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An alternative concept of the n-dimensional measure

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Dedicated to the memory of Jacek Szarski

Abstract. A coefficient $\mu_n(X)$ is introduced for every compactum X and every $n = 0, 1, \ldots$ In the case where $X = P$ is a polyhedron, $\mu_n(P)$ is the n-dimensional measure of P (in the usual elementary sense), but for arbitrary compacta, $\mu_n(X)$ differs – in general – from the n-dimensional measure of X (in the classical sense of Carathéodory or of Hausdorff). Some properties of $\mu_n(X)$ are established and some open questions are formulated.

1. Introduction. The concept of the n-dimensional measure of a polytope belongs to the elementary geometry and any essential modification of it is superfluous. But the situation is different if one considers more general classes of spaces. The classical concept of the n-dimensional measure of a metric space X, due to C. Carathéodory [2] or to F. Hausdorff [3], constitutes an essential tool adequate to the needs of the modern analysis and geometry. However, some aspects of this concept deviate from the geometric intuition. For instance, there exist spaces X with a positive n-dimensional measure which can be transformed, by an arbitrarily small deformation onto spaces with vanishing measure.

Using the well-known theorem of P. S. Alexandroff [1] which established a close connection between arbitrary compacta and polyhedra, we introduce a coefficient $\mu_n(X)$ which differs in general from the n-dimensional measure of X in the classical sense, but which for polyhedra coincides with the n-dimensional measure and consequently may be considered as an alternative concept of the n-dimensional measure. We exhibit some simple properties of this coefficient and we formulate some open questions.

2. Polyhedra. By E^ω we denote the usual Hilbert space, with points of the form $(x_1, x_2, \ldots)$, where x_i are real numbers with $\sum_{i=1}^{\infty} x_i^2 < \infty$ and with distance given by the formula

$$\varrho(x,y) = \sqrt{\sum_{i=1}^{\infty}(x_i-y_i)^2}, \qquad \text{where } x = (x_1, x_2, \ldots),\ y = (y_1, y_2, \ldots).$$

The euclidean n-dimensional space E^n may be considered as a subset of E^ω, where we identify every point $(x_1, x_2, \ldots, x_n) \in E^n$ with the point $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in E^\omega$.

If the points $a_0, a_1, \ldots, a_n \in E^\omega$ are linearly independent (i.e. if does not exist in E^ω an $(n-1)$-dimensional hyperplane H containing all points $a_0, a_1, \ldots, a_n$), then the smallest convex subset of E^ω containing all those points is said to be an *n-dimensional simplex* spanned on vertices $a_0, a_1, \ldots, a_n$. One denotes this simplex by $\Delta(a_0, \ldots, a_n)$. It is also convenient to consider the empty set as a (-1)-dimensional simplex.

If all vertices of an m-dimensional simplex Δ' belong to the set of vertices of $\Delta = \Delta(a_0, \ldots, a_n)$, then Δ' is said to be a *face* of Δ. The union of all faces Δ' of Δ with dimensions $m < n$ is said to be the *boundary* $\dot{\Delta}$ of Δ, and the set $\mathring{\Delta} = \Delta \setminus \dot{\Delta}$ is said to be the *interior* of Δ.

In the elementary geometry one assigns to every n-dimensional simplex Δ a positive number $|\Delta|_n$ called the *n-dimensional measure* of Δ.

A subset P of E^ω is said to be a *polyhedron* if it has a triangulation T, i.e. a system of simplices $\Delta_1, \Delta_2, \ldots, \Delta_k$ such that

$$P = \bigcup_{i=1}^{k} \Delta_i \tag{2.1}$$

and

$$\Delta_i \cap \Delta_j \in T \quad \text{for every two indices } i, j. \tag{2.2}$$

The dimension $\dim P$ of P is equal to the greatest of the dimensions of the simplices $\Delta_i \in T$. It is clear that the choice of the triangulation is immaterial.

Let us observe that

(2.3) For every polyhedron $P \subset E^\omega$ there is in E^ω a hyperplane H containing P.

(2.4) For two triangulations T_1 and T_2 of a polyhedron P, there exists a triangulation T of P which is a subdivision of both triangulations T_1 and T_2.

(2.5) If $P_1, P_2 \subset E^\omega$ are polyhedra, then $P_1 \cup P_2$ and $P_1 \cap P_2$ are polyhedra.

3. *n*-dimensional measure of a polyhedron. In the elementary geometry one assigns to every polyhedron P and to every $n = 0, 1, \ldots$ the *n-dimensional measure* of P defined as a number $|P|_n$ given as follows:

(3.1) If $\dim P < n$, then $|P|_n = 0$.

(3.2) If $\dim P > n$, then $|P|_n = \infty$.

(3.3) If $\dim P = n$, then one considers all n-dimensional simplices $\Delta_1, \Delta_2, \ldots, \Delta_k$ of a triangulation T of P and one sets $|P|_n = \sum_{i=1}^{k} |\Delta_i|_n$.

One sees easily (using (2.1)) that $|P|_n$ does not depend on the choice of the triangulation T. Moreover, it is clear that

(3.4) If a polyhedron P_1 is a subset of another polyhedron P_2, then $|P_1|_n \leqslant |P_2|_n$ for every $n = 0, 1, \ldots$

One infers by (2.3) and (3.3) that for two polyhedra P_1, P_2 lying in E^ω with $P_1 \cap P_2 \neq \emptyset$

$$|P_1 \cup P_2|_n = |P_1|_n + |P_2|_n - |P_1 \cap P_2|_n. \tag{3.5}$$

In particular,

(3.6) For disjoint polyhedra $P_1, P_2 \subset E^\omega$, $|P_1 \cup P_2|_n = |P_1|_n + |P_2|_n$.

Now let us prove the following

(3.7) Lemma. *For every n-dimensional simplex $\Delta \subset E^\omega$ there is a retraction $r\colon E^\omega \to \Delta$ such that for every polyhedron $P \subset E^\omega$ the set $r(P)$ is a polyhedron and $|r(P)|_n \leqslant |P|_n$.*

Proof. Assume that $n > 0$ and that Δ lies in the space $E^n \subset E^\omega$ consisting of points $x = (x_1, x_2, \ldots) \in E^\omega$ such that $x_i = 0$ for $i > n$. Let us set:

$$\varphi\big((x_1, x_2, \ldots)\big) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \qquad \text{for every point } (x_1, x_2, \ldots) \in E^\omega.$$

Now we select a point $c \in \mathring{\Delta}$ and we set for every point $x \in E^n \setminus \{c\}$:

$\psi(x) =$ point in which the ray $\overrightarrow{cx}$ intersects the boundary $\dot{\Delta}$ of Δ.

Observe that if Δ' is a face of Δ lying on Δ, then the set $\psi^{-1}(\Delta')$ is a subset of E^n which is the common part of a finite number of half-spaces of E^n (i.e., of closed subsets of E^n bounded by an $(n-1)$-dimensional hyperplane). Using (2.4), one easily sees that for every polyhedron $P \subset E^n$ there exists a triangulation T such that every simplex of it is either a subset of Δ or of one of the sets of the form $\psi^{-1}(\Delta')$. Since ψ is linear on the set $\psi^{-1}(\Delta')$, we infer that every simplex of T lies either in Δ or in one of those sets $\psi^{-1}(\Delta')$. In both cases, its image by ψ is a simplex.

Setting

$$r(x) = \psi\varphi(x) \qquad \text{for every point } x \in E^\omega,$$

we get a retraction $r\colon E^\omega \to \Delta$. If $P \subset E^\omega$ is a polyhedron, then $\varphi(P)$ is a polyhedron lying in E^n and $r(P) = \psi\varphi(P)$ is a subpolyhedron of Δ.

Since $\varrho(\varphi(x), \varphi(y)) \leqslant \varrho(x, y)$ for every $x, y \in E^\omega$, we infer that $|\varphi(P)|_n \leqslant |P|_n$.

Let $\Delta_1, \ldots, \Delta_k$ be all n-dimensional simplices of T lying in Δ, and $\Delta_{k+1}, \ldots, \Delta_l$ — lying in $E^n \setminus \mathring{\Delta}$. Then $\psi(\Delta_i) = \Delta_i$ for $i = 1, \ldots, k$ and $\psi(\Delta_j) \subset \dot{\Delta}$ for $j = k+1, \ldots, l$. It follows that

$$|r(P)|_n = |\psi\varphi(P)|_n = \sum_{i=1}^{k} |\Delta_i|_n \leqslant |\varphi(P)|_n \leqslant |P|_n.$$

Thus the proof of Lemma (3.7) is finished.

4. Case of compacta $X \subset E^\omega$. Recall the well-known theorem of P. S. Alexandroff [1]:

(4.1) THEOREM. *The dimension* $\dim X$ *of a compactum* $X \subset E^\omega$ *is* $\leqslant n$ *if and only if there exists for every* $\varepsilon > 0$ *a polyhedron* $P_\varepsilon \subset E^\omega$ *with* $\dim P_\varepsilon \leqslant n$ *and a map* $f_\varepsilon \colon X \to P_\varepsilon$ *such that*

$$\varrho(x, f_\varepsilon(x)) < \varepsilon \quad \textit{for every } x \in X.$$

Let us assign to every compactum $X \subset E^\omega$ and to every $n = 0, 1, \ldots$ a number $\mu_n(X)$ defined as follows:

(4.2) $\mu_n(X)$ *is the lower bound of the set of all numbers* a *such that for every* $\varepsilon > 0$ *there is a map* $f_\varepsilon \colon X \to E^\omega$ *with* $\varrho(x, f_\varepsilon(x)) < \varepsilon$ *for every* $x \in X$ *and that the set* $f_\varepsilon(X)$ *is a subset of a polyhedron* $P_\varepsilon \subset E^\omega$ *with* $|P_\varepsilon|_n \leqslant a$.

One can show that if X, Y are two isometric compacta lying in E^ω, then $\mu_n(X) = \mu_n(Y)$ for every $n = 0, 1, \ldots$ We omit the proof of this proposition, because it is a direct consequence of the known fact that every isometry φ of X onto Y can be extended to an isometry $\tilde{\varphi}$ mapping the whole space E^ω onto itself.

One infers by (4.1), (3.1) and (3.2) that

(4.3) *If* $\dim X < n$, *then* $\mu_n(X) = 0$.

(4.4) *If* $\dim X > n$, *then* $\mu_n(X) = \infty$.

Moreover, let us observe that

(4.5) *If* X, Y *are compacta and if* $X \subset Y \subset E^\omega$, *then* $\mu_n(X) \leqslant \mu_n(Y$ *for every* $n = 0, 1, \ldots$

Using (3.6), one gets:

(4.6) *If* X, Y *are disjoint compacta lying in* E^ω, *then* $\mu_n(X \cup Y = \mu_n(X) + \mu_n(Y)$ *for* $n = 0, 1, \ldots$

It is well known that there exists in the interval $\langle 0, 1\rangle$ a 0-dimensional compact set A with a positive 1-dimensional measure $|A|_1$ in the classical sense. Then the Cartesian n-power A^n is a 0-dimensional compactum and its n-dimensional measure $|A^n|_n$ is positive, though $\mu_n(A^n) = 0$. Consequently the coefficient μ_n does not coincide with the n-dimensional measure in the classical sense.

Formulas (4.3) and (4.4) establish a relation between $\mu_n(X)$ and $\dim X$. Let us add that E. Szpilrajn (= E. Marczewski) [5] established also some connections between the concept of dimension and the classical concept of measure. Compare also [3], p. 102.

5. Case of polyhedra. It is well known that for a polyhedron P the n-dimensional Hausdorff measure is the same as $|P|_n$. In order to show that so is also for coefficient $\mu_n(P)$, let us observe that

$$(5.1)\qquad \mu_n(P) \leqslant |P|_n \quad \text{for every polyhedron } P \subset E^\omega.$$

In fact, this follows by (4.3) and (4.4) if $\dim P \neq n$. If $\dim P = n$, observe that the inclusion map $j\colon P \to E^\omega$ satisfies the condition $\varrho(x, j(x)) < \varepsilon$ for every $\varepsilon > 0$, and we infer by (4.2) that $\mu_n(P) \leqslant |j(P)|_n = |P|_n$.

Now let us prove the following lemma.

(5.2) Lemma. *For every n-dimensional simplex $\Delta \subset E^\omega$, $\mu_n(\Delta) = |\Delta|_n$.*

Proof. One infers by (5.1) that otherwise there would exist a positive number a such that

$$(5.3)\qquad \mu_n(\Delta) < a < |\Delta|_n.$$

Consequently, for every $\varepsilon > 0$ there is a polyhedron $P_\varepsilon \subset E^\omega$ with $|P_\varepsilon|_n < a$ and a map $f_\varepsilon\colon \Delta \to P_\varepsilon$ with $\varrho(x, f_\varepsilon(x)) < \varepsilon$ for every point $x \in \Delta$. By Lemma (3.7), there exists a retraction $r\colon E^\omega \to \Delta$ such that $r(P_\varepsilon)$ is a polyhedron and that $|r(P_\varepsilon)|_n \leqslant |P_\varepsilon|_n < a$.

Let Δ' be an n-dimensional simplex lying in the interior $\mathring{\Delta}$ of Δ. Consider an $(n-1)$-dimensional cycle γ lying in the boundary $\dot{\Delta}$ of Δ which generates the group of Betti $H_{n-1}(\dot{\Delta})$. If ε is sufficiently small, the map $rf_\varepsilon\colon \Delta \to \Delta$ assigns to the cycle γ an $(n-1)$-dimensional cycle $rf_\varepsilon(\gamma)$ which is null-homologous in Δ and is homologous in $\Delta \setminus \Delta'$ to γ. It follows that $\Delta' \subset rf_\varepsilon(\Delta)$ and we infer by (3.4) that $|\Delta'|_n \leqslant |rf_\varepsilon(\Delta)|_n$. But the simplex $\Delta' \subset \mathring{\Delta}$ can be selected so that $|\Delta'|_n > a$. Consequently, $\mu_n(\Delta) > a$, contrary to (5.3). Thus the proof of Lemma (5.2) is finished.

(5.4) Theorem. *$\mu_n(P) = |P|_n$ for every polyhedron $P \subset E^\omega$.*

Proof. Because of (5.1) we have only to show that

$$(5.5)\qquad \mu_n(P) \geqslant |P|_n$$

and we can limit ourselves to the case where $\dim P = n$.

Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be all n-dimensional simplices of a triangulation T of P. Consider in the interior $\mathring{\Delta}_i$ of Δ_i an n-dimensional simplex Δ_i' and let $P' = \bigcup_{i=1}^{k} \Delta_i'$. Then $|P'|_n \leqslant |P|_n$, but for any given number $a < |P|_n$, one can select the simplices Δ_i' so that $\sum_{i=1}^{k} |\Delta_i'|_n > a$. Moreover, since simplices Δ_i' are disjoint one to another, we infer by (4.5), (4.6) and (5.2) that

$$\mu_n(P) \geqslant \mu_n(P') = \sum_{i=1}^{k} \mu_n(\Delta_i') = \sum_{i=1}^{k} |\Delta_i'|_n > a.$$

Hence $\mu_n(P) > a$ for every number $a < |P|_n$ and consequently inequality (5.5) is proved and the proof of Theorem (5.4) is finished.

6. The lengths of arcs $L \subset E^\omega$. Let

$$s\colon \langle 0, 1\rangle \to L$$

be a parametric representation of an arc $L \subset E^\omega$, i.e. s is a homeomorphism mapping the interval $0 \leqslant t \leqslant 1$ onto L. The length $|L|$ of L can be defined as the upper bound of the sums

$$\sum_{i=1}^{k} \varrho\big(s(t_i), s(t_{i+1})\big), \qquad \text{where } t_0 = 0 < t_1 < \ldots < t_k < t_{k+1} = 1.$$

Let us show that $\mu_1(L) = |L|$. First let us establish the following

(6.1) Lemma. *For every continuum $X \subset E^\omega$, the diameter $\delta(X)$ of X is $\leqslant \mu_1(X)$.*

Proof. Because of (4.2), we can limit ourselves to the case where $\dim X = 1$. Then there exist two points $a, b \in X$ such that

$$\delta(X) = \varrho(a, b).$$

If $f_\varepsilon\colon X \to E^\omega$ is a map such that $\varrho\big(f_\varepsilon(x), x\big) < \varepsilon$ for every $x \in X$ and that there is a polyhedron $P_\varepsilon \subset E^\omega$ of dimension $\leqslant 1$ such that $f_\varepsilon(X) \subset P_\varepsilon$, then there is a polygonal arc $K \subset P_\varepsilon$ joining both points $f_\varepsilon(a)$ and $f_\varepsilon(b)$. Then

$$\mu_1(P_\varepsilon) = |P_\varepsilon|_1 \geqslant |K|_1 \geqslant \varrho\big(f_\varepsilon(a), f_\varepsilon(b)\big) > \varrho(a, b) - 2\varepsilon = \delta(X) - 2\varepsilon.$$

Since the positive number ε is arbitrarily small, we infer that $\mu_1(P) \geqslant \delta(X)$ and the proof of Lemma (6.1) is finished.

(6.2) Theorem. *For every arc $L \subset E^\omega$ the length $|L| = \mu_1(L)$.*

Proof. Let $s\colon \langle 0, 1\rangle \to L$ be a parametric representation of L. It is clear that for every $\varepsilon > 0$ the system of numbers

$$0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$$

can be selected so that there exists a map $f_\varepsilon\colon L \to E^\omega$ such that $\varrho\big(x, f_\varepsilon(x)\big) < \varepsilon$ for every point $x \in L$ and that $f_\varepsilon(L)$ is a subset of a 1-dimensional polyhedron K_ε which is the union of all segments $\overline{s(t_i), s(t_{i+1})}$, where

$i = 0, 1, \ldots, k$. It follows that

$$\mu_1(L) \leqslant |L|. \tag{6.3}$$

On the other hand, there exists a positive number η so small that

$$t_i < t_i + \eta < t_{i+1} - \eta < t_{i+1} \quad \text{for } i = 0, 1, \ldots, k.$$

Then s maps the interval $\langle t_i + \eta, t_{i+1} - \eta \rangle$ onto an arc $L_i \subset L$ and $L_i \cap L_j = \emptyset$ for $i \neq j$. Consider a number $a < |L|$ and observe that if the number η is sufficiently small, then

$$\sum_{i=1}^{k} \varrho\big(s(t_i + \eta), s(t_{i+1} - \eta)\big) > a.$$

It follows by (4.5), (4.6) and (6.2) that

$$\mu_1(L) \geqslant \mu_1\Big(\bigcup_{i=1}^{k} L_i\Big) = \sum_{i=0}^{k} \mu_1(L_i) \geqslant \sum_{i=0}^{k} \delta(L_i)$$
$$\geqslant \sum_{i=0}^{k} \varrho\big(s(t_i + \eta), s(t_{i+1} - \eta)\big) > a.$$

Thus $\mu_1(L)$ is greater than every number $a < |L|$, and we infer that $\mu_1(L) \geqslant |L|$. It follows by (6.3) that $\mu_1(L) = |L|$ and the proof of Theorem (6.2) is finished.

7. Small deformations of compacta. It is well known that for every $n = 0, 1, \ldots$ there exist compacta X homeomorphic to the Cantor discontinuum for which the n-dimensional measure (in the classical sense) is arbitrarily large. Another situation is for the coefficient μ_n as follows by the

(7.1) THEOREM. *If $\mu_n(X) > 0$, then for every positive number $a < \mu_n(X)$ there is a positive number ε such that $a \leqslant \mu_n\big(f(X)\big)$ for every map $f\colon X \to E^\omega$ satisfying the condition $\varrho\big(x, f(x)\big) < \varepsilon$ for every point $x \in X$.*

Proof. Otherwise there would exist a sequence of maps $f_1, f_2, \ldots$ $\ldots : X \to E^\omega$ satisfying the condition

$$\varrho\big(x, f_k(x)\big) < 1/k \quad \text{for every } x \in X \text{ and } k = 1, 2, \ldots \tag{7.2}$$

and that

$$\mu_n\big(f_k(X)\big) < a \quad \text{for every } k = 1, 2, \ldots \tag{7.3}$$

It follows by (7.3) that for every $k = 1, 2, \ldots$ there is a map

$$\varphi_k\colon f_k(X) \to E^\omega \tag{7.4}$$

such that

$$\varrho\big(y, \varphi_k(y)\big) < 1/k \quad \text{for every point } y \in f_k(X) \tag{7.5}$$

and a polyhedron $P_k \subset E^\omega$ containing $\varphi_k(X)$ and satisfying the condition

$$|P_k|_n < a. \tag{7.6}$$

It follows by (7.2), (7.4) and (7.5) that the map $g_k = \varphi_k f_k \colon X \to E^\omega$ satisfies the condition

$$\varrho(x, g_k(x)) \leqslant \varrho(x, f_k(x)) + \varrho(f_k(x), \varphi_k f_k(x)) < 2/k \quad \text{for every } x \in X$$

and the set $g_k(X)$ lies in the polyhedron P_k. Since $|P_k|_n < a$, we infer that $\mu_n(X) \leqslant a$, contrary to our hypothesis. Thus the proof of Theorem (7.1) is finished.

8. Generalization. In the present paper we limit ourselves to the case of compacta $X \subset E^\omega$ and the definition of $\mu_n(X)$ is based on the elementary geometric concept of polyhedra. If one replaces polyhedra by the much more general class of the locally finite polytopes, then one can transfer the concept of the coefficient μ_n from compacta onto more general classes of spaces. As yet, the so generalized notion of the coefficient μ_n was not studied.

9. Some questions. Already in the domain of compacta, our knowledge of properties of μ_n is very limited. Let us formulate some questions concerning compacta $X \subset E^\omega$:

(9.1) *Is $\mu_n(X) \leqslant$ than the n-dimensional measure $|X|_n$ in the classical sense for every compactum $X \subset E^\omega$?*

(9.2) *Is it true that for every compact* ANR*-set $X \subset E^\omega$ the value of $\mu_n(X)$ coincides with the n-dimensional measure $|X|_n$ of X?*

(9.3) *Is it true that $\mu_{n+1}(X \times \langle 0, 1\rangle) = \mu_n(X)$ for every compactum $X \subset E^\omega$ and $n = 0, 1, \ldots$?*

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