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Finite difference approximation to the Cauchy problem for non-linear parabolic differential equations

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Abstract. An explicit finite difference scheme is used to approximate the solution of the initial value problem for non-linear second order parabolic equations in several independent variables. The approximated solution is allowed to belong to a "natural" class of fast increasing functions. An error estimate implying the convergence of the difference scheme is obtained.

1. Introduction. The problem of finite difference approximation to initial boundary value problems for parabolic equations in bounded domains has been investigated by many authors. In [3], [6], [9], [10] this problem is treated for non-linear equations in several independent variables, in bounded parallelepipeds. Numerical treatment of the Cauchy problem for parabolic equations is found in papers [1], [4], [5], [12], [13]. Among them only paper [1] is concerned with many independent variables. In [1], [4] the approximated solutions are assumed to be bounded. Further, in papers [5], [12], [13] the Cauchy problem for second order parabolic equations (just as in the earlier paper [8] the Cauchy problem for first order hyperbolic equations) is treated with the "longitudinal" method of lines which reduces this problem to the corresponding problem for a countable system of ordinary differential equations. However, only in [12] the approximated solution may grow as fast as in our case. On the other hand, paper [12] deals only with the case of one idependent spatial variable and its non-uniform discretization.

Considering the line method approximation to be the limit case of the finite difference scheme as the difference quotient for the time variable tends to the derivative, we see that the line method cannot be applied under our circumstances because of the assumption about the relation between the sizes of the steps for the time and space variables ($k/h^2 \geq d > 0$ in (8)).

We approximate the solution of the Cauchy problem for parabolic equations by the solution of a suitable discrete problem, using a uniform explicit difference scheme, and prove a theorem concerning the error estimate and the convergence of the scheme. We find, first of all, that the approximated solution is allowed to grow like $\exp(K|x|^2)$. This contains extensions of some the results of [1], [4] to the unbounded solutions case. Moreover, we consider non-linear equations, obtaining generalizations of the results of [3], [6], [9], [10] to the Cauchy problem.

Parabolic equations containing the mixed derivatives of the unknowns have been dealt with by A. Fitzke [3] and M. Malec [9], [10] under different assumptions on the function f occurring in the given equation. M. Malec used seven-point approximations to the mixed derivatives which enabled him to make a relatively weak assumptions on f – implying parabolicity.

In this paper we use the same approximations to the derivatives and make the same assumption of f concerning parabolicity as those introduced by M. Malec.

Let us finally mention that not long ago Z. Kowalski [7] proved the convergence of a difference scheme for non-linear elliptic equations under a more general, "almost usual" condition of ellipticity.

2. Let $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$ and $S = \{x \in R^{n+1}\colon 0 \leqslant x_0 \leqslant T,\ T > 0\}$. Consider the Cauchy problem

$$(1)\qquad u_{x_0} = f(x, u, u_x, u_{xx}) \quad \text{for } x \in S; \qquad u(x) = g(x) \quad \text{for } x_0 = 0,$$

where $u_x = (u_{x_1}, \ldots, u_{x_n})$, $u_{xx} = (u_{x_i x_j})_{i,j=1}^{n}$.

We make the following assumptions:

I. Problem (1) has a solution $u(x)$ which is of class $C^2(S)$, has continuous third order derivatives with respect to variables x_i $(i = 1, \ldots, n)$ in S and satisfies the growth conditions

$$(2)\qquad |u|,\ |u_{x_0 x_0}|,\ |u_{x_i x_j x_s}| \leqslant H(x; M, K) := M \sum_{\nu=1}^{n} \exp(K x_\nu^2)$$

for all $i, j, s = 1, \ldots, n$ and for some constants $M, K > 0$.

II. Function $f(x, u, q, r)$ is continuous for $(x, u, q, r) \in S \times R^{1+n+n^2}$ and of class C^1 in u, q, r. For any fixed pair of indices i, j $(i \neq j)$ the derivative $\partial f/\partial r_{ij}$ is always non-negative or always non-positive and $\partial f/\partial r_{ij} = \partial f/\partial r_{ji}$. Further, there exist constants $L_0, L_1 \geqslant 0$, $L_2 \geqslant \varkappa > 0$ such that

$$(3)\qquad |\partial f/\partial u| \leqslant L_0, \quad |\partial f/\partial q_i| \leqslant L_1, \quad |\partial f/\partial r_{ij}| \leqslant L_2,$$

$$(4)\qquad \partial f/\partial r_{ii} - \sum_{j=1, j\neq i}^{n} |\partial f/\partial r_{ij}| \geqslant \varkappa.$$

Furthermore

$$(5)\qquad |f(x, 0, 0, 0)| \leqslant H(x; M, K).$$

III. There exist constants $\lambda > 0$, $N > K$ and $p \in (0, 1)$ so that

$$\lambda \geqslant 4L_2 \max\{n, Np^{-1}\exp[(N/2)e^{\lambda T} + \lambda T]\}. \tag{6}$$

Assumption III is a natural restriction on K or T. It is well known that in general a solution of (1) belonging to the class of "fast increasing" functions exists only for small x_0's. We give two examples in which (6) is satisfied:

1° Arbitrary $T > 0$, $\lambda \geqslant 4nL_2$ and $p \in (0, 1)$ while K is so small that

$$Kp^{-1}\exp[(K/2)e^{\lambda T} + \lambda T] < n.$$

2° Arbitrary K, $N > K$, $p \in (0, 1)$ and

$$\lambda \geqslant 4L_2 \max\{n, Np^{-1}\exp(Ne/2 + 1)\} \quad \text{whereas } T \leqslant 1/\lambda.$$

Let A be the set of vectors (multi-indices) $a = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ such that $\alpha_0 = 0, 1, \ldots, l$; $\alpha_i = 0, \pm 1, \pm 2, \ldots$ $(1 \leqslant i \leqslant n)$, l being a positive integer. Let $A' = A \cap (\alpha_0 < l)$. In S we introduce the set S' of nodal points

$$x^a = (x_0^{\alpha_0}, x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}), \quad a \in A,$$

where

$$x_0^{\alpha_0} = \alpha_0 k \quad (k = T/l) \quad \text{and} \quad x_i^{\alpha_i} = \alpha_i h \quad (h = \text{const} > 0;\ i = 1, \ldots, n).$$

Further we define

$$i(a) = (\alpha_0, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n), \quad 0 \leqslant i \leqslant n$$

$(\alpha_0 < l$ if $i = 0)$ and

$$-i(a) = (\alpha_0, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_n), \quad 1 \leqslant i \leqslant n.$$

For a function v^a defined for $a \in A$ we define the difference operators

$$v^{a0} = (v^{0(a)} - v^a)/k, \qquad v^{ai} = (v^{i(a)} - v^{-i(a)})/2h,$$

$$v^{aij} = \frac{s(i,j)}{2h^2}(v^{i(a)} + v^{-i(a)} + v^{j(a)} + v^{-j(a)} - 2v^a - v^{i(-s(i,j)j(a))} - v^{-i(s(i,j)j(a))})$$

$(i, j = 1, \ldots, n)$, where $s(i, i) = 1$ and, for $i \neq j$,

$$s(i,j) = \begin{cases} 1 & \text{if } \partial f/\partial r_{ij} \leqslant 0, \\ -1 & \text{if } \partial f/\partial r_{ij} \geqslant 0. \end{cases}$$

Thus, in particular,

$$v^{aii} = \frac{1}{h^2}(v^{i(a)} - 2v^a + v^{-i(a)}).$$

For a function w defined on S we denote $w^a := w(x^a)$. One can show that if $w \in C^2(S)$ then $w^{aij} \to w^a_{x_i x_j}$ as $h \to 0$. Finally, let

$$v^{aI} = (v^{a1}, \ldots, v^{an}), \qquad v^{aIJ} = (v^{aij})_{i,j=1}^n.$$

We consider the following explicit difference scheme corresponding to problem (1):

$$(7)\qquad \begin{aligned} &v^{a0} = f(x^a, v^a, v^{aI}, v^{aIJ}) && \text{for } a \in A',\\ &v^a = g^a && \text{for } \alpha_0 = 0. \end{aligned}$$

Our main result is the following

THEOREM. *Let Assumptions* I, II *and* III *be satisfied. Suppose*

$$(8)\qquad h \leqslant h_0, \quad d \leqslant k/h^2 \leqslant 1/2nL_2,$$

where $d = 2/\lambda$ *and* $0 < h_0 \leqslant 2\varkappa/L_1$ *if* $L_1 > 0$ *whereas* h_0 *may be any positive number if* $L_1 = 0$. *Then*

$$(9)\qquad |u^a - v^a| \leqslant z^a \quad \textit{for } a \in A,$$

where u^a *and* v^a *are the values at the nodal points of the solutions of problems* (1) *and* (7) *respectively, and*

$$z^a = M\omega^a \Phi^a, \quad \omega^a = \mu(k, h)[\exp(L_0 \alpha_0 k) - 1]/L_0,$$

$$\mu(k, h) = k/2 + [nL_1 h^2/6 + 5n^2 L_2 h/3]\exp(K_0 h^2), \quad K_0 = NK/(N-K),$$

$$\Phi^a = \sum_{i=1}^{n} \varphi_i^a, \quad \varphi_i^a = \exp\{Ne^{\lambda\alpha_0 k}\alpha_i^2 h^2 + \gamma\alpha_0 k\},$$

$$\gamma = \max\{L_1^2 \lambda^{-1}(1-p)^{-2} N\exp[2Ne^{\lambda T}(1+h_0^2) + \lambda T],$$
$$2L_2[\exp(h_0^2 Ne^{\lambda T}) - 1]/ph_0^2, (\lambda N/2)e^{\lambda T}\}.$$

3. In this section we preserve all the assumptions of the theorem and prove several lemmas.

LEMMA 1. *We have*

$$(10)\qquad |u^a - v^a| \leqslant 2MD^{\alpha_0} \sum_{\nu=1}^{n} \exp(Ke^{\lambda\alpha_0 k}\alpha_\nu^2 h^2) \quad \textit{for } a \in A,$$

where

$$(11)\qquad D = D_0 \exp\{(K/2)e^{\lambda T}\}, \quad D_0 = 1 + 2n + h_0 L_1/2L_2 + h_0^2(L_0+1)/2nL_2.$$

Proof. We first show by induction that

$$(12)\qquad |v^a| \leqslant MD_0^{\alpha_0} \sum_{\nu=1}^{n} \exp\{K(|\alpha_\nu| + \alpha_0)^2 h^2\}, \quad a \in A.$$

If $\alpha_0 = 0$, (12) follows from the growth condition imposed on g. Suppose that (12) holds for $\alpha_0 = p$. Hence we get

$$(13)\qquad |v^{i(a)}|, |v^{-i(a)}|, |v^{i(-s(i,j)\,j(a))}|, |v^{-i(s(i,j)\,j(a))}|$$
$$\leqslant MD_0^p \sum_{\nu=1}^{n} \exp\{K(|\alpha_\nu| + 1 + p)^2 h^2\}.$$

Taking advantage of (7), (8), (3) and (13), one can easily check that

$$|v^{0(a)}| \leqslant MD_0^{p+1} \sum_{v} \exp\{K(|\alpha_v|+1+p)^2 h^2\},$$

i.e., (12) with $\alpha_0 = p+1$. Now, since

$$(|\alpha_v|+\alpha_0)^2 \leqslant e^{\lambda\alpha_0 k}\alpha_v^2 + \alpha_0^2 e^{\lambda\alpha_0 k}/(e^{\lambda\alpha_0 k}-1)$$

and $e^{\lambda\alpha_0 k}-1 \geqslant \lambda\alpha_0 k \geqslant \lambda\alpha_0 dh^2$, we get from (12)

$$|v^a| \leqslant MD^{\alpha_0} \sum_{v=1}^{n} \exp(Ke^{\lambda\alpha_0 k}\alpha_v^2 h^2). \tag{14}$$

(14) and (2) imply (10).

LEMMA 2. *For $a \in A$ we have the estimates*

$$|u_{x_0}^a - u^{a0}| \leqslant (k/2)\, H(x^a;\, M,\, K), \tag{15}$$

$$|u_{x_i}^a - u^{ai}| \leqslant (h^2/6)\, H(x^a;\, M,\, N)\exp(K_0 h^2), \tag{16}$$

$$|u_{x_i x_j}^a - u^{aij}| \leqslant (5h/3)\, H(x^a;\, M,\, N)\exp(K_0 h^2). \tag{17}$$

Proof. This is obtained by using Taylor's formula and (2).

We define

$$\eta_a(k,\, h) = f(x^a,\, u^a,\, u^{aI},\, u^{aIJ}) - u^{a0}, \qquad a \in A'. \tag{18}$$

LEMMA 3.

$$|\eta_a(k,\, h)| \leqslant \mu(k,\, h)\, H(x^a;\, M,\, N). \tag{19}$$

Proof. By (18), (1) and the mean value theorem,

$$|\eta_a(k,\, h) = |u^{a0} - f(x^a,\, u^a,\, u^{aI},\, u^{aIJ}) - [u_{x_0}^a - f(x^a,\, u^a,\, u_x^a,\, u_{xx}^a)]|$$

$$\leqslant |u^{a0} - u_{x_0}^a| + \sum_i \left|\frac{\partial f}{\partial q_i}\right| |u^{ai} - u_{x_i}^a| + \sum_{i,j} \left|\frac{\partial f}{\partial r_{ij}}\right| |u^{aij} - u_{x_i x_j}^a|.$$

Applying Lemma 2 and (3), we get (19).

LEMMA 4. *The function Φ^a satisfies the inequality*

$$\Lambda(\Phi^a) := L_2 \sum_{i,j=1}^{n} |\Phi^{aij}| + L_1 \sum_{i=1}^{n} |\Phi^{ai}| - \Phi^{a0} \leqslant 0 \qquad \textit{for } a \in A'. \tag{20}$$

Proof. It is sufficient to prove that φ_i^a satisfies the inequality

$$L_2 |\varphi_i^{aii}| + L_1 |\varphi_i^{ai}| - \varphi_i^{a0} \leqslant 0 \qquad (1 \leqslant i \leqslant n). \tag{21}$$

Setting $C = Ne^{\lambda\alpha_0 k}$, we get

$$\varphi_i^{a0}/\varphi_i^a = \frac{1}{k}\{\exp[C(e^{\lambda k}-1)\alpha_i^2 h^2 + \gamma k] - 1\}.$$

Hence, by the inequalities $e^s \geqslant 1+s$ and $k \geqslant dh^2$ we obtain

$$\varphi_i^{a0}/\varphi_i^a \geqslant \gamma \exp(2C\alpha_i^2 h^4) + \lambda C\alpha_i^2 h^2. \tag{22}$$

Further, we have

$$|\varphi_i^{ai}|/\varphi_i^a = h^{-1} e^{Ch^2} \sinh(2C|\alpha_i| h^2).$$

Since $\sinh s \leqslant se^s$ $(s \geqslant 0)$, we obtain

$$\begin{aligned} L_1 |\varphi_i^{ai}|/\varphi_i^a &\leqslant 2L_1 C|\alpha_i| h \exp[C(1+\alpha_i^2 h^4 + h^2)] \\ &\leqslant L_1^2 \lambda^{-1}(1-p)^{-1} C \exp[2C(1+\alpha_i^2 h^4 + h^2)] + \\ &\quad + \lambda(1-p) C\alpha_i^2 h^2. \end{aligned} \tag{23}$$

Comparing (23) with (22), we find, by the definition of γ,

$$L_1 |\varphi_i^{ai}| - (1-p)\varphi_i^{a0} \leqslant 0. \tag{24}$$

Let

$$\psi(\alpha_i) := (L_2 |\varphi_i^{aii}| - p\varphi_i^{a0})/\varphi_i^a, \qquad 1 \leqslant i \leqslant n.$$

It is sufficient to show that $\psi(\alpha_i) \leqslant 0$ because this and (24) yield (21). We have

$$\psi(\alpha) = \frac{2L_2}{h^2}\{e^{Ch^2}\cosh(2C\alpha h^2) - 1\} - \frac{p}{k}\{\exp[C(e^{\lambda k}-1)\alpha^2 h^2 + \gamma k] - 1\}.$$

It is easy to check that $\psi(0) \leqslant 0$. Since $\psi(-\alpha) = \psi(\alpha)$, it suffices to show that $\psi'(\alpha) \leqslant 0$ for $\alpha \geqslant 0$. Making use of the inequalities $e^s \geqslant 1+s$, $k \geqslant dh^2$, $\sinh s \leqslant se^s$ $(s \geqslant 0)$ and $s \leqslant s^2/2C + C/2$, we derive

$$\psi'(\alpha) = 4L_2 Cs\exp(s^2/2C + C/2 + Ch^2) - \lambda ps\exp(s^2/2C + \gamma dh^2),$$

where $s = 2C\alpha h^2$. Further, the definitions of λ and γ imply $\psi'(\alpha) \leqslant 0$, q.e.d.

LEMMA 5. *Function z^a satisfies the inequality*

$$A(z^a) + L_0 z^a + |\eta_a(k, h)| \leqslant 0, \qquad a \in A'. \tag{25}$$

Proof. It is easy to see that

$$z^{ai} = M\omega^a \Phi^{ai}, \qquad z^{aij} = M\omega^a \Phi^{aij}, \qquad z^{a0} = M\omega^{a0}\Phi^a + M\omega^{0(a)}\Phi^{a0},$$

$$\omega^{0(a)} \geqslant \omega^a, \qquad \Phi^{a0} \geqslant 0, \qquad \omega^{a0} \geqslant L_0\omega^a + \mu(k, h).$$

These inequalities and Lemmas 3 and 4 imply Lemma 5.

4. Proof of the theorem. We set

$$w^a = u^a - v^a, \qquad F^a = |w^a| - z^a, \qquad F_1^a = w^a - z^a, \qquad F_2^a = -w^a - z^a.$$

Let $A_\varrho = \{a \in A\colon |\alpha_i| < \varrho,\ i = 1, \ldots, n\}$. By Lemma 1 there is an integer ϱ such that $F^a \leqslant 0$ for $a \in A \setminus A_\varrho$. We shall show that $F^a \leqslant 0$ for $a \in A_\varrho$. Suppose the contrary. Then there would exist a multi-index $a \in A_\varrho$, with $\alpha_0 > 0$, such that $F^a > 0$ and $F^b \leqslant 0$ for all $b = (\beta_0, \beta_1, \ldots, \beta_n) \in A_\varrho$ with $\beta_0 < \alpha_0$. Take b so that $o(b) = a$. It is sufficient to show that $F_i^a \leqslant 0$ and $F_2^a \leqslant 0$. We have

$$F_1^a = F_1^b + kF_1^{b0} = F_1^b + k(u^{b0} - v^{b0} - z^{b0}).$$

Hence, by (18), (7) and the mean value theorem,

$$\begin{aligned} F_1^a &\leqslant F_1^b + k\left\{|\eta_b| + \left|\frac{\partial f}{\partial u}\right||w^b| + \sum_i \frac{\partial f}{\partial q_i} w^{bi} + \sum_{i,j} \frac{\partial f}{\partial r_{ij}} w^{bij} - z^{b0}\right\} \\ &= F_1^b + k\left\{\left|\frac{\partial f}{\partial u}\right| F^b + \sum_i \frac{\partial f}{\partial q_i} F_1^{bi} + \sum_{i,j} \frac{\partial f}{\partial r_{ij}} F_1^{bij}\right\} + \\ &\qquad + k\left\{|\eta_b| + \left|\frac{\partial f}{\partial u}\right| z^b + \sum_i \frac{\partial f}{\partial q_i} z^{bi} + \sum_{i,j} \frac{\partial f}{\partial r_{ij}} z^{bij} - z^{b0}\right\}. \end{aligned}$$

The expression in the last bracket is non-positive. Also $|\partial f/\partial u|\, F^b \leqslant 0$. Consequently,

$$\begin{aligned} F_1^a \leqslant F_1^b + k\Big\{ &\frac{1}{2h}\sum_i \frac{\partial f}{\partial q_i}(F_1^{i(b)} - F_1^{-i(b)}) + \frac{1}{h^2}\sum_i \frac{\partial f}{\partial r_{ii}}(F_1^{i(b)} - 2F_1^b + F_1^{-i(b)}) + \\ &+ \frac{1}{2h^2}\sum_{i,j(i\neq j)} \left|\frac{\partial f}{\partial r_{ij}}\right|(-F_1^{i(b)} - F_1^{-i(b)} - F_1^{j(b)} - F_1^{-j(b)} + \\ &\qquad + 2F_1^b + F_1^{i(-s(i,j)j(b))} + F_1^{-i(s(i,j)j(b))})\Big\}. \end{aligned}$$

Hence, rearranging the terms, we find (cf. [9])

$$\begin{aligned} F_1^a \leqslant \left(1 - \frac{2k}{h^2}\sum_{i=1}^n \frac{\partial f}{\partial r_{ii}}\right)F_1^b &+ \frac{k}{h}\sum_{i=1}^n \left[\frac{1}{h}\left(\frac{\partial f}{\partial r_{ii}} - \sum_{j=1, j\neq i}^n \left|\frac{\partial f}{\partial r_{ij}}\right|\right) + \frac{1}{2}\frac{\partial f}{\partial q_i}\right]F_1^{i(b)} + \\ &+ \frac{k}{h}\sum_{i=1}^n \left[\frac{1}{h}\left(\frac{\partial f}{\partial r_{ii}} - \sum_{j=1, j\neq i}^n \left|\frac{\partial f}{\partial r_{ij}}\right|\right) - \frac{1}{2}\frac{\partial f}{\partial q_i}\right]F_1^{-i(b)} + \\ &+ \frac{k}{2h^2}\sum_{i,j(i\neq j)} \left|\frac{\partial f}{\partial r_{ij}}\right|(2F_1^b + F_1^{i(-s(i,j)j(b))} + F_1^{-i(s(i,j)j(b))}). \end{aligned}$$

The values F_1^b, $F_1^{i(b)}$, $F_1^{-i(b)}$, $F_1^{i(-s(i,j)j(b))}$, $F_1^{-i(s(i,j)j(b))}$ are non-positive. Making use of (3), (4) and (8), we obtain $F_1^a \leqslant 0$. The inequality $F_2^a \leqslant 0$ can be shown similarly.

Note finally that if $L_1 = 0$ one can take $p = 1$ and repeat the proof, which now becomes simpler and the error less. Thus the proof is completed.

Remark 1. Our theorem related to linear equations requires the assumption that the coefficients at the second order mixed derivatives have a constant sign. However, it follows from the proof that in the linear case this assumption is superfluous.

Remark 2. The above result can easily be extended to the Cauchy problem for a weakly coupled system of the form

$$\begin{aligned} &u_{\nu,x_0} = f_\nu(x, u, u_{\nu,x}, u_{\nu,xx}) && \text{in } S, \\ &u_\nu(x) = g_\nu(x) && \text{for } x_0 = 0, \end{aligned} \qquad \nu = 1, \ldots, m,$$

where $u = (u_1, \ldots, u_m)$, $u_{\nu,x} = (u_{\nu,x_1}, \ldots, u_{\nu,x_n})$, $u_{\nu,xx} = (u_{\nu,x_i x_j})_{i,j=1}^n$.

Suppose that every function f_ν satisfies Assumption II except that the first inequality in (3) is replaced by $|\partial f_\nu/\partial u_s| \leqslant L$ ($\nu, s = 1, \ldots, m$). Then the above theorem remains valid for each component u_ν if we replace L_0 by mL. The proof merely undergoes slight obvious changes.

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