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Uniqueness of the solution of a tangential derivative problem for an infinite system of non-linear integro-differential equations of parabolic type

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Abstract. In the paper it is proved the uniqueness of the parabolic solution of the third Fourier problem for the non-linear system of integro-differential equations (1) with the initial conditions (2) and the non-linear boundary conditions (3) containing the tangential derivatives of the unknown functions. The approach is patterned on that applied in papers [2] and [3] and uses a lemma from [6].

1. The uniqueness of solutions of Fourier problems for second-order parabolic equations and systems of such equations has been examined by M. Picone, M. Krzyżański, J. Szarski, P. Besala and others (see [1]–[5], [8]–[11], [13], [15]–[20] and [23]) under the assumption that the boundary conditions of the problems do not contain the derivatives of the unknown functions in the directions not entering the closure of the domain considered.

Borzymowski [6] proved the uniqueness of the solution of the first Fourier problem for a finite system of non-linear second-order parabolic equations in a bounded domain with a non-linear boundary condition containing the tangential derivatives of the unknown functions. In this paper we prove an analogous uniqueness theorem concerning the third Fourier problem for an infinite system of second-order integro-differential equations of parabolic type in an unbounded domain. The reasoning used in the proof is adopted from P. Besala's paper [3] and is partly based on [6].

The uniqueness of solutions of boundary value problems for parabolic integro-differential equations and their finite systems, with the boundary conditions not containing the tangential derivatives, has been examined by Krzyżański [12], Łojczyk-Królikiewicz and Szarski [14], Ugowski [21], [22] and Żuk [24]. We would also like to remark that the result obtained in this paper implies the uniqueness of the solution of a boundary value problem whose existence was proved in [7] (see Remark 3 in the sequel).

2. Let R^{m+1} be the time space of points (x, t), where $x = (x_1, \ldots, x_m) \in R^m$ $(m \geqslant 2)$ and $t \geqslant 0$. In R^{m+1} we consider a non-cylindrical domain Ξ_h

whose boundary consists of m-dimensional closed domains $\bar{\Delta}_0$ and $\bar{\Delta}_h$ placed in the hyperplanes $t = 0$ and $t = h$, respectively $(0 < h < \infty)$, and an m-dimensional lateral surface σ_h situated between these hyperplanes. We denote the intersections of Ξ_h and σ_h with the hyperplane $t = \tau$ by Δ_τ and S_τ, respectively. Furthermore, we set $\Omega_\tau = R^m \setminus \bar{\Delta}_\tau$ and $D_h = (R^m \times (0, h)) \setminus \bar{\Xi}_h$.

Consider an infinite system of integro-differential equations of the form

$$(1)\quad u_t^{(r)}(x, t) = F^{(r)}\Big[x, t, u(x, t), u_x^{(r)}(x, t), \int_0^t \int_{\Omega_{x,\tau}^{(r)}} \Phi^{(r)}(x, t, y, \tau, u(y, \tau), u(x, t), u_x^{(r)}(x, t))\,dy\,d\tau, u_{xx}^{(r)}(x, t)\Big] \qquad ((x, t) \in D_h),$$

where $r = 1, 2, \ldots$; $u = (u^{(1)}, u^{(2)}, \ldots, u^{(N_r)})$ with $N_r = \max(n, r)$, n being a fixed positive integer; $u_x^{(r)}(x, t) = (u_{x_1}^{(r)}(x, t), \ldots, u_{x_m}^{(r)}(x, t))$; $u_{xx}^{(r)}(x, t) = (u_{x_1x_1}^{(r)}(x, t), u_{x_1x_2}^{(r)}(x, t), \ldots, u_{x_mx_m}^{(r)}(x, t))$; $\Phi^{(r)} = (\Phi_1^{(r)}, \Phi_2^{(r)}, \ldots)$ (¹) and $\Omega_{x,\tau}^{(r)}$ denotes a measurable subset of Ω_τ depending in general on x.

We assume that on σ_h there are defined vector fields $\{l_1\}, \ldots, \{l_q\}$ and $\{s_1\}, \ldots, \{s_{\bar{q}}\}$ (where $1 \leqslant q, \bar{q} \leqslant m-1$) so that for each point $(x, \tau) \in \sigma_h$ the vectors $s_1, \ldots, s_{\bar{q}}$ corresponding to this point are tangent to S_τ and the vectors $l_1, \ldots, l_q$ penetrate the domain Ω_τ.

An infinite system $(u^{(1)}(x, t), u^{(2)}(x, t), \ldots)$ is called a *solution of the* (F_3)-*problem* if it satisfies (1) for $(x, t) \in D_h$, possesses the derivatives

$$\left(\frac{du^{(1)}}{dl_\alpha}(x, t), \frac{du^{(2)}}{dl_\alpha}(x, t), \ldots\right) \quad \text{and} \quad \left(\frac{du^{(1)}}{ds_\beta}(x, t), \frac{du^{(2)}}{ds_\beta}(x, t), \ldots\right)$$

(where $\alpha = 1, \ldots, q$; $\beta = 1, \ldots, \bar{q}$) for $(x, t) \in \sigma_h$ and fulfils the initial condition

$$(2)\qquad u^{(r)}(x, 0) = \mathscr{C}_{(x)}^{(r)}$$

$(x \in \Omega_0$; $r = 1, 2, \ldots)$ together with the boundary condition

$$(3)\qquad \frac{du^{(r)}}{dl_\nu}(x, t) + G^{(r)}\left[x, t, u(x, t), \frac{du^{(r)}}{ds_1}(x, t), \ldots, \frac{du^{(r)}}{ds_{\bar{q}}}(x, t)\right] = 0$$

$((x, t) \in \sigma_h$; $r = 1, 2, \ldots)$, where l_ν is one of the vectors $l_1, \ldots, l_q$.

We make the following assumptions (²)

(¹) The integral of a sequence is understood as the sequence of the appropriate integrals.

(²) The symbols ζ, v, w, z, ξ and η in assumptions 3°–6° are understood as $\zeta = (\zeta_1, \ldots, \zeta_{N_r})$; $v = (v_1, \ldots, v_m)$; $w = (w_1, w_2, \ldots)$; $z = (z_{11}, z_{12}, \ldots, z_{mm})$; $\xi = (\xi_1, \ldots, \xi_{N_r})$ and $\eta = (\eta_1, \ldots, \eta_{\bar{q}})$, respectively.

1° The surface σ_h is given by the equation

(4) $$\Gamma(x, t) = 0,$$

where $\Gamma(x, t)$ is a function defined in a closed neighbourhood Π of σ_h and possessing in Π continuous derivatives $\frac{\partial \Gamma}{\partial x_i}(x, t)$, $\frac{\partial^2 \Gamma}{\partial x_i \partial x_j}(x, t)$ and $\frac{\partial \Gamma}{\partial t}(x, t)$ $(i, j = 1, 2, \ldots m)$.

We assume that

(5) $$\operatorname{grad}_x^2 \Gamma(x, t) \stackrel{\mathrm{df}}{=} \sum_{i=1}^{m} \left[\frac{\partial \Gamma}{\partial x_i}(x, t)\right]^2 \geqslant \Gamma_0^2$$

holds, where Γ_0 is a positive constant.

2° The vectors l_i $(i = 1, 2, \ldots, q)$ fulfil the inequality

(6) $$\cos(l_i, \hat{n}) \geqslant \gamma_0 > 0,$$

where γ_0 is a constant and $\hat{n}$ denotes the unit vector of the inward normal to S_t at (x, t).

3° The functions $F^{(r)}(x, t, \zeta, v, w, z)$; $r = 1, 2, \ldots$ are defined in the set

(7) $$(x, t) \in D_h; \quad \zeta_i, v_j, w_\gamma, z_{jk} \in (-\infty, +\infty)$$

and satisfy the following condition:

(8) $$F^{(r)}(x, t, \zeta, v, w, z) - F^{(r)}(x, t, \bar{\zeta}, \bar{v}, \bar{w}, \bar{z}) \leqslant L_0 \sum_{j,k=1}^{m} |z_{jk} - \bar{z}_{jk}| + (L_1 |x| + L_2)\Big(\sum_{j=1}^{m} |v_j - \bar{v}_j| + \sum_{\gamma=1}^{\infty} L_3^\gamma |w_\gamma - \bar{w}_\gamma|\Big) + (L_4 |x|^2 + L_5) \sum_{i=1}^{N_r} |\zeta_i - \bar{\zeta}_i|$$

$(\zeta_r \geqslant \bar{\zeta}_r)$, where $L_0, \ldots, L_5$ and L_3^γ $(\gamma = 1, 2, \ldots)$ are positive constants. We assume that the infinite numerical series $\sum_{\gamma=1}^{\infty} L_3^\gamma$ is convergent and we denote its sum by L_3^∞.

4° The functions $\Phi_\nu^{(r)}(x, t, y, \tau, \zeta, \xi, v)$; $\nu, r = 1, 2, \ldots$ are defined and integrable with respect to (y, τ) in the set

(9) $$(x, t) \in D_h; \quad (y, \tau) \in D_h; \quad \zeta_i, \xi_i, v_j \in (-\infty, +\infty)$$

and satisfy the following inequality

(10) $$|\Phi_\nu^{(r)}(x, t, y, \tau, \zeta, \xi, v) - \Phi_\nu^{(r)}(x, t, y, \tau, \bar{\zeta}, \bar{\xi}, \bar{v})| \leqslant \mathscr{K}_1^{(r)}(x, t; y, \tau) \sum_{i=1}^{N} |\zeta_i - \bar{\zeta}_i| + \mathscr{K}_2^{(r)}(x, t; y, \tau)\Big[\sum_{i=1}^{N} |\xi_i - \bar{\xi}_i| + \sum_{j=1}^{m} |v_j - \bar{v}_j|\Big],$$

with $\mathscr{K}_1^{(r)}(x, t; y, \tau)$ and $\mathscr{K}_2^{(r)}(x, t; y, \tau)$ being positive functions such that

(11) $$\int_0^t \int_{\Omega_{x,\tau}^{(r)}} \mathscr{K}_1^{(r)}(x, t; y, \tau) \exp\left[\frac{K}{\theta}|y|^2\right] dy d\tau \leqslant M, \qquad \int_0^t \int_{\Omega_{x,\tau}^{(r)}} \mathscr{K}_2^{(r)}(x, t; y, \tau) dy d\tau \leqslant M$$

($r = 1, 2, \ldots$), where K and M are positive constants, and θ is a constant in the interval $(0, 1)$.

5° There is a positive constant R_* such that for each point $(x, t) \in D_h$ the relations

$$\Omega_{x,\tau}^{(r)} \subset \begin{cases} \Sigma_{R_*} & \text{for } |x| \leqslant R_*, \\ \Sigma_{|x|} & \text{for } |x| > R_* \end{cases}$$

($r = 1, 2, \ldots, 0 \leqslant \tau \leqslant h$) hold, where Σ_a denotes the closed ball with centre 0 (the origin of the coordinates system) and radius a.

6° The functions $G^{(r)}(x, t, \zeta, \eta)$; $r = 1, 2, \ldots$ are defined in the set

(12) $$(x, t) \in \sigma_h; \quad \zeta_i, \eta_i \in (-\infty, +\infty)$$

and fulfil the condition

(13) $$G^{(r)}(x, t, \zeta, \eta) - G^{(r)}(x, t, \bar{\zeta}, \eta) \leqslant L_* \sum_{i=1}^{N} |\zeta_i - \bar{\zeta}_i|$$

($\zeta_r \geqslant \bar{\zeta}_r$), where L_* is a positive constant.

3. We shall prove the following theorem

THEOREM 1. *If assumptions 1°–6° are satisfied, then the (F_3)-problem possesses at most one solution parabolic, regular and of class E_2 in D_h, and of class $C^1(D_h \cup \sigma_h)$* (³).

Proof. Assume that the point 0 belongs to Ξ_h and observe that one can construct a function $\tilde{\Gamma}(x, t)$ defined and having bounded and continuous

(³) For the definition of parabolicity see Szarski [18]–[20]. By the regularity of a solution we understand that all functions $u^{(r)}(x, t)$ ($r = 1, 2, \ldots$) are continuous in $D_h \cup \Sigma$, where $\Sigma = \sigma_h \cup \bar{\Omega}_0$, and have in D_h continuous derivatives $u_x^{(r)}(x, t)$, $u_{xx}^{(r)}(x, t)$ and $u_t^{(r)}(x, t)$. A solution is said to be *of class E_2* in D_h if $|u^{(r)}(x, t)| \leqslant M_0 \exp[K_0|x|^2]$ for $(x, t) \in D_h$ and $r = 1, 2, \ldots$, where M_0 and K_0 are positive constants. In the sequel we assume that $K_0 < K$.

derivatives $\frac{\partial \tilde{\Gamma}}{\partial x_i}(x, t)$, $\frac{\partial^2 \tilde{\Gamma}}{\partial x_i \partial x_j}(x, t)$ and $\frac{\partial \tilde{\Gamma}}{\partial t}(x, t)$ $(i, j = 1, 2, \ldots, m)$ in the set $R^m \times (0, h)$, and satisfying the equality

$$(14) \qquad \tilde{\Gamma}(x, t) = \begin{cases} \Gamma(x, t) & \text{for } (x, t) \in \Pi, \\ |x| & \text{for } (x, t) \notin \Sigma_0 \times \langle 0, h \rangle, \end{cases}$$

where $\Sigma_0 \subset R^m$ is a closed ball with centre 0 and radius R_0 so large that $\Pi \subset \Sigma_0 \times \langle 0, h \rangle$.

Now, let r be an arbitrarily fixed positive integer.

Suppose that there are two solutions, $(u_1^{(1)}, u_1^{(2)}, \ldots)$ and $(u_2^{(1)}, u_2^{(2)}, \ldots)$, of the (F_3)-problem satisfying the conditions of Theorem 1.

We denote $u^{(i)} = u_1^{(i)} - u_2^{(i)}$ for $i = 1, 2, \ldots, N_r$ and we set

$$(15) \qquad u_j^{(i)} = v_j^{(i)} H^{(r)}(x, t, K)$$

$(j = 1, 2;\ i = 1, 2, \ldots, N_r)$, where

$$(16) \qquad H^{(r)}(x, t, K) = \exp\left\{\frac{K[\tilde{\Gamma}(x, t) - p]^2}{1 - \mu t} + \nu t\right\}$$

with

$$(17) \qquad p = p(K, r) = \frac{1 + L_* N_r}{2K\Gamma_0 \gamma_0};$$

$$(18) \qquad \mu = \mu(K, r) = 4KL_0 A^2 + 2L_1 A(1 + L_3^{\alpha} M) + \frac{L_4 N_r + 1}{K};$$

$$(19) \qquad \nu = \nu(K, r) = \frac{1}{\theta^2} \max\{(1 + [(L_1 p + L_2)(1 + L_3^{\alpha} M) KA + (BL_0 + C) K + pL_4 N_r + \tfrac{1}{2} L_1 L_3^{\alpha} M N_r (M_* + 1)]^2 + N_r (p^2 L_4 + L_5) + (L_1 p + L_2) L_3^{\alpha} M (M_* + 1) N_r + 2KL_0 A^2),$$
$$(1 + [(L_1 R_0 + L_2)(1 + L_3^{\alpha} M) KA + KBL_0 + KC]^2 + 2KL_0 A^2 + (L_4 R_0^2 + L_5) N_r + (L_1 R_0 + L_2) L_3^{\alpha} M (M_* + 1) N_r)\}.$$

Here K and θ are the constants appearing in (11), and A, B, C and M_* are positive constants (M_* in general depends on r) satisfying the following conditions:

$$(20) \qquad \sum_{i=1}^{m} \left|\frac{\partial \tilde{\Gamma}}{\partial x_i}(x, t)\right| \leqslant A; \quad \sum_{j,k=1}^{m} \left|\frac{\partial^2 \tilde{\Gamma}}{\partial x_j \partial x_k}(x, t)\right| \leqslant B; \quad \left|\frac{\partial \tilde{\Gamma}}{\partial t}(x, t)\right| \leqslant C;$$
$$\sup_{\Sigma_0 \times \langle 0, h \rangle} \exp\left\{\frac{K}{\theta}[\tilde{\Gamma}(y, \tau) - p]^2\right\} + \exp\left[\frac{K}{\theta} p^2\right] \leqslant M_*$$

$((x, t) \in D_h)$.

First, we shall confine ourselves to the part D_{h_0} of D_h placed in the zone $0 < t \leqslant h_0$, where

(21) $$h_0 = h_0(r) = (1-\theta)/\mu.$$

Let us consider an increasing number sequence $\{R_\alpha\}$ such that $R_\alpha > \max(R_0, R_*)$ for $\alpha = 1, 2, \ldots$ and $R_\alpha \to \infty$ when $\alpha \to \infty$. Denote by $D_{h_0}^{\alpha,\tau}$ and $\sigma_{h_0}^{\tau}$ the parts of D_{h_0} and σ_{h_0}, respectively, contained in the cylinder $\Sigma_\alpha \times (0, \tau\rangle$, where Σ_α is understood similarly as $\Sigma_{|x|}$ and Σ_0 above, and τ is arbitrarily fixed in $(0, h_0)$. Finally, consider a sequence $\{A_r^{\alpha,\tau}\}$ whose elements are defined by

(22) $$A_r^{\alpha,\tau} = \max_{1 \leqslant i \leqslant N_r} \sup_{D_{h_0}^{\alpha,\tau}} |v^{(i)}(x, t)|,$$

where $v^{(i)} = v_1^{(i)} - v_2^{(i)}$.

In order to complete the proof, it is sufficient to show that $A_r^{\alpha,\tau} = 0$ for $\alpha = 1, 2, \ldots$ which, as $\{A_r^{\alpha,\tau}\}$ is non-decreasing and $A_r^{\alpha,\tau} \geqslant 0$, reduces to proving that $A_r^{\alpha,\tau} \to 0$ when $\alpha \to \infty$.

Let us note that for each α there are a positive integer i_* $(1 \leqslant i_* \leqslant N_r)$ and a point $(x_*, t_*) \in \bar{D}_{h_0}^{\alpha,\tau}$ such that $A_r^{\alpha,\tau} = |v^{(i_*)}(x_*, t_*)|$. In the sequel we shall limit ourselves to the following two cases:

(a) $(x_*, t_*) \in D_{h_0}^{\alpha,\tau}$ and (b) $(x_*, t_*) \in \sigma_{h_0}^{\tau}$,

where $v^{(i_*)}(x_*, t_*) > 0$ (in the cases $(x_*, t_*) \in (\mathrm{Fr}\, D_{h_0}^{\alpha,\tau}) \setminus \sigma_{h_0}^{\tau}$ or $v^{(i_*)}(x_*, t_*) = 0$ the required relation $\lim_{\alpha \to \infty} A_r^{\alpha,\tau} = 0$ is straightforward and if $v^{(i_*)}(x_*, t_*) < 0$, one should apply a reasoning analogous to that for $v^{(i_*)}(x_*, t_*) > 0$ replacing $v^{(i_*)}$ by $\bar{v}^{(i_*)} = -v^{(i_*)}$).

In case (a) we shall use the relations

(23) $$\frac{\partial v^{(i_*)}}{\partial t}(x_*, t_*) \geqslant 0; \quad \frac{\partial v^{(i_*)}}{\partial x_i}(x_*, t_*) = 0;$$
$$\sum_{j,k=1}^{m} \frac{\partial^2 v^{(i_*)}}{\partial x_j \partial x_k}(x_*, t_*) \lambda_j \lambda_k \leqslant 0$$

($i = 1, 2, \ldots, m$ and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_m)$ is an arbitrary real vector) resulting from the assumption $v^{(i_*)}(x_*, t_*) > 0$. Substituting $(x, t) = (x_*, t_*)$ in the i_*-th equation of system (1), using a decomposition analogous to that in [2] (formula (3.3)) and basing subsequently on the parabolicity of the solutions $(u_j^{(1)}, u_j^{(2)}, \ldots)$ $(j = 1, 2)$, on (23) and on assumptions 3° and 4°, we get

$$(24)\quad \frac{\partial v^{(i_*)}}{\partial t}(x_*, t_*) H^{(r)}(x_*, t_*, K)$$

$$\leqslant v^{(i_*)}(x_*, t_*)\cdot\Big\{L_0 \sum_{j,k=1}^{m}\left|\frac{\partial^2 H^{(r)}}{\partial x_j \partial x_k}(x_*, t_*, K)\right| +$$

$$+(L_1|x_*|+L_2)+\Big(\sum_{i=1}^{m}\left|\frac{\partial H^{(r)}}{\partial x_i}(x_*, t_*, K)\right|+$$

$$+\sum_{\gamma=1}^{\nu} L_3^{\gamma}\Big\langle\int_0^{t_*}\int_{\Omega_{x_*,\tau}^{(i_*)}}\Big\{\mathcal{K}_1^{(i_*)}(x_*, t_*; y\tau)\cdot H^{(r)}(y, \tau, K)\cdot\sum_{i=1}^{N_{i_*}}|v^{(i)}(y, \tau)|+$$

$$+\mathcal{K}_2^{(i_*)}(x_*, t_*, y, \tau)\Big[H^{(r)}(x_*, t_*, K)\cdot\sum_{i=1}^{N_{i_*}}|v^{(i)}(x_*, t_*)|+$$

$$+v^{(i_*)}(x_*, t_*)\sum_{i=1}^{m}\left|\frac{\partial H^{(r)}}{\partial x_i}(x_*, t_*, K)\right|\Big]\Big\}dyd\tau\Big\rangle\Big)+$$

$$+(L_4|x_*|^2+L_5)N_{i_*}H^{(r)}(x_*, t_*, K)-\frac{\partial H^{(r)}}{\partial t}(x_*, t_*, K)\Big\}.$$

Let us observe that by assumption 5° and the definitions of i_* and (x_*, t_*) the relations

$$(25)\qquad |v^{(i)}(y, \tau)| \leqslant v^{(i_*)}(x_*, t_*); \quad |v^{(i)}(x_*, t_*)| \leqslant v^{(i_*)}(x_*, t_*)$$

hold for $(y, \tau)\in\Omega_{x_*,\tau}^{(i_*)}$, $0 \leqslant \tau \leqslant t_*$; $i = 1, 2, \ldots, N_r$.

Furthermore, in virtue of (16) and (21) we can assert that for $0 \leqslant \tau \leqslant t_*$ the sequence of inequalities

$$H^{(r)}(y, \tau, K) \leqslant H^{(r)}(x_*, t_*, K)\exp\Big\{\frac{K[\tilde{\Gamma}(y, \tau)-p]^2}{1-\mu\tau}-$$
$$-\frac{K[\tilde{\Gamma}(x_*, t_*)-p]^2}{1-\mu t_*}+\nu(\tau-t_*)\Big\}$$
$$\leqslant H^{(r)}(x_*, t_*, K)\exp\Big\{\frac{K[\tilde{\Gamma}(y, \tau)-p]^2}{1-\mu\tau}\Big\}$$
$$\leqslant H^{(r)}(x_*, t_*, K)\exp\Big\{\frac{K}{\theta}[\tilde{\Gamma}(y, \tau)-p]^2\Big\}$$

is valid; hence and by (11) and (20) we have

$$(26)\quad \int_0^{t_*}\int_{\Omega^{(i_*)}_{x_*,\tau}} \mathscr{K}^{(i_*)}_1(x_*, t_*; y, \tau) H^{(r)}(y, \tau, K)\, dy\, d\tau$$

$$\leqslant H^{(r)}(x_*, t_*, K)\Big(\sup_{\Sigma_0\times\langle 0,h\rangle} \exp\Big\{\frac{K}{\theta}[\tilde{\Gamma}(y, \tau)-p]^2\Big\}\cdot$$

$$\cdot\int_0^{t_*}\int_{\Omega^{(i_*)}_{x_*,\tau}\cap\Sigma_0} \mathscr{K}^{(i_*)}_1(x_*, t_*; y, \tau)\, dy\, d\tau +$$

$$+\exp\Big[\frac{K}{\theta}p^2\Big]\int_0^{t_*}\int_{\Omega^{(i_*)}_{x_*,\tau}\setminus\Sigma_0} \mathscr{K}^{(i_*)}_1(x_*, t_*; y, \tau)\cdot\exp\Big[\frac{K}{\theta}|y|^2\Big] dy\, d\tau\Big)$$

$$\leqslant M_* M H^{(r)}(x_*, t_*, K).$$

Using (24)–(26), (11), (20) and (21) and basing on the relation $N_{i_*} \leqslant N_r$ we obtain

$$(27)\quad \frac{\partial v^{(i_*)}}{\partial t}(x_*, t_*) < \frac{1}{(1-\mu t_*)^2} v^{(i_*)}(x_*, t_*)\{L_0[4K^2A^2(\tilde{\Gamma}-p)^2+$$

$$+2KA^2+2KB|\tilde{\Gamma}-p|]+(L_1|x_*|+L_2)[2KA|\tilde{\Gamma}-p|+$$

$$+L_3^\infty M(N_r(M_*+1)+2KA|\tilde{\Gamma}-p|)]+(L_4|x_*|^2+L_5)N_r+$$

$$+2KC|\tilde{\Gamma}-p|-K\mu(\tilde{\Gamma}-p)^2-\nu\theta^2\},$$

where $\tilde{\Gamma} = \tilde{\Gamma}(x_*, t_*)$.

If $|x_*| > R_0$, then we obtain from (27), on account of relations (14), (18) and (19), the following inequalities:

$$(28)\quad \frac{\partial v^{(i_*)}}{\partial t}(x_*, t_*) \leqslant \frac{1}{(1-\mu t_*)^2} v^{(i_*)}(x_*, t_*)\{-[||x_*|-p|-$$

$$-((L_1p+L_2)\cdot(1+L_3^\infty M)KA+(BL_0+C)K+pL_4N_r)-$$

$$-\tfrac{1}{2}L_1L_3^\infty MN_r(M_*+1)]^2-1\} < 0.$$

If $|x_*| \leqslant R_0$, a similar calculation yields

$$(29)\quad \frac{\partial v^{(i_*)}}{\partial t}(x_*, t_*) \leqslant \frac{1}{(1-\mu t_*)^2} v^{(i_*)}(x_*, t_*)\{-[|\tilde{\Gamma}-p|-K((L_1R_0+$$

$$+L_2)(1+L_3^\infty M)A+BL_0+C)]^2-1\} < 0.$$

Both inequalities (28), (29) contradict the first of relations (23). Q.E.D. There still remains to consider case (b) (see p. 104)

Note that in this case the inequality (see [3], p. 294)

$$(30)\quad \frac{d}{dl_\nu} v^{(i_*)}(x_*, t_*) \leqslant 0$$

holds.

Furthermore, by Lemma 1 in [6], we have

$$(31) \qquad \frac{d}{ds_j} v^{(i_*)}(x_*, t_*) = 0.$$

Lastly, the relation ([4])

$$\frac{d}{ds_j} H(x_*, t_*) = -\frac{2Kp}{1-\mu t_*} H(x_*, t_*, K) |\operatorname{grad} \Gamma(x_*, t_*)| \cdot \cos(\hat{n}, s_j) = 0$$

holds for $j = 1, 2, \ldots, \tilde{q}$.

Using the boundary condition (3) with $r = i_*$, $(x, t) = (x_*, t_*)$ and basing on (13) and (31) we obtain

$$(32) \qquad -\frac{d}{dl_v} v^{(i_*)}(x_*, t_*) H^{(i_*)}(x_*, t_*)$$

$$\leqslant v^{(i_*)}(x_*, t_*) H^{(i_*)}(x_*, t_*) \left[\frac{-2Kp}{1-\mu t_*} |\operatorname{grad} \Gamma(x_*, t_*)| \cos(\hat{n}, l_v) + L_* N_r \right]$$

$$\leqslant \frac{v^{(i_*)}(x_*, t_*)}{1-\mu t_*} (-2Kp\Gamma_0 \gamma_0 + L_* N_r) H^{(i_*)}(x_*, t_*),$$

whence, and by (17), we get

$$-\frac{d}{dl_v} v^{(i_*)}(x_*, t_*) \leqslant -\frac{1}{1-\mu t_*} v^{(i_*)}(x_*, t_*) < 0,$$

contrary to inequality (30).

Thus, the relation $u_1^{(i)}(x, t) = u_2^{(i)}(x, t)$, $i = 1, 2, \ldots, N_r$, is proved for the case of $h \leqslant h_0$. If $h > h_0$, we use the substitution $t = \tilde{t} + jh_0$ ($j = 1, 2, \ldots$) and prove the assertion successively for the parts of D_h contained in the zones $jh_0 \leqslant t \leqslant (j+1)h_0$. As r is any positive integer, the proof of Theorem 1 is completed.

Remark 1. If follows directly from the reasoning in the proof of Theorem 1 that an analogue of this theorem is valid for the interior (F_3)-problem (i.e. the (F_3)-problem for the domain Ξ_h), with the uniqueness of the solution holding in the class $C^1(\Xi_h \cup \sigma_h)$.

Remark 2. Suppose that system (1) is of the form

([4]) We choose the sign of $\Gamma(x, t)$ so that $\frac{\partial \Gamma}{\partial x_i}(x, t) = |\operatorname{grad}_x \Gamma(x, t)| \cos(\bar{n}, x_i)$; $i = 1, 2, \ldots, m$.

$$(33)\quad \sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}^{(r)}(x,t)u_{x_\alpha x_\beta}^{(r)}(x,t)-u_t^{(r)}(x,t) = \tilde{F}^{(r)}\Big[x,t,u(x,t),u_x^{(r)}(x,t), \int_0^t\int_{\Omega_{x,t}} \Phi^{(r)}(x,t,y,\tau,u(y,\tau),u(x,t),u_x^{(r)}(x,t))\,dy\,d\tau\Big]$$

and that the derivative $du^{(r)}/dl_\nu$ in the boundary condition (3) is replaced by the transversal derivative

$$\frac{du^{(r)}}{d\Gamma_x^{(r)}} = \sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}^{(r)}(x,t)u_{x_\alpha}^{(r)}(x,t)\cos(\hat{n},x_\beta).$$

If

(a) The coefficients $a_{\alpha\beta}^{(r)}(x,t)$ are bounded and continuous in $\bar{D}_h$;

(b) The characteristic forms of equations (33) are positive definite and $\sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}^{(r)}(x,t)\lambda_\alpha\lambda_\beta \geqslant g_0|\lambda|^2$ for $(x,t)\in\bar{D}_h$, where g_0 is a positive constant;

(c) The functions $\Phi_\gamma^{(r)}$ and $G^{(r)}$ $(r,\gamma = 1,2,\ldots)$ and the surface σ_h satisfy assumptions 1°, 4° and 6°, respectively, and the functions $F^{(r)}(x,t,\zeta,v,w,z) = \sum_{\alpha,\beta=1}^{m} a_{\alpha\beta}^{(r)}(x,t)\cdot z_{\alpha\beta}-\tilde{F}^{(r)}(x,t,\zeta,v,w)$ fulfil assumption 3°;

(d) Assumption 5° is satisfied,

Then Theorem 1 is valid.

To prove Remark 2 it is sufficient to repeat the proof of Theorem 1 given above with $p = (1+L_* N_r/(2K\Gamma_0 g_0)$, just replacing γ_0 by g_0 in (32).

Remark 3. It follows from the foregoing Remarks that if the assumptions of Remark 2 are satisfied and if in the problem considered in [7] the oblique derivatives in the boundary conditions are replaced by the tangential ones, then this problem possesses at most one solution.

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