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Continuity in semidynamical systems

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Abstract. We investigate the function $\boldsymbol{R}_+ \times X \ni (t, x) \mapsto F(t, x) \in \mathscr{P}(X)$ in the semidynamical system $(X, \boldsymbol{R}_+, \pi)$, where $F(t, x) = \{y \in X: \pi(t, y) = x\}$. We show that under some assumptions this function is upper semicontinuous. It is also proved that certain semidynamical systems with negative unicity are, in fact, local dynamical systems and are isomorphic to dynamical systems.

Introduction. In the semidynamical system $(X, \boldsymbol{R}_+, \pi)$ we may consider $F(t, x)$ – "the past" of a given point x. Since the function $\pi: \boldsymbol{R}_+ \times X \to X$ is continuous, it is natural to ask whether the functions $F: \boldsymbol{R}_+ \times X \to \mathscr{P}(X)$ and $F: \boldsymbol{R}_+ \times \mathscr{P}(X) \to \mathscr{P}(X)$ have properties similar to continuity (in the case of a metric space known as "upper semicontinuity"). In this paper we show that under some – not too strong – assumptions on the system we can give a positive answer to this question. Moreover, every system without start points on a locally compact, paracompact and first countable space is isomorphic to a system with the required property.

As simple consequences of these theorems we get the following results. Let us consider a semidynamical system which admits the negative unicity property (according to [1] and [3] it is said to have negative unicity); then we can define $\pi(t, x)$ for some $t < 0$ in a natural way. Then we show that every such semidynamical system without start points on a locally compact, paracompact and first countable space is in fact a *local dynamical system* and is isomorphic (as a semidynamical system) to a dynamical system – provided a suitable function is continuous.

SECTION I

1.1. Basic definitions. A semidynamical system on a topological space X is a triplet $(X, \boldsymbol{R}_+, \pi)$, where $\boldsymbol{R}_+$ is the set of all non-negative real numbers, and π is a map from $\boldsymbol{R}_+ \times X \to X$ satisfying the following conditions: $\pi(0, x) = x$ for every $x \in X$, $\pi(t, \pi(s, x)) = \pi(t+s, x)$ for every $x \in X$ and $t, s \in \boldsymbol{R}_+$

and π is continuous. Replacing $\boldsymbol{R}$ by $\boldsymbol{R}_+$ we get the definition of a dynamical system. For the definition of a local semidynamical system, which will be used in one of the theorems, the reader is referred to [1], [3], [5], [6].

Assume that a semidynamical system $(X, \boldsymbol{R}_+, \pi)$ is given. Let $A \subset \boldsymbol{R}_+$ and $M \subset X$. Let us put $F(A, M) = \{y \in X\colon \pi(t, y) \in M \text{ for some } t \in A\}$. If $A = \{t\}$ and $M = \{x\}$, we write $F(t, x)$ instead of $F(\{t\}, \{x\})$.

A point $x \in X$ is said to be a start point if $\pi(t, y) \neq x$ for any $y \in X$ and $t > 0$.

A function $\varphi\colon I \to X$ where I is a non-empty interval in $\boldsymbol{R}$ is called a solution if $\pi(t, \varphi(s)) = \varphi(t+s)$ whenever $t \in I$, $t+s \in I$ and $s \in \boldsymbol{R}_+$. If $0 \in I$ and $\varphi(0) = x$, the solution is called a solution through x. The solution φ is called a left solution through x if the maximum of the domain of φ is equal to 0 (and $\varphi(0) = x$). A solution is called a left maximal if it is a left solution and it is maximal (with respect to inclusion) relative to the property of being a left solution. It is known ([1]) that if φ is a left maximal solution and the domain of φ is equal to $(\alpha, 0]$ ($\alpha \neq -\infty$) then there are no cluster points of $\varphi(t)$ as $t \to \alpha^+$.

A semidynamical system is said to have negative unicity if for any $x, y \in X$ and $t \in \boldsymbol{R}_+$ the equality $\pi(t, x) = \pi(t, y)$ holds only if $x = y$.

Throughout this paper by a neighbourhood of x (or of the set M) we mean a set – not necessarily open! – which contains an open set containing x (or M).

Throughout this section we assume as given a semidynamical system $(X, \boldsymbol{R}_+, \pi)$ on a *locally compact* and *first countable* space X, a *compact* set $M \subset X$, $x \in X$ and $t \geqslant 0$.

1.2. Proposition ([1], 4.4). *For every neighbourhood U of M there is a neighbourhood V of M and $s > 0$ such that $F([0, s], V) \subset U$.*

1.3. Lemma ([4], 3.3). *For every $a, b \in \boldsymbol{R}_+$ ($a \leqslant b$) the sets $F([a, b], M)$ and $F(b, M)$ are closed.*

1.4. Lemma ([4], 3.4). *There exists an $\alpha > 0$ such that $F([0, t], M)$ is compact for every $t < \alpha$.*

1.5. Lemma ([4], 3.5). *Let $N \subset X$ and $I, J \subset \boldsymbol{R}_+$. Then $F(I, F(J, N)) = F(I+J, N)$.*

1.6. Lemma. *Let $F(t, M)$ be compact. Then for every neighbourhood U of $F(t, M)$ there is an $s > t$ such that $F([t, s], M) \subset U$.*

Proof. By Proposition 1.2 there exists a $\lambda > 0$ such that $F([0, \lambda], F(t, M)) \subset U$. Let us put $s = t+\lambda$; using Lemma 1.5, we finish the proof.

1.7. EXAMPLE. Let us put $X = (\boldsymbol{R} \times \{0\}) \cup (\{0\} \times (-1, 0])$ and define the function π as follows:

$$\pi(s, (z, 0)) = (s+z, 0),$$
$$\pi(s, (0, y)) = \begin{cases} (0, s+y) & \text{if } s+y \leqslant 0, \\ (s+y, 0) & \text{if } s+y \geqslant 0. \end{cases}$$

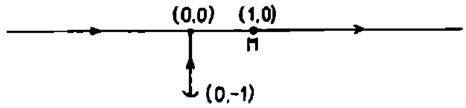

Fig. 1

It is easy to verify that a theorem analogous to Lemma 1.6 with $s < t$ is not true; let us put $M = \{(1, 0)\}$ and $t = 2$.

The same example shows that the following theorem is not true:

If $F(t, x)$ is compact, then for every neighbourhood U of $F(t, x)$ there exists a neighbourhood V of x such that $F(t, V) \subset U$.

1.8. THEOREM. *Let π have no start points. Assume that there exists an $s < t$ such that $F([s, t], M)$ is compact. Then for every neighbourhood U of $F(t, M)$ there is an $\bar{s} < t$ such that $F([\bar{s}, t], M) \subset U$.*

Proof. Suppose to the contrary that there exists a sequence $\{u_n\}$ with $u_n \to t^-$ and $F([u_n, t], M) \not\subset U$. We can find a sequence $\{s_n\}$ $(u_n \leqslant s_n \leqslant t)$ with $s_n \to t^-$ and $F(s_n, M) \not\subset U$, and so we can find nets $\{x_n\}$, $\{y_n\}$ such that every x_n belongs to M, no y_n belongs to U and $\pi(s_n, y_n) = x_n$. Let us take for every n any σ_n – left maximal solution such that $\sigma_n(-s_n) = y_n$. Let $(\alpha_n, 0]$ be the domain of σ_n. From the elementary properties of solutions we have $\alpha_n < -t$ or $\alpha_n \geqslant -s$ (as π has no start points, $\sigma_n(t_k)$ has no accumulation point with $t_k \to \alpha_n^-$ and $F([s, t], M)$ is compact). Let us put $z_n = \sigma_n(-t)$. Thus $y_n = \sigma_n(-s_n) = \sigma_n(-t+t-s_n) = \pi(t-s_n, \sigma_n(-t_n)) = \pi(t-s_n, z_n)$. But $\pi(t, z_n) = \pi(s_n+t-s_n, z_n) = \pi(s_n, y_n) = x_n \in M$ and $z_n \in F(t, M)$, and so we may assume (by the compactness of $F(t, M)$ – Lemma 1.3) that $z_n \to z \in F(t, M)$. But $t-s_n \to 0$; so $y_n = \pi(t-s_n, z_n) \to z \in F(t, M)$ and $y_n \in U$ for sufficiently large n, which is impossible.

1.9. LEMMA. *Assume that there exists a compact neighbourhood V of M with $F(t, V)$ compact. Then there is a compact neighbourhood W of M and $s < t$ such that $F([s, t], W)$ is compact.*

Proof. We first show that there exist a $\lambda > 0$ and a compact neighbourhood W of M with $\pi([0, \lambda] \times W) \subset V$. Indeed, it easily follows from the compactness of M and the tact that for every $y \in M$ there exist a compact neighbourhood W_y of y and $\lambda_y > 0$ such that $\pi([0, \lambda_y] \times W_y) \subset V$. Now put $s = t - \lambda$.

If $z \in F([s, t], W)$, then there exist an $\alpha \in [s, t]$ and a $y \in W$ with $\pi(\alpha, z) = y$. Let us notice that $t-\alpha \in [0, t-s] = [0, \lambda]$. Then $\pi(t, z) = \pi(t-\alpha, \pi(\alpha, z)) = \pi(t-\alpha, y) \in \pi([0, \lambda] \times W) \subset V$, and so $z \in F(t, V)$. By Lemma 1.3, $F([s, t], W)$ is closed, and so it is compact as a subset of $F(t, V)$.

1.10. Remark. In the sequel we shall assume that there exists a countable basis $\mathscr{B}(M)$ of M (cf. [2]) and X is first countable. One can ask if the second assumption implies the first one. The answer is negative; a counterexample (communicated to the author by Anna Waśko) is the space of *Concentric Circles* [7], Example 97 known also as *Doubled Alexandroff Circle* (see [2]). The space is locally compact and first countable but, as can easily be verified, the smaller circle is a compact set and does not have a countable basis.

However, the following lemma is true:

1.11. Lemma. *A compact set M in a second countable space X has a countable basis.*

Proof. Let $\mathscr{D}$ be a countable basis of X. Let us introduce a new family $\mathscr{B}$; $Y \in \mathscr{B}$ if and only if Y is the finite union of elements of $\mathscr{D}$ and $M \subset Y$. It follows immediately that $\mathscr{B}$ is countable. Now we show that $\mathscr{B}$ is a basis of M. Let U be an open neighbourhood of M; then $U = \bigcup_{i=1}^{\infty} V_i$, where $\{V_i\}_{i \in N}$ is a subset of $\mathscr{D}$. For every $x \in M$ there is a $V_x \in \{V_i\}_{i \in N}$ such that $x \in V_x \subset U$; by the compactness of M we may find a finite subcover $\{V_{x_j}\}_{j=1}^{n}$ of M. Thus $M \subset \bigcup_{j=1}^{n} V_{x_j} \subset U$, which finishes the proof.

1.12. Remark. Obviously in a metric space every compact set M has a countable basis.

1.13. Theorem. *Assume that there exist a countable basis $\mathscr{B}(M)$ and a compact neighbourhood $\tilde{W}$ of M with $F(t, \tilde{W})$ compact and that $(X, \boldsymbol{R}_+, \pi)$ has no start points. Then for every neighbourhood U of $F(t, M)$ there exist an $s < t$ and a neighbourhood V of M such that $F([s, t], V) \subset U$.*

Proof. By Lemma 1.9 there exist a neighbourhood W of M and an $s < t$ with $F([s, t], W)$ compact. Denote by V_n the intersections of the elements of $\mathscr{B}(M)$ with W; we may assume that $V_{n+1} \subset V_n$ and V_n is compact (for every $n \in N$). Clearly $\bigcap \{V_n: n \in N\} = M$ (as every locally compact space is T_1).

Suppose to the contrary that for every $u < t$ and a neighbourhood $\tilde{V}$ of M we have $F([u, t], \tilde{V}) \not\subset U$. Then we can find a net $\{u_n\}$, $u_n \to t^-$, such that $F([u_n, t], V_n) \not\subset U$ for every n and thus we infer that for every n there exist an $s_n \in [u_n, t]$, an $x_n \in V_n$ and a y_n with $y_n \notin U$ and $\pi(s_n, y_n) = x_n$. Clearly $s_n \to t^-$ and $s_n > s$ for sufficiently large n.

As in the proof of Theorem 1.8 we define σ_n — the left maximal solution through x_n with domain equal to $(\alpha_n, 0]$. As in Theorem 1.8, we have either $\alpha_n < -t$ or $\alpha_n \geqslant -s$. Define $z_n = \sigma_n(-t)$; moreover, $\sigma_n(-s_n) = y_n$.

Thus we get

$$y_n = \sigma_n(-s_n) = \sigma_n(-t+t-s_n) = \pi\big(t-s_n, \sigma_n(-t)\big) = \pi(t-s_n, z_n)$$

and

$$\pi(t, z_n) = \pi\big(s_n, \pi(t-s_n, z_n)\big) = \pi(s_n, y_n) = x_n \in V_n \subset W.$$

We may assume that $z_n \to z_0 \in F(t, W)$ and $x_n \to x_0 \in W$, as $F(t, W)$ and W are compact. Moreover, $x_0 \in M$ as $x_n \in V_n$, $V_{n+1} \subset V_n$ for every integer n and $\bigcap \{V_n\colon n \in N\} = M$.

So we have:

$$\pi(t, z_0) \leftarrow \pi(t, z_n) = x_n \to x_0 \in M \quad \text{and} \quad z_0 \in F(t, M).$$

Since $t-s_n \to 0$ and $z_n \to z_0$, we have $y_n = \pi(t-s_n, z_n) \to \pi(0, z_0) \in F(t, M)$, and so $y_n \in U$ for sufficiently large n, which is impossible.

1.14. Remark. Example 1.7 shows that from the compactness of $F(t, M)$ it does not follow that there exists a neighbourhood W of M with $F([0, t], W)$ compact ($M = \{(1, 0)\}$ and $t = 2$).

1.15. THEOREM. *Assume that there exist a countable basis of M and a compact neighbourhood W of M with $F(t, W)$ compact (we do not require that π should have no start points!). Then for every neighbourhood U of $F(t, M)$ there is a neighbourhood V of M such that $F(t, V) \subset U$.*

Proof. As in Theorem 1.13 we have $\{V_n\}$ – a family of compact neighbourhoods of M with $V_{n+1} \subset V_n \subset W$ and $\bigcap \{V_n\colon n \in N\} = M$.

Suppose to the contrary that $F(t, V_n) \not\subset U$ for every n. Thus we may construct sequences $\{z_n\}$, $\{x_n\}$ such that $x_n \in V_n$, $z_n \notin U$ and $\pi(t, z_n) = x_n$. By the compactness of W we may assume that $x_n \to x_0 \in W$, as in Theorem 1.13, $x_0 \in M$. Next, $z_n \in F(t, V_n)$, so $z_n \in F(t, W)$; the last set is compact, and so we can assume that $z_n \to z_0 \in F(t, W)$. Then we have

$$\pi(t, z_0) \leftarrow \pi(t, z_n) = x_n \to x_0 \in M,$$

so $z_0 \in F(t, M)$ and $z_n \in U$ for sufficiently large n, which is impossible.

1.16. LEMMA. *Assume that there exist a countable basis of M and a neighbourhood $\tilde{W}$ of M with $F(t, \tilde{W})$ compact. Then for every neighbourhood U of M there exist a neighbourhood V of M and an $s > t$ such that $F([t, s], V) \subset U$.*

Proof. By Proposition 1.2 it follows that there exist a neighbourhood W of $F(t, M)$ and a $\lambda > 0$ such that $F([0, \lambda], W) \subset U$. By Theorem 1.15 we infer that there exists a neighbourhood V of M with $F(t, V) \subset W$. Let us put $s = t+\lambda > t$. Then (by Lemma 1.5) $F([t, s], V) = F\big([0, \lambda], F(t, V)\big) \subset F([0, \lambda], W) \subset U$, which finishes the proof.

Now we state — as a corollary — the main theorem of this section, recalling once more all the assumptions.

1.17. THEOREM. *Let $(X, \boldsymbol{R}_+, \pi)$ be a semidynamical system without start points on a locally compact and first countable space; let M be a compact subset of X which has a countable basis.*

Take any $t > 0$ such that there exists a neighbourhood W of M with $F(t, W)$ compact.

Then for every neighbourhood U of $F(t, M)$ there exist a $\delta > 0$ and a neighbourhood V of M such that $F([t-\delta, t+\delta], V) \subset U$.

The proof follows immediately from Theorems 1.13 and 1.16.

SECTION II

Now we state some theorems which show that a semidynamical system without start points with negative unicity can be considered a dynamical system.

First we state some definitions and theorems. Throughout this section we assume as given a semidynamical system $(X, \boldsymbol{R}_+, \pi)$, on a *locally compact, paracompact* and *first countable* space (for instance, any locally compact metric space fulfils these conditions) *without start points*. We also assume as given $x \in X$, $t \geqslant 0$ and a *compact* set $M \subset X$.

2.1. DEFINITION. We define the negative escape time $N(x)$ as follows:

$N(x) = \inf\{s \in (0, \infty]$: $(-s, 0]$ is the domain of the left maximal solution through $x\}$.

As one can easily verify, under our assumptions this definition is equivalent to the definition given by McCann in [4]. Note that it is not equivalent to the definition given in [1].

Let us put $N(M) = \inf\{N(x)\colon x \in M\}$.

2.2. LEMMA ([4], 3.8). $N(M) = \sup\{s$: *$F([0, s], M)$ is compact*$\}$.

2.3. DEFINITION. Let $(X, \boldsymbol{R}_+, \pi)$ and $(Y, \boldsymbol{R}_+, \varrho)$ be semidynamical systems. The system $(X, \boldsymbol{R}_+, \pi)$ is said to be *isomorphic* to $(Y, \boldsymbol{R}_+, \varrho)$ if there exist a homeomorphism $h\colon X \to Y$ and a continuous mapping $\varphi\colon \boldsymbol{R}_+ \times X \to \boldsymbol{R}_+$ such that:

(i) $\varphi(0, x) = 0$ for each $x \in X$,

(ii) for each $x \in X$ the mapping $\varphi(\cdot, x)\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a homeomorphism,

(iii) $h(\pi(t, x)) = \varrho(\varphi(t, x), h(x))$ for each $(t, x) \in \boldsymbol{R}_+ \times X$.

2.4. THEOREM ([4], 4.1). *The semidynamical system $(X, \boldsymbol{R}_+, \pi)$ is isomorphic to a semidynamical system $(X, \boldsymbol{R}_+, \tilde{\pi})$ which has infinite negative escape time for each $y \in X$. Moreover, in this isomorphism the homeomorphism h from the Definition* 2.3 *is the identity* id_X ($\mathrm{id}_X\colon X \ni y \mapsto y \in X$).

2.5. DEFINITION. The semidynamical system $(X, \boldsymbol{R}_+, \pi)$ is said to *extend to the dynamical system $(X, \boldsymbol{R}, \hat{\pi})$* if $\hat{\pi}|_{\boldsymbol{R}_+ \times X} = \pi$.

2.6. LEMMA ([4], 3.10). *The function $x \mapsto N(x)$ is a lower semicontinuous function, i.e., $\liminf_{y \to z} N(y) \geqslant N(z)$ for all $z \in X$.*

2.7. THEOREM ([6], I.8.7). *Let $(X, \boldsymbol{R}_+, \varrho)$ be a local semidynamical system on the metric space X. Then there exists a semidynamical system $(X, \boldsymbol{R}_+, \hat{\varrho})$ which is isomorphic to $(X, \boldsymbol{R}_+, \varrho)$.*

2.8. PROPOSITION. *The semidynamical system $(X, \boldsymbol{R}_+, \pi)$ is isomorphic to a system $(X, \boldsymbol{R}_+, \tilde{\pi})$ with the following property:*

if there exists a countable basis $\mathscr{B}(M)$, then for every neighbourhood U of $F(t, M)$ there are a neighbourhood V of M and a $\delta > 0$ such that $F([t-\delta, t+\delta], V) \subset U$.

Proof. It is enough to use Theorem 2.4, Lemma 2.2 and Theorem 1.17.

2.9. PROPOSITION. *Let the system $(X, \boldsymbol{R}_+, \pi)$ have an infinite negative escape time for each $y \in X$. Then for every neighbourhood U of $F(t, x)$ there are a neighbourhood V of x and a $\delta > 0$ such that $F([t-\delta, t+\delta], V) \subset U$.*

The proof follows easily from Lemma 2.2, Theorem 1.17 and from the fact that X is first countable.

2.10. Remark. In the case of a metric space Proposition 2.9 means that the function $F\colon \boldsymbol{R}_+ \times X \to \mathscr{P}(X)$ is upper semicontinuous, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $F(s, y) \subset B_\varepsilon(F(t, x))$ for every $s \in B_\delta(t)$ and $y \in B_\delta(x)$; the symbol $B_\sigma(p)$ denotes a ball of radius σ centred at p. It is natural to ask if this function is lower semicontinuous (and thereby continuous), i.e., if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $F(t, x) \subset B_\varepsilon(F(s, y))$ for every $s \in B_\delta(t)$ and $y \in B_\delta(x)$ (see [6]). The answer is negative, as is shown by the following counterexample.

2.11. EXAMPLE. Let us put $X = \boldsymbol{R}^2$ and function π as follows:

$$\pi(s, (z, y)) = \begin{cases} (s+z, y) & \text{if } y = 0, \\ (z+s-y, 0) & \text{if } y > 0,\ y-s < 0, \\ (z, y-s) & \text{if } y > 0,\ y-s \geqslant 0, \\ (z+s+y, 0) & \text{if } y < 0,\ s+y < 0, \\ (z, s+y) & \text{if } y < 0,\ s+y \geqslant 0. \end{cases}$$

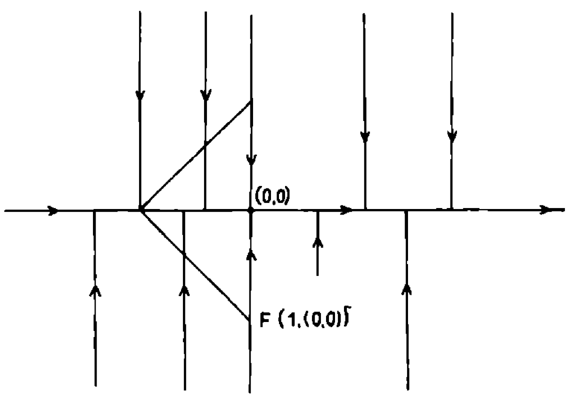

Fig. 2

Let us put $t = 1$ and $x = (0, 0)$. As one can easily verify, in every neighbourhood of (0, 0) there is a point p of the upper half-plane for which $F(t, p)$ contains only one element (which is an element of the upper half-plane as well). Nevertheless, the set $F(1, x) = \{(z, y) \in \boldsymbol{R}^2\colon |y|+|z| = 1,\ z \leqslant 0\}$ contains the points of the lower half-plane. This implies that the condition from the definition of lower semicontinuity does not hold.

2.12. Theorem. *Let $N(x) = +\infty$ for every $x \in X$ and let $(X, \boldsymbol{R}_+, \pi)$ have negative unicity. Define $\hat{\pi}\colon \boldsymbol{R} \times X \to X$ as follows:*

$$\hat{\pi}(s, y) = \begin{cases} \pi(s, y) & \text{if } s \geqslant 0, \\ \text{the unique element of } F(-s, y) & \text{if } s < 0. \end{cases}$$

Then $\hat{\pi}$ is a continuous function and $(X, \boldsymbol{R}, \hat{\pi})$ is a dynamical system (i.e., the semidynamical system $(X, \boldsymbol{R}_+, \pi)$ extends to the dynamical system $(X, \boldsymbol{R}, \hat{\pi})$).

Proof. We have to show that for each $y \in X$, $s \geqslant 0$ and a neighbourhood U of $\pi(s, y)$ (and $F(s, y)$) there exist a neighbourhood V of x and a $\delta > 0$ with $\pi\big((s-\delta, s+\delta) \times V\big) \subset U$ (and $F\big((s-\delta, s+\delta), V\big) \subset U$).

The continuity of $\hat{\pi}$ for $s \geqslant 0$ follows immediately by the continuity of π; for $s < 0$ we obtain it by Theorem 1.17 (the assumptions of this proposition are fulfilled as X is first countable).

To show the whole theorem notice that the first condition from the definition of the dynamical system is obvious and the second condition follows immediately by Lemma 1.5.

2.13. Corollary. *If the semidynamical system $(X, \boldsymbol{R}_+, \pi)$ has negative unicity, then it is isomorphic (as a semidynamical system, with $h = \mathrm{id}_X$) to the dynamical system $(X, \boldsymbol{R}, \hat{\pi})$.*

The proof follows easily by Theorem 2.4 and Theorem 2.12.

2.14. Corollary. *Let $(X, \boldsymbol{R}_+, \pi)$ be a semidynamical system with negative unicity on the metric space X. Then there exists a dynamical system which is isomorphic (as a semidynamical system) to $(X, \boldsymbol{R}_+, \pi)$ (i.e. $(X, \boldsymbol{R}_+, \pi)$ is isomorphic to a semidynamical system which extends to a dynamical system).*

Proof. It is enough to use Theorem 2.4 and Corollary 2.13.

2.15. Lemma. *Assume that π has negative unicity and $F(t, y) \neq \emptyset$ for every $y \in M$ (recall that $t \geqslant 0$ is fixed). Then $N(M) > t$ and for every neighbourhood U of $F(t, M)$ there is a $\delta > 0$ such that $F([t-\delta, t+\delta], M) \subset U$.*

Proof. We first show that $N(M) > t$. It is enough to show that there exists an $\alpha > 0$ with $N(y) \geqslant t+\alpha$ (for every $y \in M$). Of course $N(y) > t$ for every $y \in M$, as $F(t, y) \neq \emptyset$ and $(X, \boldsymbol{R}_+, \pi)$ has no start points.

Suppose to the contrary that there exists a sequence $\{x_n\} \subset M$ with $N(x_n) < t + 1/n$. By the compactness of M we may assume that $x_n \to x_0 \in M$.

Thus $\liminf_{n\to\infty} N(x_n) \leqslant \liminf_{n\to\infty}(t+1/n) = t$, but from Lemma 2.6 $\liminf_{n\to\infty} N(x_n) \geqslant N(x_0) > t$, which is a contradiction.

To show the second part of the lemma notice that by Lemma 2.2 and Lemma 1.3 the set $F([0, t], M)$ is compact (as $N(M) > t$) and we can use Lemma 1.6 and Theorem 1.8.

2.16. THEOREM. *Let $(X, \boldsymbol{R}_+, \pi)$ have negative unicity. Assume that there exists a countable basis of M and $F(t, y) \neq \emptyset$ for every $y \in M$. Then for every neighbourhood U of $F(t, M)$ there exist a $\delta > 0$ and a neighbourhood V of M such that $F([t-\delta, t+\delta], V) \subset U$.*

Proof. By Theorem 1.17 it is enough to show that there exists a neighbourhood V of M with $F([0, t], V)$ compact. By Lemma 2.2 it is sufficient to show that $N(V) > t$.

We shall show that there exist an $\alpha > 0$ and a neighbourhood V of M with $N(y) \geqslant t+\alpha$, for each $y \in V$. Suppose to the contrary that there exist a net $\{V_n\}$ of compact neighbourhoods of M such that $V_{n+1} \subset V_n$, $\bigcap V_n = M$, and a net $\{x_n\}$ such that $x_n \in V_n$, $N(x_n) < t+1/n$.

We may assume that $x_n \to x_0 \in V_1$, as V_1 is compact, and that $x_0 \in V_k$ for every integer k, as $x_n \in V_k$ for each $n \geqslant k$ and V_k is compact; so $x_0 \in M$. Thus, by Lemma 2.6, $N(x_0) \leqslant \liminf_{n\to\infty} N(x_n) \leqslant \liminf_{n\to\infty}(t+1/n) = t$, but by the previous lemma $N(x_0) \geqslant N(M) > t$, which is a contradiction.

2.17. COROLLARY. *Let $(X, \boldsymbol{R}_+, \pi)$ have negative unicity and $F(t, x) \neq \emptyset$. Then for every neighbourhood U of $F(t, x)$ there exist a neighbourhood V of x and a $\delta > 0$ such that $F([t-\delta, t+\delta], V) \subset U$.*

The proof follows immediately by Theorem 2.16, as X is first countable.

2.18. COROLLARY. *Let $(X, \boldsymbol{R}_+, \pi)$ have negative unicity. Let us put $f(t, x)$ as the unique element of $F(t, x)$ – if $F(t, x) \neq \emptyset$. Then the function f is continuous on its domain.*

This is immediate by Corollary 2.17.

2.19. Remark. As one can easily verify (Corollary 2.17), in the case of a system with negative unicity on a metric space the function F is lower semicontinuous (cf. Remark 2.10).

2.20. THEOREM. *Let $(X, \boldsymbol{R}_+, \pi)$ have negative unicity; let us define $\hat{\pi}$ as in Theorem* 2.12 (*if $F(t, x) \neq \emptyset$*). *Then $(X, \boldsymbol{R}, \hat{\pi})$ is a local dynamical system ($\hat{\pi}$ is continuous). Moreover, for every $y \in X$, the positive escape time of y* (see [1]) *ω_y equals $+\infty$.*

Proof. The only difficulty is to show the openness of the domain of $\hat{\pi}$; the remaining conditions follow directly as in the proof of Theorem 2.12 (we use Corollary 2.18). The positive escape time is $\omega_y = +\infty$ since the semidynamical system is global.

The openness of the domain $\hat{\pi}$ is equivalent to the lower semicontinuity of the function $N\colon X \ni x \mapsto N(x) \in \boldsymbol{R}_+$ (cf. [5], Lemma 1.3.1, Remark 1.3.4), which is true by Lemma 2.6.

2.21. Remark. In the case where X is a manifold, Theorem 2.20 was stated by Hajek in [3] (Proposition VI.4.1).

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