\sin(n-k+\frac{1}{2})t}{(n-k)\log(n-k+1)} \times \right.$$
$$\left. \times \sqrt{\sin\tfrac{1}{2}(\theta-t)}\, df\{\cos(\theta-t)\} \right|$$
$$= I_1 + I_2 ,$$

say. Since for $k \geqslant [n/2]$, $p_n - p_k = O(1)$, we have

(4.11)

$$I_1 = O\left[\sum_{T+1}^{\infty} \frac{1}{p_n p_{n-1}} \left| \int_{c/np_n^2}^{\theta-c/\sqrt{n}} \sum_{k=[n/2]}^{n-1} \frac{p_n-p_k}{(k+1)} \cdot \frac{1}{(n-k)\log(n-k+1)}\, df\{\cos(\theta-t)\} \right| \right]$$
$$= O\left[\sum_{T-1}^{\infty} \frac{1}{np_n p_{n-1}} \sum_{k=[n/2]}^{n-1} \frac{1}{(n-k)\log(n-k+1)} \int_{c/np_n^2}^{\theta-c/\sqrt{n}} |df\{\cos(\theta-t)\}| \right]$$
$$= O\left(\sum_{T+1}^{\infty} \frac{\log\log n}{n\log^2 n} \right)$$
$$= O(1) .$$

Using Abel's transformation again, and in view of the relations

$$\sum_{a}^{b} \sin nt = O(1/t); \qquad \sum_{a}^{b} \cos nt = O(1/t) ,$$

we have

$$(4.12) \qquad I_2 = O\left[\sum_{T+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}} \sum_{k=[n/2}^{n-2} \frac{1}{t} \left| \Delta \frac{p_k}{k+1} \cdot \frac{1}{\log(n-k+1)} \right| \right] +$$
$$+ O\left(\frac{1}{t}\right) \left[\sum_{T+1}^{\infty} \frac{1}{(n+1)p_n p_{n-1}} \cdot \frac{p_{n-1}}{n} \right]$$
$$= O\left(\frac{1}{t}\right) \left[\sum_{T+1}^{\infty} \frac{1}{n^2 \log n} \right]$$
$$= O\left(\frac{1}{t}\right) \left(\frac{1}{T\log T} \right)$$
$$= O(1).$$

Similarly for $c/\sqrt{n} \leqslant \varphi \leqslant \theta - c/np_n^2$, we have

$$J_{2,1} = O(1) \quad \text{and} \quad J_{2,3} = O(1)\,.$$

Thus, we get

$$J_2 = O(1)\,. \tag{4.13}$$

J_4 and J_5 can also be disposed off in exactly the same way as J_2 and J_1 respectively.

This completes the proof.

I am thankful to Dr. D. P. Gupta for his valuable suggestions and encouragement during the preparation of this paper.

References

[1] G. H. Hardy and W. W. Rogosinski, *Notes on Fourier series IV. Summability* (R_2), Proc. Cambridge Philos. Soc. 43 (1947), pp. 10-25.

[2] L. McFadden, *Absolute Nörlund Summability*, Duke Math. Journ. 9 (1942), pp. 168-207.

[3] R. Mohanty, *On the absolute Riesz summability of a Fourier series and allied series*, Proc. London Math. Soc., Ser. 2, 52 (1951), pp. 295-320.

[4] N. Obrechkoff, *Formules asymptotiques pour les polynomes de Jacobi et sur les séries suivant les mêmes polynomes*, Annuaire de l'universite de Sofia, Faculté Physico-Mathématique, 1 (1936), pp. 39-133.

[5] G. Sonsone, *Orthogonal Functions* (English Edition), New York 1959.

[6] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., New York 1959.

[7] E. C. Titchmarsh, *The theory of functions*, Oxford 1939.

[8] O. P. Varshney, *On the absolute harmonic summability of a series related to a Fourier series*, Proc. Amer. Math. Soc. 10 (1959), pp. 784-789.

DEPARTMENT OF MATHEMATICS
VIKRAM UNIVERSITY
UJJAIN, INDIA

Reçu par la Rédaction le 25. 4. 1966