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Remarks on a three-point boundary value problem in a differential equation of the third order

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Abstract. In this paper it is shown that under certain conditions on the coefficients $A = A(x, \lambda, \mu)$, resp. $b = b(x, \lambda, \mu)$ of the differential equation

(a) $$y''' + 2A(x, \lambda, \mu)y' + [A'(x, \lambda, \mu) + q(x, \lambda, \mu)]y = 0$$

of the third order, it is possible to choose parameters λ and μ so that there exists a non-trivial solution y of the differential equation (a) satisfying the boundary conditions

$$y(a) = y'(a) = y(b) = y(c) = 0,$$

where $a < b < c$.

1. Consider the differential equation of the third order

(1) $$y''' + 2A(x, \lambda, \mu)y' + [A'(x, \lambda, \mu) + q(x, \lambda, \mu)]y = 0,$$

where $A(x, \lambda, \mu)$, $A'(x, \lambda, \mu) = \frac{d}{dx} A(x, \lambda, \mu)$, $q(x, \lambda, \mu)$ are continuous functions of $x \in \langle a, c\rangle$, $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$.

The problem is to determine sufficient conditions on q, resp. A under which it is possible to choose parameteres λ and μ so that there exists a non-trivial solution y of the differential equation (1) satisfying the boundary conditions

(2) $$y(a) = y'(a) = y(b) = y(c) = 0,$$

where $a < b < c$.

We will show that under certain conditions on q resp. A there exist an integer N, an infinite number of pairs $(\lambda_{N+p}, \mu_{N+p})$, $p = 0, 1, 2, \ldots$, and a sequence of functions $\{y_{N+p}\}_{p=0}^{a}$, such that $y_{N+p} = y(x, \lambda_{N+p}, \mu_{N+p})$ is a solution of equation (1) and fulfils the boundary conditions (2).

A similar problem for an equation of the second order is solved in paper [1] and for an equation of the third order of the form $y''' + q(x, \lambda, \mu)y = 0$ it is solved in paper [2]

2. Consider a differential equation of the third order of the form

(3) $$y''' + 2A(x, \lambda)y' + [A'(x, \lambda) + q(x, \lambda)]y = 0,$$

where

$$A(x, \lambda), \quad A'(x, \lambda) = \frac{d}{dx} A(x, \lambda), \quad q(x, \lambda)$$

are continuous functions of $x \in \langle a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$.

Let $q(x, \lambda) \geqslant 0$ for $x \in \langle a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and suppose that $q(x, \lambda)$ does not vanish identically in any subinterval of $\langle a, \infty)$.

LEMMA 1. *Let $y(x, \lambda)$ be the solution of the differential equation* (3) *with the property $y(\alpha, \lambda) = 0$, where $a \leqslant \alpha < \infty$.*

Then the null-points of the solution $y(x, \lambda)$ of (3), *lying right to α, are continuous functions of the parameter $\lambda \in (\Lambda_1, \Lambda_2)$.*

The proof is given in [3].

OSCILLATION THEOREM. *Let* 1° $|A(x, \lambda)| \leqslant k$, $|A'(x, \lambda)| \leqslant k$ *and let* $\lim\limits_{\lambda \to \Lambda_2} q(x, \lambda) = +\infty$ *uniformly with respect to x in $\langle a, \infty)$, or* 2° $\lim\limits_{\lambda \to \Lambda_2} A(x, \lambda) = +\infty$ *uniformly with respect to x in $\langle a, \infty)$; let further $q(x, \lambda) \geqslant 0$ for all x and λ and suppose that $q(x, \lambda)$ does not vanish identically in any subinterval of $\langle a, \infty)$. Finally, let $b > a$ be a given number and let $y(x, \lambda)$ be the solution of the differential equation* (3) *with the property $y(a, \lambda) = 0$.*

Then with the increasing $\lambda \to \Lambda_2$ the number of null-points of the solution $y(x, \lambda)$ in the interval $\langle a, b\rangle$ tends to infinity and at the same time the maximum distance between two neighbouring null-points tends to zero.

The proof is given in [3]

Note. The assertion of the oscillation theorem is also true when $\lim\limits_{\lambda \to \Lambda_2} q(x, \lambda) = \infty$ or $\lim\limits_{\lambda \to \Lambda_2} A(x, \lambda) = +\infty$ uniformly with respect to x of $\langle \alpha, b\rangle$, where $a < \alpha < b$, but now the maximum distance of two neighbouring null-points tends to zero for $\lambda \to \Lambda_2$ only in the interval $\langle \alpha, b\rangle$.

3. Now we are in a position to state and to prove the main result concerning the problem (1), (2).

THEOREM 1. *Let $a < b < c$ be real numbers. Let $A(x, \lambda)$, $A'(x, \lambda) = \frac{d}{dx} A(x, \lambda)$ be continuous functions of $x \in \langle a, c\rangle$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and let $|A(x, \lambda)| \leqslant k$ and $|A'(x, \lambda)| \leqslant k$ for all x, λ, where $k > 0$. Let $q(x, \lambda, \mu)$ be of the form*

$$q(x, \lambda, \mu) = q_\lambda(x, \lambda) + q_\mu(x, \mu),$$

where $q_\lambda(x, \lambda)$ is a continuous function of $x \in \langle a, c\rangle$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and let

$$q_\mu(x, \mu) = \begin{cases} r(x) & \text{for} \quad x \in \langle a, b\rangle, \\ s(x, \mu) & \text{for} \ x \in \langle b, c\rangle, \mu \in (M_1, M_2) \end{cases}$$

$(s(b,\mu) = r(b)$ for $\mu \in (M_1, M_2))$, where r, s are continuous functions of $x \in \langle a, c\rangle$ and $\mu \in (M_1, M_2)$. Further, let $q(x, \lambda, \mu)$ be non-negative for all $x \in \langle a, c\rangle$, $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$ and let $\lim_{\lambda \to \Lambda_2} q_\lambda(x, \lambda) = +\infty$ uniformly with respect to $x \in \langle \beta, c\rangle$, where $b < \beta < c$.

Then there exists an integer $N > 0$ and sequences $\{\lambda_{N+p}\}_{p=0}^{\infty}$, $\{\mu_{N+p}\}_{p=0}^{\infty}$, $\{y_{N+p}\}_{p=0}^{\infty}$ such that $y_{N+p} = y(x, \lambda_{N+p}, \mu_{N+p})$ is the solution of equation (1), which fulfils boundary conditions (2) and has exactly $N+p$ null-points in $\langle a, b\rangle$.

Proof. We extend the functions $A(x,\lambda)$, $A'(x,\lambda)$, $q(x,\lambda,\mu)$ onto $\langle c, \infty) \times (\Lambda_1, \Lambda_2)$, resp. $\langle c, \infty) \times (\Lambda_1, \Lambda_1) \times (M_1, M_2)$ as follows:

$$A(x,\lambda) = A(c,\lambda), \quad A'(x,\lambda) = A'(c,\lambda), \quad q(x,\lambda,\mu) = q(c,\lambda,\mu)$$

for $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$.

Let $y(x,\lambda,\mu)$ be the solution of the differential equation (1) with the property that $y(a,\lambda,\mu) = y'(a,\lambda,\mu) = 0$, $y''(a,\lambda,\mu) \neq 0$ for $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$. From the oscillation theorem it follows, that there exist such $\bar\lambda$ and $\bar\mu$ that $y(x, \bar\lambda, \bar\mu)$ has in (a, b) certain number of null-point. Denote this number by N. Then we have

$$x_N(\bar\lambda, \bar\mu) < b \leqslant x_{N+1}(\bar\lambda, \bar\mu),$$

where x_N is the N-th null-point of the solution $y(x, \bar\lambda, \bar\mu)$ in (a, b).

From the oscillation theorem it follows that there exists such $\lambda^* \in (\bar\lambda, \Lambda_2)$ for which $x_{N+1}(\lambda^*, \bar\mu) < b$. Then, by Lemma 1, there exists a λ_N, $\bar\lambda \leqslant \lambda_N < \lambda^*$, for which $y(b, \lambda_N, \bar\mu) = 0$ and $y(x, \lambda_N, \bar\mu)$ has in (a, b) exactly N null-points.

For $\lambda = \lambda_N$, $\mu = \bar\mu$, let $y(x, \lambda_N, \bar\mu)$ have on (b, c) exactly ν null-points. Then evidently the following inequality is true:

$$\xi_\nu(\lambda_N, \bar\mu) < c \leqslant \xi_{\nu+1}(\lambda_N, \bar\mu),$$

where ξ_ν is the ν-th null-point of the solution $y(x, \lambda_N, \bar\mu)$ in the interval (b, c). From the oscillation theorem it follows that there exists a $\mu^* \in (\bar\mu, M_2)$ for which $\xi_{\nu+1}(\lambda_N, \mu^*) < c$.

From Lemma 1 follows the existence of $\mu_N \in \langle \bar\mu, \mu^*)$ such that

$$y(c, \lambda_N, \mu_N) = 0.$$

Writing $y_N = y(c, \lambda_N, \mu_N)$, we obtain the existence of the first members of the sequences

$$\{\lambda_{N+p}\}, \quad \{\mu_{N+p}\}, \quad \{y_{N+p}\}.$$

Continuing in the same way we prove the existence of further members of those sequences.

THEOREM 2. *Let $a < b < c$ be real numbers. Let $q(x, \lambda) \geqslant 0$ be a continuous function of $x \in \langle a, c\rangle$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and suppose that $q(x, \lambda)$ does not vanish identically in any subinterval of $\langle a, c\rangle$. Let $A(x, \lambda, \mu) \geqslant 0$, $A'(x, \lambda, \mu) = \frac{d}{dx} A(x, \lambda, \mu)$ be continuous functions of $x \in \langle a, c\rangle$, $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$ and let $A(x, \lambda, \mu)$ be of the form*

$$A(x, \lambda, \mu) = A_\lambda(x, \lambda) + A_\mu(x, \mu),$$

where $A_\lambda(x, \lambda)$, $A'_\lambda(x, \lambda)$ are continuous functions of $x \in \langle a, c\rangle$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and

$$A_\mu(x, \mu) = \begin{cases} A_1(x) & \text{for } x \in \langle a, b\rangle, \\ A_2(x, \mu) & \text{for } x \in \langle b, c\rangle,\ \mu \in (M_1, M_2) \end{cases}$$

$(A_2(b, \mu) = A_1(b)$, $A'_2(b, \mu) = A'_1(b)$ for all $\mu \in (M_1, M_2))$. Let $A_\mu(x, \mu)$, $A'_\mu(x, \mu)$ be continuous functions of $x \in \langle a, c\rangle$ and $\mu \in (M_1, M_2)$ Further, let $\lim_{\lambda \to \Lambda_2} A_\lambda(x, \lambda) = +\infty$ uniformly with respect to $x \in \langle a, c\rangle$ and $\lim_{\mu = M_2} A_\mu(x, \mu) = +\infty$ uniformly with respect to $x \in \langle \beta, c\rangle$, where $b < \beta < c$.

Then there exists a number N and sequences

$$\{\lambda_{N+p}\}_{p=0}^{\infty}, \quad \{\mu_{N+p}\}_{p=0}^{\infty}, \quad \{y_{N+p}\}_{p=0}^{\infty},$$

such that $y_{N+p} = y(x, \lambda_{N+p}, \mu_{N+p})$ is the solution of equation (1), which fulfils boundary conditions (2) and has exactly $N+p$ null-points in $\langle a, b\rangle$.

The proof of Theorem 2 is similar to the proof of Theorem 1.

References

[1] M. Greguš, F. Neuman and F. M. Arscott, *Three-point boundary value problem in differential equations*, J. London Math. Soc. (2), 3 (1971), p. 429–436.

[2] M. Greguš, *Three-point boundary value problem in a differential equation of the third order*, Proceedings, EQUADIFF 3, Brno 1972, p. 115–118.

[3] — *Über die lineare homogene Differentialgleichung dritter Ordnung*, Wiss Z. Univ. Halle, Math.-Nat. XII/3 (1963), p. 265–286.

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