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On partial differential inequalities of the first order

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This note deals with systems of inequalities of the form

$$u_x^i \leqslant f^i(x, Y, U, u_Y^i) \quad (i = 1, \ldots, m),$$

where $Y = (y_1, \ldots, y_n)$, $U = (u^1, \ldots, u^m)$, $u_Y^i = (u_{y_1}^i, \ldots, u_{y_n}^i)$. The theorem proved below is a generalization of a differential inequalities theorem concerning the case when strong initial inequalities and weak differential inequalities imply strong inequalities between functions in question in a suitable domain. This case is treated in the monography *Differential inequalities* by J. Szarski ([2], Th. 59.2, p. 179 - 181), by means of the characteristics method, under considerably stronger assumptions.

For any two vectors $U = (u^1, \ldots, u^m)$ and $V = (v^1, \ldots, v^m)$ we shall write

$$U \leqslant V \text{ if } u^j \leqslant v^j \ (j = 1, \ldots, m)$$

and

$$U < V \text{ if } u^j < v^j \ (j = 1, \ldots, m).$$

The index i being fixed we write

$$U \overset{i}{\leqslant} V \text{ if } u^j \leqslant v^j \ (j = 1, \ldots, m) \text{ and } u^i = v^i.$$

Definition. A region D in the space $(x, Y, U, Q) = (x, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n)$ will be called *positive* with respect to U if whenever $(x, Y, U, Q) \in D$ and $V \geqslant U$, then $(x, Y, V, Q) \in D$.

Condition W. *System $f^i(x, Y, U, Q)$ $(i = 1, \ldots, m)$ of functions defined in a domain D positive with respect to U is said to satisfy condition* W *with respect to U if for any fixed i, $U \overset{i}{\leqslant} V$ implies $f^i(x, Y, U, Q) \leqslant f^i(x, Y, V, Q)$.*

CONDITION C. *A function $f^i(x, Y, U, Q)$ is said to satisfy condition C with respect to u^i if $u^i \leqslant \tilde{u}^i$ implies*

$$(1)\qquad f^i(x, Y, u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^m, Q) - \\ -f^i(x, Y, u^1, \ldots, u^{i-1}, \tilde{u}^i, u^{i+1}, \ldots, u^m, Q) \leqslant \sigma(x-x_0, u^i-\tilde{u}^i),$$

where the function $\sigma(t, z)$ has the following properties:

1° *$\sigma(t, z)$ is continuous and non-negative in the half-strip $t \in \langle 0, a)$, $z \leqslant 0$, and $\sigma(t, 0) \equiv 0$,*

2° *the left-hand minimum solution of the equation*

$$(2)\qquad \frac{dz}{dt} = \sigma(t, z)$$

satisfying the condition $\lim_{t\to a-} z(t) = 0$ is $z(t) \equiv 0$ (¹).

By an argument similar to that used in the proof of Lemma 14.1 in [2] one can prove the following

LEMMA 1. *If $\sigma(t, z)$ has property 1°, then the right-hand minimum solution $\omega_+(t; t_0, z_0)$ of the equation*

$$(3)\qquad \frac{dz}{dt} = -\sigma(t, -z)$$

through (t_0, z_0), $0 \leqslant t_0 < a$, $z_0 \geqslant 0$, exists and is non-negative in $\langle t_0, a)$. Moreover, $\omega_+(t; t_0, 0) \equiv 0$.

LEMMA 2. *If $\sigma(t, z)$ has properties 1° and 2°, and $z_0 > 0$, then the right-hand minimum solution $\omega_+(t; 0, z_0)$ of (3) is positive in $\langle 0, a)$.*

Proof. $\omega_+(t; 0, z_0)$ is positive in a right-hand neighbourhood of $t = 0$. Suppose that for some $t' \in (0, a)$, $\omega_+(t'; 0, z_0) = 0$. Then, by Lemma 1, $\omega_+(t; 0, z_0) \equiv 0$ for $t \in (t', a)$. Thus we would have a solution such that $\lim_{t\to a-} \omega_+(t; 0, z_0) = 0$ and which takes on some positive values in $\langle 0, a)$. On the other hand, by assumption 2°, $z(t) \equiv 0$ is the left-hand maximum solution of (3) satisfying $\lim_{t\to a-} z(t) = 0$. This contradiction completes the proof.

LEMMA 3. *Let $\sigma(t, z)$ satisfy 1° and 2°. For z_0, $\delta > 0$ denote by $\omega_+(t; 0, z_0, \delta)$ the right-hand minimum solution of the equation*

$$(4)\qquad \frac{dz}{dt} = -\sigma(t, -z) - \delta$$

through $(0, z_0)$. If $z_0 > 0$ is fixed, then to every $\varepsilon > 0$ there corresponds $\delta_0(\varepsilon) > 0$

(¹) A similar condition was introduced in [1].

such that for $0 < \delta < \delta_0$ *the solution* $\omega_+(t; 0, z_0, \delta)$ *exists and is positive in* $\langle 0, a-\varepsilon \rangle$.

Proof of Lemma 3 follows from Lemma 2 and the continuous dependence of solutions on the right-hand side of equation (cf. [2]).

THEOREM. *We make the following assumptions:*

1° *The functions* $f^i(x, Y, U, Q) = f^i(x, y_1, \ldots, y_n, u^1, \ldots, u^m, q_1, \ldots, q_n)$ $(i = 1, \ldots, m)$ *are defined in a region which is positive with respect to* U *and whose projection on the space of points* (x, Y) *contains the pyramid*

$$(5) \qquad D: 0 \leqslant x - x_0 < a, \quad |y_k - \mathring{y}_k| \leqslant a_k - L(x - x_0) \quad (k = 1, \ldots, n),$$

where $0 \leqslant L < +\infty$, $0 < a_k < +\infty$, $a = \min(a_k/L)$.

Furthermore they satisfy condition W *with respect to* U *and the Lipschitz condition with respect to* Q

$$(6) \qquad |f^i(x, Y, U, Q) - f^i(x, Y, U, \tilde{Q})| \leqslant L \sum_{k=1}^{n} |q_k - \tilde{q}_k| \qquad (i = 1, \ldots, m).$$

2° *Every function* f^i *satisfies condition* C *with respect to* u^i.

3° $U(x, Y) = (u^1(x, Y), \ldots, u^m(x, Y))$ *and* $V(x, Y) = (v^1(x, Y), \ldots, v^m(x, Y))$ *are continuous in* D *and satisfy the initial inequalities*

$$(7) \qquad U(x_0, Y) < V(x_0, Y).$$

4° *Define* $E = \{(x, Y) \in D: U(x, Y) \leqslant V(x, Y)\}$ *and let functions* $U(x, Y)$, $V(x, Y)$ *have first order derivatives for* $(x, Y) \in E$ *and, moreover, Stolz's differentials if* (x, Y) *belongs to the side surface of* D, *and satisfy on* E *the systems of differential inequalities*

$$\begin{array}{ll}(8) & u^i_x(x, Y) \leqslant f^i\big(x, Y, U(x, Y), u^i_Y(x, Y)\big) \\ (9) & v^i_x(x, Y) \geqslant f^i\big(x, Y, V(x, Y), v^i_Y(x, Y)\big)\end{array} \qquad (i = 1, \ldots, m).$$

Under these assumptions the inequalities

$$(10) \qquad U(x, Y) < V(x, Y)$$

are satisfied in the pyramid D.

Proof. In Lemma 3 we choose $0 < z_0 < \min\{\min_j \min_Y [v^j(x_0, Y) - u^j(x_0, Y)]\}$ and δ so that $\omega_+(t; 0, z_0, \delta) \geqslant 0$ in $\langle 0, a-\varepsilon \rangle$. We shall write shortly $\omega(t)$ instead of $\omega_+(t; 0, z_0, \delta)$. Denote $\Omega(t) = (\omega(t), \ldots, \omega(t))$, $\tilde{u}^i(x, Y) = u^i(x, Y) + \omega(x - x_0)$, $\tilde{U}(x, Y) = U(x, Y) + \Omega(x - x_0)$. Observe that

$$(11) \qquad \tilde{U}(x_0, Y) < V(x_0, Y).$$

We define $\tilde{E}^i = \{(x, Y) \in D: \tilde{U}(x, Y) \overset{i}{\leqslant} V(x, Y)\}$. It is evident that if a point (x, Y) belongs to $\tilde{E}^i$, it belongs to E too.

Consequently, for every i inequalities (8) and (9) hold true on $\tilde{E}^i$. Now we add (8) and (4) (with $z = \omega$) and apply successively conditions O and W to obtain

$$\begin{aligned}\tilde{u}_x^i &\leqslant f^i(x, Y, U, u_Y^i) - \sigma\big(x - x_0, -\omega(x-x_0)\big) - \delta \\ &\leqslant f^i(x, Y, u^1, \ldots, u^{i-1}, \tilde{u}^i, u^{i+1}, \ldots, u^m, \tilde{u}_Y^i) - \delta \\ &\leqslant f^i(x, Y, \tilde{U}, \tilde{u}_Y^i) - \delta.\end{aligned}$$

Since $\delta > 0$ we finally get

$$\tilde{u}_x^i < f^i(x, Y, \tilde{U}, \tilde{u}_Y^i) \quad \text{for } (x, Y) \in \tilde{E}^i. \tag{12}$$

In view of (12), (9), (11) we can employ the theorem on strong differential inequalities ([2], p. 169) to deduce

$$\tilde{U}(x, Y) < V(x, Y)\cdot$$

which, because of $\Omega(x - x_0) \geqslant 0$, implies (10) for $(x, Y) \in D$, $0 \leqslant x - x_0 \leqslant a - \varepsilon$. Therefore, since ε is arbitrary, inequality (10) holds true in D.

Remark 1. If all the assumptions of the theorem on weak differential inequalities are satisfied (see [2], Th. 59.1) with the exception that one of the weak differential inequalities (of the form (8) or (9)) is replaced by the strong inequality (for $(x, Y) \in D$, $x > x_0$), then (10) holds for $(x, Y) \in D$, $x > x_0$. This statement can be proved by applying the weak differential inequalities theorem and then repeating the argument used in the proof of the theorem on strong differential inequalities (see [2], Th. 57.1).

Remark 2. The above result can be extended to the overdetermined systems of first order partial differential inequalities considered in [2] in an evident way.

References

[1] A. Lasota, *Sur l'effet épidermique extérieur et intérieur pour les inégalités différentielles ordinaires*, Ann. Polon. Math. 6 (1959), p. 259–264.

[2] J. Szarski, *Differential inequalities*, Warszawa 1965.

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