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On a boundary value problem for an equation of the type of non-stationary filtration

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Abstract. In this paper we consider a boundary value problem with non-continuous boundary data for the equation $u_t = [\varphi(u)]_{xx}$. We examine the existence, uniqueness and regularity properties for a certain class of weak solutions of that problem.

Introduction. Let S_T denote the half-stripe $(0, \infty)\times(0, T]$ in the (x, t)-plane for some fixed $T > 0$.

By $C^{l+\alpha}$, for non-negative integers l and $\alpha\in(0, 1)$ we shall denote:

(a) In the case of functions of variables x, t — the space of functions f whose derivatives of the form $(\partial/\partial x)^s(\partial/\partial t)^r f$, where $0 \leqslant 2r+s \leqslant l$, are bounded in the maximum and Hölder norms (cf. [3]).

(b) In the case of functions of one variable — the space of functions, whose derivatives up to order l are bounded and the l-th derivative is Hölder continuous with the exponent α.

We shall consider the following problem:

$$u_t = [\varphi(u)]_{xx} \quad \text{in } S_T, \tag{1}$$

$$u(x, 0) = u_0(x) \quad \text{for } x\in(0, \infty), \tag{2}$$

$$u(0, t) = u_1(t) \quad \text{for } t\in(0, T], \tag{3}$$

where:

(I) for some $\alpha\in(0, 1)$ we have $\varphi\in C^1([0, \infty])\cap C^{2+\alpha}([s_1, s_2])$ for every $0 < s_1 < s_2$, $\varphi(0) = \varphi'(0) = 0$, $\varphi(s) > 0$, $\varphi_1'(s) > 0$, $\varphi''(s) > 0$ for $s > 0$,

(II) $\int\limits_0^1 [\varphi'(s)/s]\,ds < \infty$,

(III) $u_0\in L^\infty((0, \infty))$, $\quad \operatorname*{inf\,ess}\limits_{x\in(0,\infty)} u_0(x) \geqslant 0$, $\quad u_1\in L^\infty((0, T])\cap C((0, T])$, $u_1 \geqslant 0$.

Equation (1) is a degenerate parabolic equation: it is parabolic for

$u > 0$, but it is not if $u = 0$. When $\varphi(s) = s^m$, $m > 1$, equation (1) describes flows of gases or fluids in a porous medium.

Problem (1)–(3) was studied by many authors: see [8], [9], [11] and references therein. In [11], Oleĭnik, Kalašnikov and Yuĭ-Lin examined a class of continuous in $\bar{S}_T$ weak solutions of problem (1)–(3). In this paper we study a class of weak solutions, which are continuous only in $[0, \infty) \times (0, T]$.

Definition. A function u will be called a *weak solution of problem* (1)–(3) if:

(i) u is non-negative, bounded and continuous in $[0, \infty) \times (0, T]$,

(ii) u satisfies the identity

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} [f_t u + f_{xx}\varphi(u)]\,dx\,dt - \int_{x_1}^{x_2} fu\Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} f_x\varphi(u)\Big|_{x_1}^{x_2} dt = 0 \tag{4}$$

for all $0 \leqslant x_1 < x_2$, $0 < t_1 < t_2 \leqslant T$ and for all $f \in C^{2,1}([x_1, x_2] \times [t_1, t_2])$ such that $f|_{x=x_1} = f|_{x=x_2} = 0$,

(iii) $u(0, t) = u_1(t)$ for $t \in (0, T]$ and

$$\lim_{t \to 0^+} u(x, t) = u_0(x) \tag{5}$$

for almost all $x \in (0, \infty)$.

Note that classical solutions of problem (1)–(3) are also weak solutions of problem (1)–(3) in the sense of the above definition.

Example. Consider the following problem

$$\big(\varphi'(h)h'\big)' + \tfrac{1}{2}\eta h' = 0 \quad \text{for } 0 < \eta < \infty, \tag{6}$$

$$h(0) = c, \quad \lim_{\eta \to \infty} h(\eta) = 0, \tag{7}$$

where the mark $'$ denotes differentiation, $h = h(\eta)$, $c > 0$, and φ satisfies assumptions (I) and (II).

It was proved in [2] that there exists a unique weak (in a suitably defined sense) solution h of problem (6)–(7), which has the following properties: (i) h is bounded, continuous and non-negative on $[0, \infty)$, (ii) $H(\eta) = \varphi(h(\eta))$ has a continuous derivative H' on $[0, \infty)$, (iii) there exists a positive constant $a = a(c)$ such that $h(\eta) > 0$ for $0 \leqslant \eta < a$ and $h(\eta) = 0$ for $a \leqslant \eta < \infty$, (iv) h is a classical solution of equation (6) in a neighbourhood of any point where $h > 0$.

Put

$$\tilde{u}_c(x, t) = h(xt^{-1/2}) \tag{8}$$

for $x \geqslant 0$ and $0 < t \leqslant T$. The function $\tilde{u}_c$ is a classical solution of equation (1) in $S_T - \{(x, t): xt^{-1/2} = a\}$. It is easy to verify that $\tilde{u}_c$ is a weak solution of problem (1)–(3) satisfying the boundary data

$$u_0(x) = 0 \quad \text{for } 0 < x < \infty, \quad u_1(t) = c \quad \text{for } 0 < t \leqslant T.$$

The following lemma will be useful in Section 3.

LEMMA 1. *Let h be the solution of problem* (6)–(7) *mentioned above. Then* $h'(\eta) < 0$ *for* $0 < \eta < a$.

Proof. The result follows immediately from the formula

$$\varphi'(h(\eta))h'(\eta) = -\tfrac{1}{2}\Big(\eta h(\eta) + \int_\eta^a h(\sigma)\,d\sigma\Big) \quad \text{for } 0 < \eta < a.$$

In the paper, in Section 2 we prove the uniqueness of a weak solution of problem (1)–(3). The existence of a solution of problem (1)–(3) in the considered class of weak solutions is proved in Section 3. We also obtain some regularity properties of this solution. In Section 4 we estimate from below $\sup\{x \in (0, \infty): u(x, t) > 0\}$ for $t \in (0, T]$.

2. Uniqueness. We shall prove the following theorem:

THEOREM 1. *Let hypotheses* (I) *and* (IV) *of Section* 1 *be satisfied. Then there exists at most one weak solution of problem* (1)–(3).

Proof. Let u and u^* be two weak solutions of problem (1)–(3) and let $g \in C_0^\infty(S_T)$. We shall show that $\int_0^\infty \int_0^T (u-u^*)g\,dx\,dt = 0$. Choose a number r_0 such that $g = 0$ for $x \geqslant r_0 - 1$ and set in identity (4): $x_1 = 0$, $x_2 = r$, $t_1 = \tau$, $t_2 = T$, where $r \geqslant r_0$, $0 < \tau < T$. Then for each $f \in C^{2,1}([0, r] \times [\tau, T])$ such that $f|_{x=0} = f|_{x=r} = 0$ we have

$$\text{(9)} \quad \int_0^r \int_\tau^T (u-u^*)[f_t + a(x, t)f_{xx}]\,dx\,dt - \int_0^r f(u-u^*)\Big|_{t=\tau}^{t=T} dx - \int_0^T f_x[\varphi(u) - \varphi(u^*)]\Big|_{x=r} dt = 0,$$

where

$$\text{(10)} \qquad a(x, t) = \begin{cases} \dfrac{\varphi(u) - \varphi(u^*)}{u - u^*} & \text{if } u \neq u^*, \\ \varphi'(u) & \text{if } u = u^* \end{cases}$$

for $(x, t) \in [0, \infty) \times (0, T]$. Note that $a = a(x, t)$ is non-negative, bounded and continuous on $[0, \infty) \times (0, T]$.

Let a_n for $n = 1, 2, \ldots$ be positive $C^\infty(\bar{S}_T)$ functions such that $a_n \to a$ if $n \to \infty$ uniformly on every compact subset of $[0, \infty)\times(0, T]$ and $a_n \geqslant a$ in $[0, r]\times[\tau, T]$. Putting, in (9), $a = (a-a_n)+a_n$ for $n = 1, 2, \ldots$ we obtain

$$(11)\quad \int_0^r\int_\tau^T (u-u^*)[f_t + a_n(x, t) f_{xx}]\,dxdt + \int_0^r\int_\tau^T (u-u^*)(a-a_n) f_{xx}dxdt - \\ - \int_0^r f(u-u^*)\Big|_{t=\tau}^{t=T} dx - \int_\tau^T f_x[\varphi(u)-\varphi(u^*)]\Big|_{x=r} dt = 0$$

for $r \geqslant r_0$, $n = 1, 2, \ldots$ and for all $f \in C^{2,1}([0, r]\times[\tau, T])$ such that $f|_{x=0} = f|_{x=r} = 0$.

Consider, for given $r \geqslant r_0$ and $n \geqslant 1$, the following problem:

$$(12)\qquad f_t + a_n(x, t) f_{xx} = g \quad \text{in } [0, r]\times[\tau, T],$$

$$(13)\qquad f|_{x=0} = f|_{x=r} = f|_{t=T} = 0.$$

It follows from Theorem 7 of [3], p. 65, and Theorem 10 of [3], p. 72, that for each $r \geqslant r_0$ and $n \geqslant 1$ there exists a unique solution $f^{n,r} = f^{n,r}(x, t)$ of problem (12)–(13) such that $f^{n,r} \in C^\infty([0, r]\times[\tau, T])$. Moreover, the following estimates hold:

$$(14)\qquad |f^{n,r}(x, t)| \leqslant C_1 e^{-x} \quad \text{for } 0 \leqslant x \leqslant r,\ \tau \leqslant t \leqslant T,$$

$$(15)\qquad |f_x^{n,r}(r, t)| \leqslant C_2 e^{-r} \quad \text{for } t \in [\tau, T],$$

$$(16)\qquad \int_0^r\int_\tau^T a_n(x, t)(f_{xx}^{n,r})^2\,dxdt \leqslant C_3,$$

where the positive constants C_1–C_3 do not depend on n, r, τ. The proof of estimates (14)–(16) is analogous to the one given in [6] and we omit it.

If we apply identity (11) to the functions $f^{n,r}$ for $r \geqslant r_0$ and $n = 1, 2, \ldots$ instead of f, we obtain

$$(17)\quad \int_0^r\int_\tau^T (u-u^*)g\,dxdt + \int_0^r\int_\tau^T (u-u^*)(a-a_n) f_{xx}^{n,r}\,dxdt + \int_0^r f^{n,r}(u-u^*)\Big|_{t=\tau} dx + \\ + \int_\tau^T f_x^{n,r}[\varphi(u)-\varphi(u^*)]\Big|_{x=r} dt = 0$$

for $r \geqslant r_0$ and $n = 1, 2, \ldots$ Hence, using estimates (14), (15) and (16) and the Schwartz inequality, we have

$$(18)\quad \left|\int_0^r\int_\tau^T (u-u^*)g\,dxdt\right| \leqslant \left(\int_0^r\int_\tau^T (u-u^*)^2 \frac{(a-a_n)^2}{a_n}\,dxdt\right)^{1/2} C_3^{1/2} + \\ + C_1\int_0^r e^{-x}|u(x, \tau)-u^*(x, \tau)|\,dx + C_2 e^{-r}\int_\tau^T \big|\varphi(u(r, t))-\varphi(u^*(r, t))\big|\,dt$$

for $r \geqslant r_0$ and $n = 1, 2, \ldots$ Let $n \to \infty$ in (18). Then, since $(a-a_n)^2/a_n \leqslant 2 \sup_{[0,r]\times[\tau,T]} |a-a_n|$ we have

$$(19) \quad \Big|\int_0^r \int_\tau^T (u-u^*) g\,dxdt\Big| \leqslant C_1 \int_0^r e^{-x}|u(x,\tau)-u^*(x,\tau)|\,dx + C_2 e^{-r} \int_\tau^T |\varphi(u(r,t))-\varphi(u^*(r,t))|\,dt$$

for $r \geqslant r_0$ Hence, in the limit (as $r \to \infty$),

$$(20) \quad \Big|\int_0^\infty \int_\tau^T (u-u^*)\, g dxdt\Big| \leqslant C_1 \int_0^\infty e^{-x}|u(x,\tau)-u^*(x,\tau)|\,dx.$$

Inequality (20) holds for all $0 < \tau < T$. It follows from condition (5) that $\lim_{t\to 0^+} \big(u(x,t)-u^*(x,t)\big) = 0$ for almost all $x \in (0, \infty)$. Then, from (20),

$$\int_0^\infty \int_0^T (u-u^*) g\,dxdt = 0,$$

what holds for each $g \in C_0^\infty(S_T)$. Hence $u \equiv u^*$ in S_T.

Remark. It follows from inequality (20) that Theorem 1 remains true if we assume, instead of (5), that

$$(5') \quad \lim_{t\to 0^+} \int_J |u(x,t)-u_0(x)|\,dx = 0$$

for every finite interval $J \subset (0, \infty)$.

3. Existence. In this section we shall use the following transformations of non-negative classical solutions u of equation (1), introduced in [11]:

$$v = \varphi(u), \quad w = \psi(u),$$

where $\psi(s) = \int_0^s [\varphi'(\xi)/\xi]\,d\xi$ for $s \geqslant 0$.

In view of hypotheses (I) and (II) of Section 1 there exist the inverse transformations Φ, Ψ, such that $\varphi(\Phi(v)) = v$, $\psi(\Psi(w)) = w$ and $\Phi, \Psi \in C([0,\infty)) \cap C^2((0,\infty))$.

THEOREM 2. *Let assumptions* (I)–(III) *of Section* 1 *be satisfied. Moreover, assume that u_0 is a piecewise continuous function and $u_1(t) \geqslant c > 0$ for some constant c and for $t \in (0, T]$, or $u_1(t) = 0$ for $t \in (0, T]$. Then there exists a weak solution u of problem* (1)–(3). *It is a classical solution of equation* (1) *in a neighbourhood of any point $(x_0, t_0) \in S_T$, where $u(x_0, t_0) > 0$.*

Proof. We follow the constructive method given in [11]. Putting $v = \varphi(u)$, we transform equation (1) to the form

$$v_{xx} = \Phi'(v)\, v_t.$$

Write $v_0 = \varphi(u_0)$, $v_1 = \varphi(u_1)$ and let $M \geqslant \max\{\text{sup ess } v_0, \sup u_1\}$. Without loss of generality we may assume that v_0 is upper semi-continuous (in view of

Theorem 1 we can suitably redefine the function v_0 at discontinuity points).

In view of the assumptions imposed on u_0 and u_1 it is possible to construct sequences of functions $\{v_{0,k}\}$ and $\{v_{1,k}\}$ such that

(i) $v_{0,k} \in C^\infty([0, \infty))$, $v_{1,k} \in C^\infty([0, T])$ for $k = 2, 3, \ldots$,

(ii) $v_{0,k} \to v_0$, $v_{1,k} \to v_1$ as $k \to \infty$,

(iii) $v_{0,k+1} \leqslant v_{0,k}$, $v_{1,k+1} \leqslant v_{1,k}$ for $k = 2, 3, \ldots$,

(iv) $k^{-1} \leqslant v_{0,k} \leqslant M$, $k^{-1} \leqslant v_{1,k} \leqslant M$ for $k = 2, 3, \ldots$,

(v) $v_{0,k} = M$ on $[k-1, k]$ for $k = 2, 3, \ldots$,

(vi) $v_{0,k}(0) = v_{1,k}(0)$ for $k = 2, 3, \ldots$,

(vii) $\left(\dfrac{d^j}{dx^j} v_{0,k}\right)(0) = \left(\dfrac{d^j}{dt^j} v_{1,k}\right)(0)$ for $j = 1, 2$ and $k = 2, 3, \ldots$

For a given integer $k \geqslant 2$ we consider the following problem:

$$v_{xx} = \Phi'(v)v_t \quad \text{in } Q_k = (0, k) \times (0, T], \tag{21}$$

$$v(0, t) = v_{1,k}(t) \quad \text{for } t \in [0, T], \tag{22}$$

$$v(x, 0) = v_{0,k}(x) \quad \text{for } x \in [0, \infty), \tag{23}$$

$$v(k, t) = M \quad \text{for } t \in [0, T]. \tag{24}$$

Since the boundary data (22)–(24) are strictly positive, it follows from the theory of non-degenerate parabolic equations [10], p. 640, that for each integer $k \geqslant 2$ there exists a unique solution v_k of problem (21)–(24) such that $v_k \in C(\bar{Q}_k) \cap C^{2+\alpha}(D)$ for every compact subset $D \subset Q_k$. Moreover, we obtain from the maximum principle that $k^{-1} \leqslant v_k \leqslant M$ and $v_{k+1} \leqslant v_k$ in $\bar{Q}_k$, for $k = 2, 3, \ldots$

Put $u_k = \Phi(v_k)$ for $k = 2, 3, \ldots$ The functions u_k, for $k = 2, 3, \ldots$, satisfy equation (1) and $\Phi(k^{-1}) \leqslant u_k \leqslant \Phi(M)$, $u_{k+1} \leqslant u_k$ in $\bar{Q}_k$, for $k = 2, 3, \ldots$ Hence, for every $(x, t) \in \bar{S}_T$, there exists $\lim_k u_k(x, t) = u(x, t)$. It is clear that u is non-negative and bounded on $\bar{S}_T$, u satisfies identity (4), $u(x, 0) = u_0(x)$ for almost all $x \in (0, \infty)$ and $u(0, t) = u_1(t)$ for $t \in (0, T]$.

In order to prove the continuity of u in S_T consider the functions $w_k = \psi(u_k)$ for $k = 2, 3, \ldots$ The functions w_k satisfy the equation

$$(w_k)_t = A_k(x, t)(w_k)_{xx} + B_k(x, t)(w_k)_x$$

in Q_k, for $k = 2, 3, \ldots$, respectively, where $A_k(x, t) = \varphi'(\Psi(w_k(x, t)))$ and $B_k(x, t) = (w_k)_x(x, t)$ for $(x, t) \in Q_k$ and $k = 2, 3, \ldots$ Let $k \geqslant 2$ be a fixed integer and let $D \subset Q_k$ be a closed rectangle. We have A_k, B_k, $(A_k)_x$, $(B_k)_x \in C^{0+\beta}(D)$ for some $\beta \in (0, 1)$. Hence, by Theorem 10 of [3], p. 72, we obtain $(w_k)_{xxx} \in C^{0+\beta}(D)$. Thus $(w_k)_{xxx} \in C(Q_k)$ for $k = 2, 3, \ldots$ It follows from the proof of Theorem 2 of [7], p. 66, that if $\delta > 0$ and $0 < \tau \leqslant T$, then

$$|(w_k)_x(x, t)| \leqslant C \tag{25}$$

for $(x, t) \in Q_{k-1} \cap [\delta, \infty) \times [\tau, T]$ and $k = 2, 3, \ldots$, where the constant C depends only on δ, τ, $\psi(\Phi(M))$ and $\varphi'(\Phi(M))$. Hence, if $\delta > 0$ and $0 < \tau \leqslant T$, then

$$|w_k(x, t) - w_k(x', t)| \leqslant C|x - x'| \tag{26}$$

for (x, t), $(x', t) \in Q_{k-1} \cap [\delta, \infty) \times [\tau, T]$ and $k = 2, 3, \ldots$, and, by the result of [4],

$$|w_k(x, t) - w_k(x, t')| \leqslant C'|t - t'|^{1/2} \tag{27}$$

for (x, t), $(x, t') \in Q_{k-1} \cap [\delta, \infty) \times [\tau, T]$ and $k = 2, 3, \ldots$, where the constant C' does not depend on k. Since (26) and (27) holds also in the limit as $k \to \infty$, then $w = \psi(u)$ is continuous in S_T and the same is true for u.

Now we prove that u is continuous up to the boundary $\{0\} \times (0, T]$.

Let $t_0 \in (0, T]$ and $u_1(t_0) = 0$. Since $u \geqslant 0$ and $u_k \geqslant u$ for $k = 2, 3, \ldots$, then $\varliminf_{(x,t)\to(0,t_0)} u(x, t) \geqslant 0$ and $\varlimsup_{(x,t)\to(0,t_0)} u(x, t) \leqslant \lim_{(x,t)\to(0,t_0)} u_k(x, t) = \Phi(v_{1,k}(t_0))$ for $k = 2, 3, \ldots$ Hence, in the limit (as $k \to \infty$), we obtain $\varlimsup_{(x,t)\to(0,t_0)} u(x, t) \leqslant 0$ and therefore $\lim_{(x,t)\to(0,t_0)} u(x, t) = 0$.

Assume that $u_1(t) \geqslant c > 0$ for $t \in (0, T]$. Then the following lemma is valid.

LEMMA 2. *Let $0 < c' < c$ and let $\tilde{u}_{c'}$ be the weak solution of problem* (1)–(3) *defined by* (8). *Then $\tilde{u}_{c'} \leqslant u_k$ for $k = 2, 3, \ldots$*

Proof. Write, for simplicity, $\tilde{u} = \tilde{u}_{c'}$ and suppose that $\tilde{u}(x_0, t_0) > u_k(x_0, t_0)$ at some point $(x_0, t_0) \in Q_k$ for some integer $k \geqslant 2$. Since $u_{k'} \leqslant u_k$ if $k' \geqslant k$, we can assume that $k \geqslant a(c')T^{1/2}$. Then, in view of the continuity of $\tilde{u}$ and u_k at (x_0, t_0), there exists $\varrho > 0$ such that $\tilde{u}(x, t) > u_k(x, t)$ for $(x, t) \in R_\varrho = [x_0 - \varrho, x_0 + \varrho] \times [t_0 - \varrho, t_0 + (T - t_0)\varrho]$. We have $\tilde{u}(0, t) = c'$ and $u_k(0, t) \geqslant c > c'$ for $t \in (0, T]$. Moreover, in view of Lemma 1, $\tilde{u}(x, t) \leqslant c'$ for $(x, t) \in \bar{S}_T$. It follows from the uniform continuity of u_k that we can choose a number $0 < \tau < t_0 - \varrho$ such that $u_k(x, \tau) > c'$ for $x \in [0, a(c')\tau^{1/2}]$. Since $\tilde{u}(x, \tau) = 0$ for $x \geqslant a(c')\tau^{1/2}$, we have $u_k(x, \tau) - \tilde{u}(x, \tau) \geqslant 0$ for $x \in [0, k]$.

As in the proof of Theorem 1, for each $f \in C^{2,1}([0, k] \times [\tau, T])$ such that $f|_{x=0} = f|_{x=k} = 0$ we have

$$\int_0^k \int_\tau^T (u_k - \tilde{u})[f_t + a_{k,n}(x, t) f_{xx}]\,dxdt - \int_0^k f(u_k - \tilde{u})\Big|_{t=\tau}^{t=T} dx + \int_\tau^T f_x[\varphi(u_k) - \varphi(\tilde{u})]\Big|_{x=0} dt = \int_\tau^T f_x[\varphi(u_k) - \varphi(\tilde{u})]\Big|_{x=k} dt + \int_0^k \int_\tau^T (u_k - \tilde{u})(a_{k,n} - a_k) f_{xx}\,dxdt, \tag{28}$$

where

$$a_k(x,t) = \begin{cases} \dfrac{\varphi(u_k)-\varphi(\tilde{u})}{u_k-\tilde{u}} & \text{if } u_k \neq \tilde{u}, \\ \varphi'(\tilde{u}) & \text{if } u_k = \tilde{u} \end{cases}$$

for $(x, t) \in \bar{Q}_k$ and $a_{k,n}$ for $n = 1, 2, \ldots$ are positive $C^\infty([0, k) \times [\tau, T])$-functions such that $a_{k,n} \searrow a_k$ if $n \to \infty$. Instead of (12)–(13) we consider, for a given $n \geq 1$, the problem

$$f_t + a_{k,n}(x,t) f_{xx} = g \quad \text{in } [0, k] \times [\tau, T], \tag{29}$$

$$f|_{x=0} = f|_{x=k} = f|_{t=T} = 0, \tag{30}$$

where: $g \in C_0^\infty(Q_k)$ is a non-positive function such that $g < 0$ in $R_{\varrho/2}$, $g = 0$ in $\bar{Q}_k - R_{\varrho/2}$. Denote by $f^{k,n}$ the solution of problem (29)–(30) and $n \geq 1$.

Note that for $f^{k,n}$, $n \geq 1$, estimate (16) in which $r = k$ and $a_n = a_{k,n}$ hold true. Moreover, from the maximum principle [3], p. 34, we have $f^{k,n} \geq 0$ for $n \geq 1$. Since $f^{k,n}(0, t) = 0$, $f^{n,k}(k, t) = 0$ for $t \in [\tau, T]$ and $n \geq 1$, then $f_x^{k,n}(0, t) \geq 0$ and $f_x^{n,k}(k, t) \leq 0$ for $t \in [\tau, T]$, $n \geq 1$. Putting in identity (28) the functions $f^{k,n}$, for $n = 1, 2, \ldots$, instead of f we obtain

$$\begin{aligned} \int_0^k \int_\tau^T (u_k - \tilde{u}) g \, dx dt + \int_0^k f^{k,n}(u_k - \tilde{u})\Big|_{t=\tau} dx + \int_\tau^T f_x^{k,n} [\varphi(u_k) - \varphi(\tilde{u})]\Big|_{x=0} dt \\ = \int_\tau^T f_x^{k,n} [\varphi(u_k) - \varphi(\tilde{u})]\Big|_{x=k} dt + \int_0^k \int_\tau^T (u_k - \tilde{u})(a_{k,n} - a_n) f_{xx}^{k,n} \, dx dt \end{aligned} \tag{31}$$

for $n = 1, 2, \ldots$ Since $\int_0^k f^{k,n}(u_k - \tilde{u})\Big|_{t=\tau} dx > 0$, $\int_\tau^T f_x^{k,n} [\varphi(u_k) - \varphi(\tilde{u})]\Big|_{x=0} dt \geq 0$ and $\int_\tau^T f_x^{n,k} \varphi(u_k)|_{x=k} dt \leq 0$ for $n = 1, 2, \ldots$, then, from (31) and (16),

$$\int_0^k \int_\tau^T (u_k - \tilde{u}) g \, dx dt < C_3^{1/2} \left(\int_0^k \int_\tau^T (u_k - \tilde{u})^2 \frac{(a_{k,n} - a_k)^2}{a_{k,n}} dx dt \right)^{1/2}$$

for $n = 1, 2, \ldots$

Hence, passing to infinity with n we obtain $\int_0^k \int_\tau^T (u_k - \tilde{u}) g \, dx dt \leq 0$. But in view of the choice of the function g we have $\int_0^k \int_\tau^T (u_k - \tilde{u}) g \, dx dt > 0$. This contradiction ends the proof of the lemma.

Let $t_0 \in (0, T]$. It follows from Lemma 2 that u is strictly positive in some neighbourhood of the point $(0, t_0)$. Therefore, using the standard barrier method for non-degenerate parabolic equations ([12], p. 123), we obtain that u is continuous at the point $(0, t_0)$.

Using the fact that $u_k \searrow u$ we can prove, by the barrier method, that u is continuous up to the boundary $t = 0$ at every point of continuity of u_0, which proves condition (5). The proof is analogous to the one given in [5] and we omit the details.

Finally, let $u(x_0, t_0) > 0$ at some $(x_0, t_0) \in S_T$. Then $u_k(x, t) \geqslant \frac{1}{2} u(x_0, t_0)$ for $(x, t) \in R_\sigma = [x_0 - \sigma, x_0 + \sigma] \times [t_0 - \sigma, t_0 + (T - t_0)\sigma]$ for some $\sigma > 0$. It follows from Theorem 1.1 of [10], p. 476, that there exist constants $C'' > 0$ and $\gamma \in (0, 1)$, which do not depend on k, such that

$$|u_k(x, t) - u_k(x', t')| \leqslant C''(|x - x'|^\gamma + |t - t'|^{\gamma/2})$$

for $(x, t), (x', t') \in R_{\sigma/2}$. Then, by Theorem 15 of [3], p. 80, the sequence $\{v_k\}$ is compact in $C^{2,1}(R_{\sigma/4})$. Hence u is a classical solution of equation (1) in $R_{\sigma/4}$.

Remark 1. If Ψ is uniformly Hölder continuous with an exponent $\nu \in (0, 1]$ on $[0, \infty)$, we obtain from (26) and (27) that if $\delta > 0$ and $0 < \tau < T$, then

$$|u(x, t) - u(x', t')| \leqslant \bar{C}(|x - x'|^\nu + |t - t'|^{\nu/2}) \tag{32}$$

for $(x, t), (x', t') \in [\delta, \infty) \times [\tau, T]$ and some constant $\bar{C}$. In particular, if $\varphi(s) = s^m$, $m > 1$, then $\psi(s) = [m/(m-1)] s^{m-1}$, Ψ is uniformly Hölder continuous with the exponent $\nu = \min\{1, (m-1)^{-1}\}$ in $[0, \infty)$, and u satisfies (32). In this case inequality (32) follows also from the results of [1] and [4].

We shall prove the following regularity theorem:

THEOREM 3. *Let the assumptions of Theorem 2 be fulfilled and let u be the weak solution of problem (1)–(3). Then the derivative $\varphi(u)_x$ exists and is continuous in S_T. Moreover, if $\delta > 0$ and $0 < \tau < T$, then*

$$|\varphi(u)_x(x, t)| \leqslant Cu(x, t) \tag{33}$$

for $(x, t) \in [\delta, \infty) \times [\tau, T]$, where the constant C depends only on δ, τ and $\psi(\Phi(M))$, $\varphi'(\Phi(M))$ arising in the construction of u from Theorem 2.

Proof. Let u be the weak solution of problem (1)–(3). By Theorem 2, $\varphi(u)_x$ exists and is continuous in a neighbourhod of any point $(x, t) \in S_T$, where $u(x, t) > 0$. Moreover, in view of the uniqueness Theorem 1 and the construction given in Theorem 2, if $(x, t) \in S_T$ and $u(x, t) > 0$, then

$$\varphi(u)_x(x, t) = \lim_k \varphi(u_k)_x(x, t),$$

where u_k, for $k = 2, 3, \ldots$, are strictly positive classical solutions of equation (1) such that $u_k \to u$ as $k \to \infty$.

Let $\delta > 0$ and $0 < \tau < T$. Since $\varphi(u_k)_x = \varphi\big(\Psi(\psi(u_k))\big)_x$, by (25) we have

$$(34) \qquad |\varphi(u_k)_x(x, t)| \leqslant C u_k(x, t)$$

for $(x, t) \in Q_k \cap [\delta, \infty] \times [\tau, T]$ and $k = 2, 3, \ldots$, where the constant C depends only on δ, τ, and $\psi(\Psi(M))$, $\varphi'(\Phi(M))$ from the construction of u given in Theorem 2. Hence, if $(x_0, t_0) \in S_T$ and $u(x_0, t_0) = 0$, then $\lim \varphi(u_k)_x(x_0, t_0) = 0$.

Put

$$F(x, t) = \begin{cases} \varphi(u)_x(x, t) & \text{if } u(x, t) > 0, \\ 0 & \text{if } u(x, t) = 0 \end{cases}$$

for $(x, t) \in S_T$. It follows from (34) that if $\delta > 0$ and $0 < \tau < T$, then

$$(35) \qquad |F(x, t)| \leqslant C u(x, t) \quad \text{for } (x, t) \in [\delta, \infty) \times [\tau, T].$$

Hence F is continuous at the points where $u(x, t) = 0$ and, consequently, F is continuous in S_T.

Let $(x_0, t_0) \in S_T$, $u(x_0, t_0) = 0$ and let $0 < \sigma_1 < x_0 < \sigma_2$. For $x \in [\sigma_1, \sigma_2]$ and $k \geqslant \sigma_2$ we have

$$(36) \qquad \varphi\big(u_k(x, t_0)\big) = \varphi\big(u_k(\sigma_1, t_0)\big) + \int_{\sigma_1}^{x} \varphi(u_k)_x(\sigma, t_0)\, d\sigma.$$

It follows from (34) that $|\varphi(u_k)_x(\sigma, t_0)| \leqslant M'$ for $\sigma \in [\sigma_1, \sigma_2]$, where the constant M' does not depend on k. Taking $k \to \infty$ in (36) we obtain, by the Lebesgue dominated convergence theorem,

$$\varphi\big(u(x, t_0)\big) = \varphi\big(u(\sigma_1, t_0)\big) + \int_{\sigma_1}^{x} F(\sigma, t_0)\, d\sigma \quad \text{for } x \in [\sigma_1, \sigma_2].$$

Hence, in view of the continuity of F, the derivative $\varphi(u)_x(x_0, t_0)$ exists and $\varphi(u)_x(x_0, t_0) = F(x_0, t_0)$. Inequality (33) follows now from (35).

4. Two comparison theorems. The following theorem holds:

THEOREM 4. *Let assumptions* (I)–(III) *of Section* 1 *be satisfied. Moreover, assume that* u_0 *is piecewise continuous and* $u_1(t) \geqslant c > 0$ *for* $t \in (0, T]$ *and for some constant* c. *Let* u *be the weak solution of problem* (1)–(3). *If* $0 < c' < c$ *and* $\tilde{u}_{c'}$ *is the weak solution of problem* (1)–(3) *defined by* (8), *then*

$$\tilde{u}_{c'}(x, t) \leqslant u(x, t) \quad \text{for } (x, t) \in S_T.$$

Proof. The result follows immediately from Theorem 1, Theorem 2 and Lemma 2.

Assume additionally that

(IV) $\varphi(u_0)$, $\varphi(u_1)$ are Lipschitz continuous, $u_0(0+) = u_1(0+)$, and

(V) $u_0(x) > 0$ for $x < x_0$ and $u_0(x) = 0$ for $x \geqslant x_0$ and for some $x_0 > 0$.

Let u be the weak solution of problem (1)–(3). In view of Theorem 22 of [11] we have $\xi(t) = \sup\{x \in (0, \infty): u(x, t) > 0\} < \infty$ for all $t \in (0, T]$. As a consequence of Theorem 4 we obtain the following result:

THEOREM 5. *Let assumptions* (I)–(III) *of Section* 1 *and* (IV)–(V) *of Section* 4 *be fulfilled, let* $u_1(t) \geqslant c > 0$ *for* $t \in (0, T]$ *and for some constant* c, *and let* u *be the weak solution of problem* (1)–(3). *Then there exists a positive constant* $A = A(c)$ *such that*

$$At^{1/2} \leqslant \xi(t) \quad \textit{for } t \in (0, T].$$

Proof. Let $0 < c' < c$ and let $\tilde{u}_{c'}$ be the weak solution of problem (1)–(3) defined by (8). It follows from Theorem 4 that $\tilde{u}_{c'} \leqslant u$ in S_T. From the definition of $\tilde{u}_{c'}$ we have

$$\tilde{u}_{c'}(x, t) > 0 \quad \text{if } x < at^{1/2} \text{ and } t \in (0, T],$$

$$\tilde{u}_{c'}(x, t) = 0 \quad \text{if } x \geqslant at^{1/2} \text{ and } t \in (0, T]$$

for some positive constant $a = a(c')$. Hence

$$a(c')t^{1/2} \leqslant \xi(t)$$

for $t \in (0, T]$ and $0 < c' < c$, and therefore

$$A(c)t^{1/2} \leqslant \xi(t)$$

for $t \in (0, T]$, where $A(c) = \sup\limits_{0 < c' < c} a(c')$.

Added in proof.

Remark 2. Let C_{u_0} denotes the set of discontinuity points of u_0. It follows from the construction given in Section 3 that Theorem 2 remains true if we assume the following condition:

$$C_{u_0} = \bar{C}_{u_0} \quad \text{and} \quad \text{meas } C_{u_0} = 0$$

instead of "u_0 is a piecewise continuous function".

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