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A difference method for a system of second order ordinary differential equations

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Abstract. The paper concerns a system of second order ordinary differential equations of elliptic type, together with a two-point boundary condition. This boundary value problem is approached by a difference one and the proof of convergence is the our main goal.

Introduction. The present paper concerns a system of nonlinear ordinary differential equations of second order

$$f_\alpha(t, x_1, \ldots, x_p, x'_\alpha, x''_\alpha) = 0 \quad (t \in [0, \tau],\ \alpha = 1, \ldots, p),$$

together with the two-point boundary condition

$$x_\alpha(0) = a_\alpha, \quad x_\alpha(\tau) = b_\alpha, \quad \alpha = 1, \ldots, p.$$

We will approach this system with a system of suitable difference equations (with two-side differences).

Our goal is to give a proof of convergence of the difference method and the technique we use is that of difference inequalities. An inequality like ours (with $p = 1$) can be found in [1]. It comes from partial differential equations of elliptic type. In our case the proof becomes simpler and the resulting error estimate is better than that of [1] (cf. the last section).

1. A difference inequality

1.1. Let h be fixed positive number. Let $u = (u_0, \ldots, u_{n+1}) \in \boldsymbol{R}^{n+2}$ be a system of $n+2$ real numbers which we call a *discrete function*.

The central two-side differences are defined as follows:

$$u_i^{(1)} = \frac{1}{2h}(u_{i+1} - u_{i-1}), \tag{1.1}$$

$$u_i^{(2)} = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) \quad (i = 1, \ldots, n). \tag{1.2}$$

The following simple observation will be crucial for our purposes.

LEMMA 1. *Suppose discrete function u attains its maximum for some different from* 0 *and* $n+1$, *i.e.*

$$u_i \leqslant u_{i_0} \quad \textit{for every } i = 0, 1, \ldots, n+1.$$

Then

$$u_{i_0}^{(2)} \leqslant 0, \tag{1.3}$$

$$2|u_{i_0}^{(1)}| \leqslant -hu_{i_0}^{(2)}. \tag{1.4}$$

Proof. (1.3) follows immediately from (1.2). Indeed,

$$u_{i_0}^{(2)} = \frac{1}{h^2}(u_{i_0+1}-u_{i_0}) + \frac{1}{h^2}(u_{i_0-1}-u_{i_0}).$$

Both differences are nonpositive and this establishes (1.3).

By (1.1) and (1.2),

$$2hu_i^{(1)} + h^2 u_i^{(2)} = 2(u_{i+1}-u_i), \tag{1.5}$$

$$2hu_i^{(1)} - h^2 u_i^{(2)} = 2(u_i - u_{i-1}) \quad (i = 1, \ldots, n). \tag{1.6}$$

Condition (1.5) implies

$$2u_{i_0}^{(1)} = -hu_{i_0}^{(2)} + \frac{2}{h}(u_{i_0+1}-u_{i_0}) \leqslant -hu_{a_0}^{(2)},$$

because

$$u_{i_0+1}-u_{i_0} \leqslant 0.$$

Condition (1.6) implies

$$2u_{i_0}^{(1)} = h_{i_0}^{(2)} + \frac{2}{h}(u_{i_0}-u_{i_0-1}) \geqslant hu_{i_0}^{(2)},$$

because

$$u_{i_0+1}-u_{i_0-1} \geqslant 0.$$

Both above inequalities give (1.4).

1.2. We will consider the following difference inequalities

$$a_i u_i^{(2)} + b_i u_i^{(1)} + c_i u_i \geqslant -\varepsilon \tag{1.7}$$

$(i = 1, \ldots, n)$ for $\varepsilon > 0$, together with the boundary condition

$$u_0 = u_{n+1} = 0. \tag{1.8}$$

In the sequel we will need the following assumption concerning a_1, b_i and c_i:

ASSUMPTION A_1. There exists constants ω, λ and η such that

$$a_i \geqslant \omega > 0 \quad (i = 1, \ldots, n), \tag{1.9}$$

$$|b_i| \leqslant 2\lambda \quad (i = 1, \ldots, n), \tag{1.10}$$

$$c_i \leqslant \eta < 0 \quad (i = 1, \ldots, n), \tag{1.11}$$

$$\omega - \lambda h \geqslant 0. \tag{1.12}$$

Now we can prove the basic result concerning our difference inequality.

THEOREM 1. *Suppose Assumption* A_1 *holds. Let* $u = (u_0, \ldots, u_{n+1})$ *be a discrete function satisfying* (1.8) *and* $u_{i_0} = \max\{u_j \colon j = 0, \ldots, n+1\} > 0$. *If the function* u *satisfies for* $i = i_0$ *the inequality*

$$a_{i_0} u_{i_0}^{(2)} + b_{i_0} u_{i_0}^{(1)} + c_{i_0} u_{i_0} \geqslant -\varepsilon, \tag{1.13}$$

then

$$u_i \leqslant \mu_1 \quad (i = 0, \ldots, n+1), \tag{1.14}$$

where

$$\mu_1 = -\eta^{-1}\varepsilon > 0. \tag{1.15}$$

Proof. Suppose on the contrary that there exists an i such that $u_i > \mu_1$. Then also

$$u_{i_0} > \mu_1. \tag{1.16}$$

We claim that

$$a_{i_0} u_{i_0}^{(2)} + b_{i_0} u_{i_0}^{(1)} \leqslant (\omega - \lambda h)\, u_{i_0}^{(2)}, \tag{1.17}$$

$$c_{i_0} u_{i_0} < \eta\mu_1. \tag{1.18}$$

Indeed (1.17) follows from Lemma 1. More precisely, comparing (1.15), (1.16) and (1.18) we see that $i_0 \neq 0$ and $i_0 \neq n+1$. Thus (1.3) and (1.9) imply

$$a_{i_0} u_{i_0}^{(2)} \leqslant \omega u_{i_0}^{(2)}. \tag{1.19}$$

Moreover, (1.4) and (1.10) imply

$$|b_{i_0} u_{i_0}^{(1)}| \leqslant -\lambda h u_{i_0}^{(2)}. \tag{1.20}$$

Now, (1.19) and (1.20) give (1.17).

Inequality (1.18) follows from (1.11) and (1.16). More precisely, (1.11) and (1.16) imply

$$c_{i_0} u_{i_0} \leqslant \eta u_{i_0}.$$

Using once again (1.11) and (1.16) we infer that

$$\eta u_{i_0} < \eta \mu_1 .$$

The above inequalities together imply (1.18). Summing (1.17) and (1.18) we obtain

$$(1.21) \qquad a_{i_0} u_{i_0}^{(2)} + b_{i_0} u_{i_0}^{(1)} + c_{i_0} u_{i_0} < (\omega - \lambda h) u_{i_0}^{(2)} + \eta \mu_1 .$$

The ingredient $(\omega - \lambda h) u_{i_0}^{(2)}$ is nonpositive, by (1.12) and (1.3) and consequently we can drop it making the right-hand side of (1.21) greater. Moreover, $\eta \mu_1 = -\varepsilon$, by (1.15), we get

$$(1.22) \qquad a_{i_0} u_{i_0}^{(2)} + b_{i_0} u_{i_0}^{(1)} + c_{i_0} u_{i_0} < -\varepsilon .$$

Since, as we have checked, $i_0 \neq 0$ and $i_0 \neq n+1$, and $u_{i_0} > 0$, inequality (1.13) contradicts (1.22), and this completes the proof.

2. A system of difference inequalities

2.1. Now we deal with a system of p discrete functions $u_\alpha = (u_{\alpha,0}, \ldots, u_{\alpha,n+1})$ $(\alpha = 1, \ldots, p)$ and a system of inequalities

$$(2.1) \qquad a_{\alpha i} u_{\alpha i}^{(2)} + b_{\alpha i} u_{\alpha i}^{(1)} + \sum_{\beta=1}^{p} c_{\alpha\beta i} u_{\beta i} \geqslant -\varepsilon \qquad (i = 1, \ldots, n)$$

$\alpha = 1, \ldots, p$, $\varepsilon > 0$, together with the boundary conditions

$$(2.2) \qquad u_{\alpha 0} = u_{\alpha, n+1} = 0 \qquad (\alpha = 1, \ldots, p).$$

(Note that in (2.1) the difference quotients are taken with respect to i, while the Greek letters α, β indicate the number of a discrete function.)

Instead of Assumption A_1 we need the following:

ASSUMPTION A_p. There are ω, λ, η and δ such that

$$(2.3) \qquad a_{\alpha i} \geqslant \omega > 0 \qquad (i = 1, \ldots, n,\ \alpha = 1, \ldots, p),$$

$$(2.4) \qquad |b_{\alpha i}| \leqslant 2\lambda \qquad (i = 1, \ldots, n,\ \alpha = 1, \ldots, p),$$

$$(2.5) \qquad c_{\alpha\alpha i} \leqslant \eta < 0 \qquad (i = 1, \ldots, n,\ \alpha = 1, \ldots, p),$$

$$(2.6) \qquad \omega - \lambda h \geqslant 0,$$

$$(2.7) \qquad 0 \leqslant c_{\alpha\beta i} \leqslant \delta \qquad (i = 1, \ldots, n,\ \alpha = 1, \ldots, p,\ \beta = 1, \ldots, p,\ \alpha \neq \beta),$$

$$(2.8) \qquad \eta + (p-1)\delta < 0.$$

Notice that, by (2.3), inequality (2.6) holds automatically when h is sufficiently small — this is important for of the limit procedure.

Using Theorem 1 we prove a theorem concerning the above of difference inequalities.

THEOREM 2. *Suppose Assumption* A_p *is satisfied.*

Let $u_\alpha = (u_{\alpha,0}, \ldots, u_{\alpha,n+1})$ $(\alpha = 1, \ldots, p)$ *be a system of p discrete function satisfying* (2.2) *such that if* $u_{\alpha_0 i_0} > 0$ *for some* α_0, i_0; *then*

$$a_{\alpha i}\, u_{\alpha i}^{(2)} + b_{\alpha i}\, u_{\alpha i}^{(1)} + \sum_{\beta=1}^{p} c_{\alpha\beta i}\, u_{\beta i} \geqslant -\varepsilon, \qquad \varepsilon > 0 \tag{2.9}$$

for $\alpha = \alpha_0$ *and* $i = i_0$.

Then

$$u_{\alpha i} \leqslant \mu_p \qquad (i = 0, \ldots, n+1,\ \alpha = 1, \ldots, p), \tag{2.10}$$

where

$$\mu_p = -(\eta + (p-1)\delta)^{-1}\varepsilon \qquad (\mu_p > 0). \tag{2.11}$$

Proof. Let i_α be such that

$$u_{\alpha i_\alpha} = \max\{u_{\alpha i}\colon i = 0, \ldots, n+1\}, \tag{2.12}$$

$$u_{\alpha_0 i_0} = \max\{u_{\alpha i_\alpha}\colon \alpha = 1, \ldots, p\}. \tag{2.13}$$

Notice that (2.2) implies

$$u_{\alpha i_\alpha} \geqslant 0 \qquad \text{for all } \alpha. \tag{2.14}$$

Since $\mu_p > 0$ (cf. (2.11) with (2.8)), the only nontrivial case is that of $u_{\alpha_0 i_0} > 0$. Then, by our assumption,

$$a_{\alpha_0 i_0}\, u_{\alpha_0 i_0}^{(2)} + b_{\alpha_0 i_0}\, u_{\alpha_0 i_0}^{(1)} + \sum_{\beta=1}^{p} c_{\alpha_0\beta i_0}\, u_{\beta i_0} \geqslant -\varepsilon. \tag{2.15}$$

Since for $\beta \neq \alpha_0$, $c_{\alpha_0\beta i_0} \geqslant 0$ by (2.7), and, moreover, $u_{\beta i_0 i_0} \leqslant u_{\alpha_0 i_0}$ by (2.13), it follows that

$$c_{\alpha_0\beta i_0}\, u_{\beta i_0} \leqslant c_{\alpha_0\beta i_0}\, u_{\alpha_0 i_0} \qquad \text{for } \beta \neq \alpha_0. \tag{2.16}$$

By (2.7), $c_{\alpha_0\beta i_0} \leqslant \delta$. Since $u_{\alpha_0 i_0} > 0$ we get

$$c_{\alpha_0\beta i_0}\, u_{\alpha_0 i_0} \leqslant \delta u_{\alpha_0 i_0} \qquad \text{for } \beta \neq \alpha_0. \tag{2.17}$$

Both inequalities (2.16) and (2.17) imply

$$c_{\alpha_0\beta i_0}\, u_{\beta i_0} \leqslant \delta u_{\alpha_0 i_0} \qquad \text{for } \beta \neq \alpha_0. \tag{2.18}$$

Inequalities (2.18) and (2.5) yield

$$a_{\alpha_0 i_0} u^{(2)}_{\alpha_0 i_0} + b_{\alpha_0 i_0} u^{(1)}_{\alpha_0 i_0} + \sum_{\beta=1}^{p} c_{\alpha_0 \beta i_0} u_{\beta i_0}$$

$$= a_{\alpha_0 i_0} u^{(2)}_{\alpha_0 i_0} + b_{\alpha_0 i_0} u^{(1)}_{\alpha_0 i_0} + c_{\alpha_0 \alpha_0 i_0} u_{\alpha_0 i_0} + \sum_{\substack{\beta=1 \\ \beta \neq \alpha_0}}^{p} c_{\alpha_0 i_0 \beta i_0} u_{\beta i_0}$$

$$\leqslant a_{\alpha_0 i_0} u^{(2)}_{\alpha_0 i_0} + b_{\alpha_0 i_0} u^{(1)}_{\alpha_0 i_0} + \eta u_{\alpha_0 i_0} + (p-1)\delta u_{\alpha_0 i_0}.$$

Comparing this with (2.15) we obtain

$$a_{\alpha_0 i_0} u^{(2)}_{\alpha_0 i_0} + b_{\alpha_0 i_0} u^{(1)}_{\alpha_0 i_0} + \big(\eta + (p-1)\delta\big) u_{\alpha_0 i_0} \geqslant -\varepsilon. \tag{2.19}$$

This is a difference inequality for a single discrete function

$$u_{\alpha_0} = (u_{\alpha_0,0}, u_{\alpha_0,1}, \ldots, u_{\alpha_0,n+1}).$$

Thus we are able to use Theorem 1. Conditions (2.3), (2.4), (2.6) imply (1.9), (1.10), (1.12) respectively (2.19).

Moreover, since all c_i are equal to $\eta + (p-1)\delta$ and (2.8) is satisfied, so is (1.11). Theorem 1 gives us the following estimate:

$$u_{\alpha_0 i_0} \leqslant -\big(\eta + (p-1)\delta\big)^{-1} \varepsilon$$

which combined with (2.13) and (2.12) yields (2.10) and completes the proof of Theorem 2.

Remark 1. Let us point out that for $p = 1$, A_p coincides with A_1 of the preceding section, and so in that case Theorem 2 is just Theorem 1.

2.2. It is plain that by replacing each u_α by $-u_\alpha$. Theorem 2 gives the following

THEOREM 3. *Suppose Assumption* A_p *holds. Let* $u_\alpha = (u_{\alpha,0}, \ldots, u_{\alpha,n+1})$, $\alpha = 1, \ldots, p$ *be a system of* p *discrete functions satisfying* (2.2) *and such that if* $u_{\alpha_1 i_1} < 0$ *for some* α_1, i_1, *then*

$$a_{\alpha i} u^{(2)}_{\alpha i} + b_{\alpha i} u^{(1)}_{\alpha i} + \sum_{\beta=1}^{p} c_{\alpha\beta i} u_{\beta i} \leqslant \varepsilon \tag{2.20}$$

for $\alpha = \alpha_1$ *and* $i = i_1$. *Then*

$$u_{\alpha i} \geqslant -\mu_p \quad (i = 0, \ldots, n+1,\ \alpha = 1, \ldots, p) \tag{2.21}$$

(see (2.11) for the definition of μ_p).

3. Convergence of the difference method

Now we consider a system of p ordinary differential equation of second order

(3.1) $$f_\alpha(t, x_1, \ldots, x_p, x'_\alpha, x''_\alpha) = 0 \quad (\alpha = 1, \ldots, p)$$

together with the boundary conditions

(3.2) $$x_\alpha(0) = a_\alpha, \quad x_\alpha(\tau) = b_\alpha \quad (\alpha = 1, \ldots, p).$$

We will need the following

ASSUMPTION CA_p. The functions $f_\alpha(t, x_1, \ldots, x_p, y, z)$, $t \in [0, \tau]$, $(x_1, \ldots, x_p, y, z) \in \boldsymbol{R}^{p+2}$, are continuously differentiable in $[0, \tau] \times \boldsymbol{R}^{p+2}$ and satisfy the following conditions:

(3.3) $$\frac{\partial}{\partial x_\alpha} f_\alpha \leqslant \eta < 0 \quad (\alpha = 1, \ldots, p),$$

(3.4) $$0 \leqslant \frac{\partial}{\partial x_\beta} f_\alpha \leqslant \delta \quad (\beta \neq \alpha,\ \beta = 1, \ldots, p,\ \alpha = 1, \ldots, p),$$

(3.5) $$\left|\frac{\partial}{\partial y} f_\alpha\right| \leqslant 2\lambda \quad (\alpha = 1, \ldots, p),$$

(3.6) $$0 < \omega \leqslant \frac{\partial}{\partial z} f_\alpha \quad (\alpha = 1, \ldots, p),$$

(3.7) $$\eta + (p-1)\delta < 0,$$

where ω, η, λ, δ are some constants.

We wish to approach a solution of the problem (3.1), (3.2) by a solution of

(3.8) $$f_\alpha(t_i, v_{1i}, \ldots, v_{pi}, v_{\alpha i}^{(1)}, v_{\alpha i}^{(2)}) = 0 \quad (i = 1, \ldots, n,\ \alpha = 1, \ldots, p),$$

(3.9) $$v_{\alpha,0} = a_\alpha, \quad v_{\alpha,n+1} = b_\alpha \quad (\alpha = 1, \ldots, p),$$

where $t_i = ih$, $i = 0, 1, \ldots, (n+1)$, $h = \tau(n+1)^{-1}$.

We show that solutions of (3.8), (3.9) converge to a solution of (3.1), (3.2) as $h \to 0$.

THEOREM 4. *Suppose Assumption* CA_p *holds. Let* C^2 *functions* x_α ($\alpha = 1, \ldots, p$) *be solutions of* (3.1) *satisfying* (3.2). *Then discrete functions* $v_\alpha = (v_{\alpha,0}, \ldots, v_{\alpha,n+1})$ ($\alpha = 1, \ldots, p$) *which are solutions of* (3.8), (3.9) *satisfy, with* $t_i = ih$, $h = \tau(n+1)^{-1}$,

(3.10) $$|x_\alpha(t_i) - v_{\alpha i}| \leqslant -\varepsilon\big(\eta + (p-1)\delta\big)^{-1}$$

for $i = 0, \ldots, n+1$, $\alpha = 1, \ldots, p$, *provided* h *is small enough, i.e.,*

(3.11) $$0 \leqslant h \leqslant \omega\lambda^{-1}.$$

The number $\varepsilon \in \varepsilon(h)$ (defined below, cf. (3.15)) *is such that*

(3.12) $$\varepsilon(h) \to 0 \quad \text{as } h \to 0.$$

Proof. Put $x_{\alpha i} = x_\alpha(t_i)$. The discrete functions $x_\alpha = (x_{\alpha,0}, \ldots, x_{\alpha,n+1})$, $\alpha = 1, \ldots, p$, satisfy a system

(3.13) $$f_\alpha(t_i, x_{1i}, \ldots, x_{pi}, x_{\alpha i}^{(1)}, x_{\alpha i}^{(2)}) = \varepsilon_{\alpha i}(h)$$

for $\alpha = 1, \ldots, p$, $i = 1, \ldots, n$. Continuity of f_α as well as of x'_α and x''_α implies that

(3.14) $$\varepsilon(h) \to 0 \quad \text{as } h \to 0,$$

where

(3.15) $$\varepsilon(h) = \max\{|\varepsilon_{\alpha i}(h)|: \alpha = 1, \ldots, p, \ i = 1, \ldots, n\}.$$

Set

(3.16) $$r_{\alpha i} = x_{\alpha i} - v_{\alpha i} \quad (i = 0, \ldots, n+1, \ \alpha = 1, \ldots, p).$$

By the Mean Value Theorem, the functions $r_\alpha = (r_{\alpha,0}, \ldots, r_{\alpha,n+1})$ satisfy the following system of difference equations:

(3.17) $$\sum_{\beta=1}^{p} \frac{\partial f_\alpha}{\partial x_\beta}(A_i) r_{\beta i} + \frac{\partial f_\alpha}{\partial y}(A_i) r_{\alpha i}^{(1)} + \frac{\partial f_\alpha}{\partial z}(A_i) r_{\alpha i}^{(2)} = \varepsilon_{\alpha i}$$

(A_i denotes here a suitable intermediate point depending on i).

Denote the derivatives involved in (3.17) by $c_{\alpha\beta i}$, $b_{\alpha i}$, $a_{\alpha i}$ respectively. System (3.17) takes now the form

(3.18) $$a_{\alpha i} r_{\alpha i}^{(2)} + b_{\alpha i} r_{\alpha i}^{(1)} + \sum_{\beta=1}^{p} c_{\alpha\beta i} r_{\beta i} = \varepsilon_{\alpha i}.$$

Moreover, by (3.2) and (3.9)

(3.19) $$r_{\alpha,0} = r_{\alpha,n+1} = 0 \quad (\alpha = 1, \ldots, p).$$

Notice that Assumption CA_p and (3.11) imply that the coefficients $a_{\alpha i}$, $b_{\alpha i}$ and $c_{\alpha\beta i}$ satisfy A_p.

Suppose for some α_0, i_0

(3.20) $$r_{\alpha_0 i_0} > 0.$$

The discrete functions r_α satisfy, by (3.18) and (3.15), the inequality

(3.21) $$a_{\alpha_0 i_0} r_{\alpha_0 i_0}^{(2)} + b_{\alpha_0 i_0} r_{\alpha_0 i_0}^{(1)} + \sum_{\beta=1}^{p} c_{\alpha\beta i_0} r_{\alpha_0 i_0} \geqslant -\varepsilon.$$

Using (3.20) and (3.21) we can apply Theorem 2, getting right away

$$r_{\alpha i} \leqslant \mu_p \tag{3.22}$$

for all indices α and i, where μ_p is defined by (2.11).

Suppose now that for some α_1, i_1

$$r_{\alpha_1 i_1} < 0. \tag{3.23}$$

(3.18) and (3.15) yield

$$a_{\alpha_1 i_1} r^{(2)}_{\alpha_1 i_1} + b_{\alpha_1 i_1} r^{(1)}_{\alpha_1 i_1} + \sum_{\beta=1}^{p} c_{\alpha_1 \beta i_1} r_{\beta i_1} \leqslant \varepsilon. \tag{3.24}$$

Theorem 3 now gives the estimate

$$r_{\alpha_i} \geqslant -\mu_p \quad \text{for all } \alpha \text{ and } i. \tag{3.25}$$

Comparing (3.22) and (3.25) we obtain

$$|r_{\alpha i}| \leqslant \mu_p = -\varepsilon\big(\eta+(p-1)\delta\big)^{-1}, \tag{3.26}$$

which is (3.10). This completes the proof.

4. Concluding remarks

As we have mentioned in the introduction, Pliś and Kowalski [1] considered the difference inequality of elliptic type

$$\sum_{i,j} a^M_{ij} u^{Mij} + \sum_j b^M_j u^{Mj} + c^M u^M \geqslant -\varepsilon, \tag{4.1}$$

where u^{Mj} and u^{Mij} $(i, j = 1, \ldots, n)$ denote the difference quotients of the partial derivatives $\partial u/\partial x_j$ and $\partial^2 u/\partial x_i \partial x_j$ at a suitable nodal point M.

Inequality (4.1) in the case of a function of a single variable is just (1.7).

Pliś and Kowalski got the following estimate

$$u^M \leqslant -\eta^{-1}\big(\varepsilon + n\lambda h \Lambda + E(h)\big), \tag{4.2}$$

where a positive constant Λ (depending on n) dominates all difference quotients of second order, $E(h)$ depends on h, Λ and on how repidly all second order difference quotients change, the remainder has the same meaning as in this paper. The explicit form of $E(h)$ is too complicated to be stated here; it depends essentially on the quadratic form $\sum a^M_{ij} \xi_i \xi_j$.

In [2] we have been able to eliminate, in our context, the term $E(h)$. In the present paper it turns out that, for h sufficiently small, i.e., satisfying (1.12), the term $n\lambda h\Lambda$ is needless too, and the right-hand side of (4.2) is as in (1.15).

The paper is based on the author's Ph.D. dissertation [Jagiellonian University, 1975].

References

[1] A. Pliś, Z. Kowalski, *Difference inequalities of the elliptic type*, Ann. Polon. Math. 26 (1972), 239–251.
[2] K. Szafraniec, *A note on a difference inequality*, Zeszyty Nauk. Uniw. Jagiellon. 17 (1975), 143–145.

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