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On differentiable solutions of some systems of functional equations of p-th order

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Abstract. On the basis of the results of [3], a theorem on the existence of a differentiable solution Φ of the system of functional equations

$$\Phi(f^p(x)) = G(x, \Phi(x), \ldots, \Phi(f^{p-1}(x)))$$

is proved in this paper. Under some additional conditions, regularity of the solution Φ at the fixed point of the function f is investigated. Moreover, an example is discussed.

1. Introduction. The purpose of the present paper is to prove some theorems concerning the existence of differentiable solutions of the system of m functional equations of order p

$$\Phi(f^p(x)) = G[x, \Phi(x), \ldots, \Phi(f^{p-1}(x))], \tag{1}$$

where $\Phi \subset R \times R^m$ is an unknown function and functions $f \subset R \times R$ and $G \subset R^{mp+1} \times R^m$ are given.

The problem of the existence and uniqueness of differentiable solutions of the system of functional equations

$$\varphi_i(x) = h_i(x, \varphi_1(f_1(x)), \ldots, \varphi_m(f_1(x)), \ldots, \varphi_m(f_1(x)), \ldots, \varphi_m(f_n(x))),$$

where φ_i are unknown functions, was investigated by Z. Kominek in [1]. Most of the results of that paper have been obtained with use of fixed-point theorems. Here we are not able to apply these methods.

All the theorems proved in this paper concern the case of non-uniqueness.

Section 2 contains a lemma on the equivalence between system (1) and a certain system (2) of functional equations of first order, involving R^{mp}-functions:

$$\varphi(f(x)) = g(x, \varphi(x)), \tag{2}$$

where φ is the unknown function and the functions f and g are given. This section contains also a theorem on the existence of a C^r-solution of system (1) in an interval (a, b).

In Section 3 we formulate some sufficient conditions for a C^r-solution of system (2) in the interval (a, b) to be continuous at the point b.

The theorem on the existence of a C^r-solutions of system (1) in the interval $(a, b\rangle$ is the subject of Section 4.

Finally, in Section 5 we show an example illustrating the theorems proved in the previous sections.

Our considerations are based on the theory of differentiable solutions of equation (2) contained mainly in [4], and the results of [3].

2. C^r-solution in (a, b). The investigation of system (1) can be reduced to the investigation of a certain system of equations of first order. This is a consequence of the following

LEMMA 1 (cf. [4], p. 246, also [2], p. 54). *If the function f maps some number interval I into itself, then the equation*

$$\Phi\big(f^p(x)\big) = G\big[x, \Phi(x), \Phi\big(f(x)\big), \ldots, \Phi\big(f^{p-1}(x)\big)\big]$$

is equivalent to the equation

$$\varphi\big(f(x)\big) = g\big(x, \varphi(x)\big),$$

where the function

$$g = (g_1, \ldots, g_p), \quad g_i \subset R^{mp+1} \times R^m, \quad i = 1, \ldots, p,$$

is defined by

$$(3) \qquad g_i(x, y_1, \ldots, y_p) = y_{i+1}, \quad g_p(x, y_1, \ldots, y_p) = G(x, y_1, \ldots, y_p),$$
$$x \in I,\ y_i \in R^m,\ i = 1, \ldots, p-1.$$

This equivalence to be understood in the following sense: if a function $\Phi\colon I \to R^m$ satisfies in I equation (1), then the function $\varphi\colon I \to R^{mp}$ defined by

$$(4) \qquad \varphi(x) = \Big(\Phi(x), \Phi\big(f(x)\big), \ldots, \Phi\big(f^{p-1}(x)\big)\Big), \quad x \in I,$$

satisfies in I equation (2) with g defined by (3). Conversely, if a function $\varphi = (\varphi_1, \ldots, \varphi_p)$, $\varphi_i(x) \in R^m$, $i = 1, \ldots, p$, satisfies in I equation (2) with g defined by (3), then the function $\Phi = \varphi_1$ satisfies in I equation (1).

We start with quoting a theorem which was proved in [3]. First we remind the hypotheses of this theorem.

Let

$$g\colon R^{N+1} \supset \Omega \to R^N, \quad \Omega_x = \{y\colon (x, y) \in \Omega\},$$
$$\langle a, b\rangle \subset \{x\colon \Omega_x \neq \emptyset\}, \quad \Gamma_x = g(x, \Omega_x), \quad \Gamma = \bigcup_{x \in \langle a,b\rangle} \{x\} \times \Gamma_x,$$

and let r be a fixed positive integer. We assume that:

(I) $f\colon \langle a, b\rangle \to \langle a, b\rangle$, $f(a) = a$, $f(b) = b$, $f(x) > x$ for $x \in (a, b)$; $f \in C^r(\langle a, b\rangle)$, and $f'(x) > 0$ in $\langle a, b\rangle$,

(II) $g \in C^r(\Omega)$(¹) and for every $x \in \langle a, b\rangle$ the function $y \mapsto g(x, y)$ is invertible,

(III) $h \in C^r(\Gamma)$, where h denotes the function inverse to the function $y \mapsto g(x, y)$,

(IV) there exist sets $A_i \subset R^{q-1}$, $i = 1, 2, \ldots$, $B_j \subset R^{q-1}$, $j = 1, 2, \ldots$, where $q < N$, and functions

$$u_i\colon \langle a, b\rangle \times A_i \to R^N, \quad u_i \in C^1(\langle a, b\rangle \times A_i), \quad i = 1, 2, \ldots,$$

$$v_j\colon \langle a, b\rangle \times B_j \to R^N, \quad v_j \in C^1(\langle a, b\rangle \times B_j), \quad j = 1, 2, \ldots$$

such that

$$\Gamma_x - \Omega_{f(x)} = \bigcup_{i=1}^{\infty} u_i(x, A_i), \qquad \Omega_{f(x)} - \Gamma_x = \bigcup_{j=1}^{\infty} v_j(x, B_j);$$

moreover, the set $\Omega \cup \Gamma$ is a region in the space R^{N+1},

(V) there exist a point $(x_0, \eta) \in \Omega$, $x_0 \in (a, b)$ such that $\eta = g(x_0, \eta)$ and a $\varrho_0 > 0$ such that

$$\langle x_0, f(x_0)\rangle \times S_0 \subset \Omega \cup \Gamma,$$

where S_0 is the sphere

$$S_0 = \{y \in R^N\colon |y - \eta| \leqslant \varrho_0\}.$$

Then we have

THEOREM 1 (cf. [3], Theorem 2). *If hypotheses* (I)–(V) *are fulfilled, then for every ϱ with $0 < \varrho \leqslant \varrho_0$ and for every system of elements $l^k \in R^N$, $k = 1, \ldots, r$, there exists a function φ with the following properties:*

$$\varphi \in C^r\big((a, b)\big), \tag{5}$$

$$\varphi \textit{ satisfies system } (2) \textit{ in } (a, b), \tag{6}$$

$$|\varphi - \eta| \leqslant \varrho \quad \textit{for every } x \in \langle x_0, f(x_0)\rangle, \tag{7}$$

$$\varphi^{(k)}(x_0) = l^k, \quad k = 1, \ldots, r. \tag{8}$$

Now we are going to formulate a theorem concerning the properties of system (1) resulting from Lemma 1 and from Theorem 1. For this purpose we accept some hypotheses regarding the function G.

We assume that:

(II_G) $G\colon R^{mp+1} \supset \Omega \to R^m$, $G \in C^r(\Omega)$, and for every admissible $(x, y_2, \ldots, y_p)$ the function $y_1 \to G(x, y_1, y_2, \ldots, y_p)$ is invertible.

(III_G) $H \in C^r(\Gamma)$, where H denotes the inverse function to the function $y_1 \mapsto G(x, y_2, \ldots, y_p)$.

(¹) In the whole of this paper we understand the notion of C^r-class of a function in the global sense.

Here

$$\Gamma = \bigcup_{x\in\langle a,b\rangle} \bigcup_{(y_2,\dots,y_p)\in\Omega''_x} \{x\}\times\{(y_2,\dots,y_p)\}\times G(x,\Omega_x,y_2,\dots,y_p),$$

where

$$\Omega_x = \{(y_1,\dots,y_p):\ (x,y_1,\dots,y_p)\in\Omega\},$$
$$\Omega'_x = \{y_1:\ \exists(y_2,\dots,y_p):\ (y_1,\dots,y_p)\in\Omega_x\},$$

and

$$\Omega''_x = \{(y_2,\dots,y_p):\ \exists y_1:\ (y_1,\dots,y_p)\in\Omega_x\},$$
$$x\in\langle a,b\rangle,\ y_i\in R^m,\ i=1,\dots,p,$$

are non-empty subsets of the spaces R^{mp}, R^m, $R^{m(p-1)}$ respectively.

As consequence of Theorem 1 we get the following

THEOREM 1_G. *If hypotheses* (I), (II_G), (III_G) *are fulfilled and if hypotheses* (IV) *and* (V) *are fulfilled for the function* g *defined by* (3), *then for every* ϱ *with* $0<\varrho\leqslant\varrho_0$ *and for every system of elements* $L^k\in R^m$, $k=1,\dots,r$, *there exists a function* Φ *with the following properties:*

(9) $$\Phi\in C^r(\langle a,b)),$$

(10) Φ *satisfies system* (1) *in* (a,b),

(11) $$|\Phi-\eta|\leqslant\varrho \quad \textit{for every } x\in\langle x_0,f(x_0)\rangle,$$

(12) $$\Phi^{(k)}(x_0)=L^k,\quad k=1,\dots,r.$$

Proof. According to Lemma 1 it suffices to find a solution of equation (2) with g defined by (3). Putting $N=mp$ we see that hypothesis (II) results from hypothesis (II_G). The function h inverse to the function g given by (3) is defined by

(13) $$h_1(x,z_1,\dots,z_p)=H(x,z_1,\dots,z_p),\qquad h_i(x,z_1,\dots,z_p)=z_{i-1},\quad i=2,\dots,p.$$

Hypothesis (III_G) and formula (13) imply hypothesis (III). Consequently, the hypotheses of Theorem 1 are fulfilled. As elements l^k in Theorem 1 we take

$$l^k=(L^k,L_2^k,\dots,L_p^k),$$

where L_i^k, $i=2,\dots,p$, $k=1,\dots,r$, are arbitrary elements of the space R^m. By Theorem 1 there exists the function $\varphi=(\varphi_1,\dots,\varphi_p)$ with properties (5)–(8) and hence it follows, by Lemma 1, that the function $\Phi=\varphi_1$ fulfils conditions (9)–(12).

This completes the proof.

3. Continuity of a regular solution at the fixed point. Now we are going to formulate some sufficient conditions for the continuity of a C^r-

solution of system (2) in (a, b) at the point b (b is the fixed point of the function f). For this purpose we assume some additional hypotheses:

(VI) There exists a point $\boldsymbol{d} \in \Omega_b$ such that $\boldsymbol{g}(b, \boldsymbol{d}) = \boldsymbol{d}$.

(VII) There exist $\delta > 0$, $\varrho > 0$, and points $x_0 \in (b-\delta, b)$, $\boldsymbol{\eta} = \Omega_{x_0}$ such that

$$\boldsymbol{g}(x_0, \boldsymbol{\eta}) = \boldsymbol{\eta} \quad \text{and} \quad |\boldsymbol{\eta}-\boldsymbol{d}| \leqslant \varrho/2.$$

We are now in a position to prove the following

THEOREM 2. *If hypotheses* (I)–(IV), (VI), (VII), *and*

$$\text{(VIII)} \qquad \left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}(b, \boldsymbol{d}) \right\| < 1 \text{ (}^2\text{)}$$

are fulfilled, then there exists a C^r-solution of system (2) *which is continuous at the point b.*

Proof. The theorem will be proved if we show that the function $\boldsymbol{g}$ fulfils a Lipschitz condition with respect to $\boldsymbol{y}$ with a constant less than 1 in a neighbourhood of $(b, \boldsymbol{d})$. Indeed, hypothesis (VII) implies hypothesis (V); thus Theorem 1 implies the existence of a C^r-solution of system (2) in (a, b) which, in particular, fulfils condition (7). Then, to complete the proof, we may repeat the same argument as in the proof of Theorem 12.9 ([4], p. 252, cf. also [4], p. 75, Theorem 3.6) in which an essential role is played by the Lipschitz condition.

On account of hypothesis (II) and the continuity of the function

$$(x, \boldsymbol{y}) \mapsto \left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}(x, \boldsymbol{y}) \right\| \tag{14}$$

there exist $\delta > 0$, $\varrho > 0$ and $0 < \vartheta < 1$ such that

$$\left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}(x, \boldsymbol{y}) \right\| < \vartheta \quad \text{for } (x, \boldsymbol{y}) \in V \cap \Omega, \tag{15}$$

where

$$V = (b-\delta, b\rangle \times \{\boldsymbol{y}\colon |\boldsymbol{y}-\boldsymbol{d}| \leqslant \varrho\} \subset \Omega \cup \Gamma.$$

Further, we remark that we are not able to apply the mean-value theorem to the function $\boldsymbol{g}$ (according to hypothesis (IV), a required segment need not be included in a correspondent part of the domain of $\boldsymbol{g}$), but we can omit this inconvenience in the following way:

(²) The norm of a matrix $A = [a_{ij}]$, $i, j = 1, \ldots, k$, is to be understand as the operator norm

$$\|A\| = \sup_{|\boldsymbol{u}|=1} |A\boldsymbol{u}|, \quad \text{where } \boldsymbol{u} \in R^k.$$

From hypothesis (II) it follows that there exist a neighbourhood U of the point (b, d) and a function $\bar{g}\colon U \to R^N$ such that $\bar{g} \in C^r(U)$ and $\bar{g}|_{U\cap\Omega} = g$.

We suppose additionally that V has been chosen in such a way that $V \subset U$. For arbitrary points $(x, y) \in V\cap\Omega$ and $(x, \bar{y}) \in V\cap\Omega$ we have the equality

$$\bar{g}(x, y) - \bar{g}(x, \bar{y}) = \overline{\mathfrak{M}}(y - \bar{y}), \tag{16}$$

where

$$\overline{\mathfrak{M}} = \left[\frac{\partial \bar{g}_i}{\partial y_j}(x, \bar{\xi}_{ij})\right], \quad (x, \bar{\xi}_{ij}) \in V, \quad i, j = 1, \ldots, N.$$

In the sequel we make use of the set identity

$$V = (V\cap\Omega)\cup\big(V\cap(\Gamma\setminus\Omega)\big). \tag{17}$$

On account of hypothesis (IV) and the Sard theorem ([6], [5], cf. also [3]) we have

$$m_{N+1}(\Gamma - \Omega) = m_{N+1}\Big(\bigcup_{i=1}^{\infty} \bigcup_{x\in\langle a,b\rangle} \{x\} \times u_i\big(f^{-1}(x), A_i\big)\Big) = 0$$

($m_{N+1}(A)$ denotes the $(N+1)$-dimensional Lebesgue measure of the set A); thus the set $V\cap(\Gamma\setminus\Omega)$ has no interior in the set V. From this by (17) we conclude that the set $V\cap\Omega$ is dense in V. Hence, by the continuity of function (14) for every $0 < \varepsilon < 1-\vartheta$ (ϑ from (15)) there exists a point $(x, \xi_{ij}) \in V\cap\Omega$ such that

$$\|\overline{\mathfrak{M}}\| \leqslant \|\mathfrak{M}\| + \varepsilon, \tag{18}$$

where

$$\mathfrak{M} = \frac{\partial g_i}{\partial y_j}(x, \xi_{ij}) = \frac{\partial \bar{g}_i}{\partial y_j}(x, \xi_{ij}), \quad i, j = 1, \ldots, N.$$

By (16), (18) and the properties of the matrix norm we have

$$\begin{aligned} |g(x, y) - g(x, \bar{y})| &= |\bar{g}(x, y) - \bar{g}(x, \bar{y})| = |\overline{\mathfrak{M}}(y - \bar{y})| \\ &\leqslant \|\overline{\mathfrak{M}}\|\,|y - \bar{y}| \leqslant (\|\mathfrak{M}\| + \varepsilon)\,|y - \bar{y}| \leqslant (\theta + \varepsilon)\,|y - \bar{y}| \\ &\qquad \text{for } (x, y) \in V\cap\Omega \text{ and } (x, \bar{y}) \in V\cap\Omega, \end{aligned}$$

where $\vartheta + \varepsilon < 1$, because $0 < \varepsilon < 1-\vartheta$.

This means that the function g fulfils a contractive Lipschitz condition in y. This completes the proof.

Now we are going to formulate another theorem giving sufficient condition for the continuity of C^r-solution of equation (2) in (a, b) at the point b.

Replacing in Theorem 2 hypothesis (VIII) by the hypothesis

$$0 < \lambda_0 < 1, \tag{IX}$$

where $\lambda_0 = \max_{1\leqslant \varkappa \leqslant N} |\lambda_\varkappa|$, and $\lambda_\varkappa$ are the characteristic roots of the matrix $\frac{\partial g}{\partial y}(b, d)$ we obtain

THEOREM 3. *If hypotheses* (I)–(IV) *and* (VI), (VII), (IX) *are fulfilled, then there exists a C^r-solution of system* (2), *in* (a, b), *which is continuous at the point* b.

Proof. From a result by A. Ostrowski ([7], p. 151, also [2], p. 67, Lemma 7) it follows that for an $0 < \varepsilon < 1-\lambda_0$ there exists a non-singular matrix A such that

$$\left\| A \frac{\partial g}{\partial y}(b, d) A^{-1} \right\| \leqslant \lambda_0 + \varepsilon < 1.$$

Now we consider the linear map T: $R^N \to R^N$ defined by $y^* = Ay$ and the function $g^*(x, y^*) = Ag(x, A^{-1}y^*)$. We remark that

$$\frac{\partial g^*}{\partial y^*}(b, d^*) = A \frac{\partial g}{\partial y}(b, d) A^{-1};$$

thus the function g^* fulfils hypothesis (VIII).

Further, we see that equation (2) is equivalent to the equation

$$(2^*) \qquad \varphi^*\big(f(x)\big) = g^*\big(x, \varphi^*(x)\big), \quad \text{where } \varphi^*(x) = A\varphi(x),$$

in the following sense: if a function φ satisfies system (2) in $(a, b\rangle$, then the function φ^* satisfies system (2^*) in $(a, b\rangle$, and conversely, if a function φ^* satisfies system (2^*) in $(a, b\rangle$, then the function $\varphi = A^{-1}\varphi^*$ satisfies system (2) in $(a, b\rangle$.

On account of hypotheses (I)–(IV), (VI), (VII) and the properties of the map T the function g^* fulfils analogous hypotheses involving the sets $\Omega^*, \Gamma^*, \Omega^*_x, \Gamma^*_x$, respectively.

Finally, we may apply Theorem 2 to system (2^*), which is equivalent to system (2) in the sense just described. This completes the proof.

In virtue of Lemma 1 we may formulate analogous theorems for system (1).

For example, hypothesis (IX) obtains the form

$$(\mathrm{IX}_G) \qquad 0 < \lambda_0 < 1,$$

where $\lambda_0 = \max_{1\leqslant \varkappa \leqslant mp} |\lambda_\varkappa|$ are the characteristic roots of the matrix

$$\begin{bmatrix} 0 & I \\ \frac{\partial G}{\partial y} & (b, d^p) \end{bmatrix},$$

and $\mathbf{0}$ denotes the zero $(p-1)m\times m$-matrix, I denotes the unit $(p-1)m\times(p-1)m$-matrix, $\boldsymbol{d}^p = \big((\boldsymbol{d},\ldots,\boldsymbol{d})\colon p \text{ times}\big)$.

As a consequence of Theorem 3 we have the following

THEOREM 3_G. *If hypotheses* (I), (II_G), (III_G), (IX_G) *are fulfilled, and if hypotheses* (IV), (VI), *and* (VII) *are fulfilled for the function* $\boldsymbol{g}$ *defined by* (3), *then there exists a* C^r*-solution of system* (1) *in* (a, b), *which is continuous at the point* b.

4. C^r-solutions in the interval $(a, b\rangle$. In this section we formulate a theorem on the existence of a C^r-solution of system (1) in the interval $(a, b\rangle$.

Because the k-th derivative of the function satisfying system (1) is a solution of some linear functional equation of p-th order, we first prove a lemma on continuous solutions in $(a, b\rangle$ of the equation

$$(\mathrm{L}^p)\qquad \boldsymbol{\Phi}\big(f^p(x)\big) = \boldsymbol{F}(x) + \sum_{i=0}^{p-1} \boldsymbol{a}_i(x)\,\boldsymbol{\Phi}\big(f^i(x)\big),$$

where the functions $\boldsymbol{F} \subset R\times R^m$, $\boldsymbol{a}_i \subset R\times R^{m^2}$ (the values of $\boldsymbol{a}_i$ are square matrices of rank m), $i = 0,\ldots,p-1$, $f\subset R\times R$ are known, and the function $\boldsymbol{\Phi}\subset R\times R^m$ is unknown.

LEMMA 2. *If the known functions in equation* (L^p) *are continuous in* $(a, b\rangle$, *the function* f *fulfils hypothesis* (I) *with* $r = 0$ *and*

$$0<\sigma_0<1,$$

where $\sigma_0 = \max\limits_{1\leqslant\varkappa\leqslant mp} |\sigma_\varkappa|$, $\sigma_\varkappa$ *are the characteristic roots of the matrix*

$$\begin{bmatrix} \mathbf{0} & & I \\ \boldsymbol{a}_0(b) & \ldots & \boldsymbol{a}_{p-1}(b)\end{bmatrix}$$

($\mathbf{0}$ *and* I *as in hypothesis* (IX_G)), *then any solution of equation* (L^p) *which is continuous in the interval* (a, b) *is also continuous at the point* b.

Proof. Let $\boldsymbol{\Phi}$ be a continuous solution of equation (L^p) in (a, b). Then by Lemma 1 the function $\boldsymbol{\varphi}$ defined by (4) is a continuous solution of the equation

$$(\mathrm{L})\qquad \mathfrak{a}\big(f(x)\big) = \mathfrak{a}(x)\,\mathfrak{a}(x) + \mathfrak{b}(x),$$

where

$$\mathfrak{a}(x) = \underbrace{\begin{bmatrix} \mathbf{0} & & I \\ \boldsymbol{a}_0(x) & \ldots & \boldsymbol{a}_{p-1}(x)\end{bmatrix}}_{mp}\Bigg\}mp$$

($\mathbf{0}$ and $\boldsymbol{I}$ as above) and

$$\mathfrak{b}(x) = \left[\begin{matrix}\mathbf{0}\\ \underbrace{\boldsymbol{F}(x)}_{m}\end{matrix}\right]\Big\}{mp}, \qquad x \in (a, b).$$

On account of A. Ostrowski's result ([7], p. 151) we choose a non-singular matrix $\boldsymbol{A}$ such that

$$(*) \qquad \|\mathfrak{a}^*(b)\| < 1,$$

where

$$\mathfrak{a}^*(x) = \boldsymbol{A}\,\mathfrak{a}(x)\boldsymbol{A}^{-1}.$$

Now, the function

$$\boldsymbol{\varphi}^*(x) = \boldsymbol{A}\boldsymbol{\varphi}(x)$$

is in (a, b) a continuous solution of the equation

$$(\mathrm{L}^*) \qquad \boldsymbol{a}^*(f(x)) = \mathfrak{a}^*(x)\,\boldsymbol{a}^*(x) + \mathfrak{b}^*(x),$$

where

$$\mathfrak{b}^*(x) = \boldsymbol{A}\mathfrak{b}(x).$$

From condition $(*)$ it follows that there exists exactly one $\boldsymbol{d}^* \in R^{mp}$ such that

$$\boldsymbol{d}^* = \mathfrak{a}^*(b)\boldsymbol{d}^* + \mathfrak{b}^*(b).$$

Putting

$$\boldsymbol{\varphi}^*(b) = \boldsymbol{d}^*,$$

we prove as in [4], Theorem 2.9, p. 57, that the function $\boldsymbol{\varphi}^*$ is a continuous solution of equation (L*) in (a, b), continuous at the point b. Thus the function $\boldsymbol{\varphi} = \boldsymbol{A}^{-1}\boldsymbol{\varphi}^*$ is a continuous solution of equation (L) in $(a, b\rangle$. Finally, by Lemma 1, we obtain that $\boldsymbol{\Phi}$ is a continuous solution of equation (L^p) in $(a, b\rangle$.

In the sequel we assume the inequality

$$(\mathrm{IX}_G^r) \qquad 0 < \lambda_0[f'(b)]^{-r} < 1,$$

where λ_0 is defined in hypothesis (IX_G).

We aim at proving the following

THEOREM 3_G^r. *If hypotheses* (I), (II_G), (III_G), (IX_G^r) *are fulfilled, and if hypotheses* (IV), (VI), (VII) *are fulfilled for the function* $\boldsymbol{g}$ *defined by* (3), *then there exists a* C^r*-solution of system* (1) *in* $(a, b\rangle$.

Proof. From the accepted hypotheses it follows that the assumptions of Theorem 3_G are fulfilled. In particular, from (IX_G^r) we obtain (IX_G), because $0 < f'(b) \leqslant 1$. Thus, by Theorem 3_G, there exists a function $\boldsymbol{\Phi}$ in (a, b) which is a C^r-solution of system (1) and is continuous at the point b.

Observe that the k-th derivative of the function Φ satisfies the linear equation of p-th order

$$(19)\quad \Phi^{(k)}\big(f^p(x)\big) = \sum_{i=1}^{p} \frac{\partial G}{\partial y_i}\big[x, \Phi(x), \ldots, \Phi\big(f^{p-1}(x)\big)\big]\Big[\prod_{j=i}^{p} f'\big(f^{j-1}(x)\big)\Big]^{-k} \Phi^{(k)}\big(f^{i-1}(x)\big) + F_k\big[x, \Phi(x), \ldots, \Phi\big(f^{p-1}(x)\big); \Phi'(x), \ldots, \Phi'\big(f^{p-1}(x)\big); \ldots \ldots; \Phi^{k-1}(x), \ldots, \Phi^{(k-1)}\big(f^{p-1}(x)\big)\big]$$

for every $x \in (a, b)$.

Now we make use of Lemma 2. For this purpose we have to study characteristic roots of the matrix

$$(20)\qquad \mathfrak{N} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline \mathfrak{M}_1[f'(b)]^{-kp} & \mathfrak{M}_2[f'(b)]^{-k(p-1)} \ \ldots\ \mathfrak{M}_p[f'(b)]^{-k} \end{array}\right],$$

where

$$\mathfrak{M}_j = \frac{\partial G}{\partial y_j}(b, d^p), \quad j = 1, \ldots, p,$$

$$\frac{\partial G}{\partial y_j}(b, d^p) = \left[\frac{\partial G_i}{\partial y_{j,\nu}}(b, \underbrace{d, \ldots, d}_{mp})\right], \quad i, \nu = 1, \ldots, m,$$

$\mathbf{0}$ denotes the $m(p-1) \times m$ zero matrix, $\mathbf{I}$ the unit matrix of order $m(p-1)$.

We notice that if λ is an eigenvalue of the matrix

$$\mathfrak{P} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline \mathfrak{M}_1 & \mathfrak{M}_2, \ldots, \mathfrak{M}_p \end{array}\right],$$

then $\lambda[f'(b)]^{-k}$ is an eigenvalue of matrix (20), and conversely. This fact follows from the equality

$$\det(\mathfrak{P} - \lambda\mathbf{I}) = \big(f'(b)\big)^{-kpm}\det\big(\mathfrak{N} - \lambda[f'(b)]^{-k}\mathbf{I}\big).$$

Writing

$$\lambda_0[f'(b)]^{-k} = \max_{1 \leqslant \varkappa \leqslant mp} |\lambda_\varkappa|\,[f'(b)]^{-k},$$

from (IX_G^r) and by hypothesis (I) we obtain the inequality

$$(21)\qquad 0 < \lambda_0[f'(b)]^{-k} < 1.$$

It is easy to observe, by (21), that the numerical system of equations

$$d_k = \sum_{i=1}^{p} \mathfrak{M}_i[f'(b)]^{-k(p-i+1)} d_k + c_k,$$

where c_k are defined by formulae

$$c_1 = \frac{\partial G}{\partial x}(b, d^p)[f'(b)]^{-p},$$

$$c_{k+1} = F_k(b, d^p, d_1^p, \ldots, d_{k-1}^p), \quad k = 1, \ldots, r,$$

has exactly one solution.

Now we assume that

$$\Phi^{(k)}(b) = d_k, \quad k = 1, \ldots, r.$$

For $k = 1$, by hypotheses (I), (II_G), by inequality (21), from the fact that $\Phi \in C^0(\langle a, b\rangle)$ and that Φ' satisfies equation (19), on account of Lemma 2 we obtain $\lim_{x\to b} \Phi'(x) = d_1$. In the sequel we assume that for fixed $k = s-1$ the functions $\Phi, \Phi', \ldots, \Phi^{s-1}$ are continuous in $(a, b\rangle$. The function $\Phi^{(s)}$ satisfies equation (19) for $k = s$. From the inductive assumption it follows that the coefficients in equation (19) are continuous functions. Condition (21) is also fulfilled; thus by Lemma 2 $\lim_{x\to b_-} \Phi^{(s)}(x) = d_s$. Consequently the function Φ is a C^r-solution of system (1) in $(a, b\rangle$. This completes the proof.

5. An example. Consider a linear equation of the form

$$(\mathrm{L}^p) \qquad \sum_{i=0}^{p} a_i(x)\Phi\big(f^i(x)\big) = F(x).$$

We know (cf. [4], p. 259) that it is possible to reduce the order of equation (L^p) by the substitution

$$\Psi_{(x)} = \Phi\big(f(x)\big) - \Lambda(x)\Phi(x),$$

provided that the function Λ satisfies the equation

$$(\mathrm{N}^{p-1}) \qquad a_0(x) + \sum_{i=1}^{p} a_i(x)\prod_{j=0}^{i-1} \Lambda\big(f^{j-1}(x)\big) = 0.$$

Then, for the function Ψ we obtain a linear equation of order $p-1$.

Equation (N^{p-1}) (Λ is unknown) is of a lower order than (L^p) but it is not linear. The known theorems do not apply in this case. From theorems proved in the present paper we get some information about equation (N^{p-1}) for complex-valued function of the real variable.

Let the functions a_i map $\langle a, b\rangle$ into C, C — the set of complex number, $a_i \in C^r(\langle a, b\rangle)$, $i = 0, \ldots, p$, and let the function f fulfil hypothesis (I).

We assume that $a_p(x) \neq 0$ and $a_0(x) \neq 0$ for every $x \in \langle a, b\rangle$. Then denoting

$$b_i(x) = -\frac{a_i(x)}{a_p(x)}, \quad i = 0, \ldots, p-1,$$

we may write equation (N^{p-1}) in the form

$$(1^{\Lambda})\qquad \Lambda\big(f^{p-1}(x)\big) = \sum_{i=0}^{p-2} \frac{b_i(x)}{\prod_{j=i}^{p-2} \Lambda\big(f^j(x)\big)} + b_{p-1}(x).$$

On account of Lemma 1, equation (1^{Λ}) is equivalent to the system

$$(1^{\lambda})\qquad \lambda\big(f(x)\big) = g\big(x, \lambda(x)\big),$$

where the known function g is of the form

$$g\colon \Omega \ni (x, y_0, \ldots, y_{p-2}) \mapsto \Bigg(y_1, \ldots, y_{p-2}, \sum_{i=0}^{p-2} \frac{b_i(x)}{\prod_{j=i}^{p-2} y_j} + b_{p-1}(x)\Bigg),$$

$$y_i \in C,\ i = 0, \ldots, p-2,$$

$$\Omega = \langle a, b\rangle \times E,$$

and

$$E = \{(y_0, \ldots, y_{p-2})\colon\ y_i \in C,\ y \neq 0 \text{ for every } i = 0, \ldots, p-2\}.$$

Observe that $\Omega_x = E$, and

$$\Gamma_x = \Bigg\{(z_0, \ldots, z_{p-2})\colon\ z_i \in C,\ z_i \neq 0 \text{ for every } i = 0, \ldots, p-3,\ z_{p-2} \neq \sum_{i=1}^{p-2} \frac{b_i(x)}{\prod_{j=i-1}^{p-3} z_j} + b_{p-1}(x)\Bigg\};$$

thus we may write

$$\Gamma_x - \Omega_{f(x)} = u(x, C^{p-2})$$

and

$$\Omega_{f(x)} - \Gamma_x = v(x, B),$$

where the functions u and v are defined by

$$u\colon \langle a, b\rangle \times C^{p-2} \ni (x, y_0, \ldots, y_{p-3}) \mapsto (y_0, \ldots, y_{p-3}, 0) \in C^{p-1},$$

$$v\colon \langle a, b\rangle \times B \ni (x, y_0, \ldots, y_{p-3}) \mapsto \Bigg(y_0, \ldots, y_{p-3}, \sum_{i=1}^{p-2} \frac{b_i(x)}{\prod_{j=i-1}^{p-3} y_j} + b_{p-1}(x)\Bigg) \in C^{p-1}$$

and

$$B = \{(y_0, \ldots, y_{p-3})\colon\ y_i \in C,\ y_i \neq 0 \text{ for every } i = 0, \ldots, p-3\}.$$

In this case, $q = 2p-3$, $N = 2p-2$ and, in fact, $q < N$ (q, N from (IV)). Moreover, u and v are of required regularity.

Hypothesis (V) is fulfilled, as well, because the condition $\boldsymbol{g}(x_0, \boldsymbol{\eta}) = \boldsymbol{\eta}$, $\boldsymbol{\eta} = (\eta_0, \ldots, \eta_{p-2})$ is now equivalent to the condition

$$\sum_{i=0}^{p} a_i(x_0)\eta_0^i = 0.$$

Thus from Theorem 2 it follows that there exists a function Λ of class C^r which satisfies equation (N^{p-1}) in (a, b). (In fact, there exist infinitely many such functions.)

Consequently there exists a C^r-solution of equation (N^{p-1}) in $(a, b\rangle$.

One can easily verify that hypothesis (VI) is also fulfilled. Further, since the functions a_i, $i = 0, \ldots, p$, are continuous and since the roots of an equation of p-th order depend continuously upon the coefficients of the equation (cf. [8], Theorem 76, p. 211–218), it follows that hypothesis (VII) is fulfilled too.

Suppose, moreover, that the roots of the equation

$$(\omega) \qquad \det\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{y}}(b, \boldsymbol{d}) - \omega I\right) = \sum_{j=0}^{p-1}\left(\sum_{i=0}^{j} a_i(b)\,d^i\right)\omega^j = 0$$

are less in absolute value than 1; here $\boldsymbol{d}$ satisfies the equation $\boldsymbol{g}(b, \boldsymbol{d}) = \boldsymbol{d}$, i.e. $\sum_{i=0}^{p} a_i(b)\,d^i = 0$. Then, on account of Theorem 3_G, there exists a C^r-solution of equation (N^{p-1}) in (a, b) such that $\lim_{x\to b-} \Lambda(x) = d$.

Finally, if we accept the inequality

$$0 < \omega_0[f'(b)]^{-r} < 1,$$

where $\omega_0 = \max_{1\leqslant \varkappa\leqslant p-1} |\omega_\varkappa|$, $\omega_\varkappa$ are the roots of equation (ω), then we may apply Theorem 3_G^r to equation (N^{p-1}) thus obtaining the existence of a C^r-solution of this equation in the interval $(a, b\rangle$.

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