ANNALES
POLONICI MATHEMATICI
XXX (1974)

The existence of periodic solutions of an autonomous second order non-linear differential equation

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Abstract. Necessary and sufficient conditions are sought for the existence of (infinitely many) periodic solutions of the equation

$$x''+f(x)h(x'^2)+g(x) = 0,$$

where the "damping" term h is non-linear and the "restoring force" term g is free of the usual sign restrictions.

Additional conditions are found under which the equation admits solutions with different periods.

1. A number of authors, [4], [7], [8], [11] have recently considered the equation

$$x''+f(x)x'^{2n}+g(x) = \mu p(t), \tag{1.1}$$

where f, g, p are continuous, $xg(x) > 0$ for $x \neq 0$, $p(t)$ is periodic with period ω, $|\mu|$ is sufficiently small and $n \geqslant 1$. The object has been to find sufficient conditions for (1.1) to admit at least one periodic solution with period ω. (Henceforth, we shall mean a non-constant periodic solution whenever we refer to a periodic solution.)

The method of approach to this problem is normally to proceed by considering the autonomous equation

$$x''+f(x)x'^{2n}+g(x) = 0, \tag{1.2}$$

the link between (1.1) and (1.2) being the following Lemma of Bernstein and Halanay [1].

Lemma 1.1. *If* (1.2) *has a periodic solution with period* $\omega_0 \neq \omega$, *then, for* $|\mu|$ *sufficiently small,* (1.1) *has a periodic solution of period* ω.

We should also mention here that in [7], Ràb has investigated the more general equation

$$x'' = \varphi(x, x')+\mu\psi(x, x', t, \mu). \tag{1.3}$$

* Research supported by National Research Council of Canada grant NRC-A-8130.

Our main concern in this paper is to obtain necessary and sufficient conditions for the existence of infinitely many periodic solutions of an equation with a more general damping term than that of (1.2) and without the usual sign restriction of the "restoring force" term $g(x)$.

The problem of further resolving those equations for which there are periodic solutions with different periods seems to be rather difficult; we shall give some partial results in this direction which slightly extend a result of Heidel [4], and in one special case we are able to give a fairly complete answer to the problem.

For an indication of the physical significance of (1.2), we refer to [9], [10].

2. The equations to be considered are

$$x''+f(x)h(x'^2)+g(x) = \mu p(t), \tag{2.1}$$

$$x''+f(x)h(x'^2)+g(x) = 0, \tag{2.2}$$

where f, g, h and p are continuous, p is periodic with period ω, $|\mu|$ is sufficiently small, and $h(u)$ is increasing for $u > 0$. For convenience, we shall also assume that $h(u)$ is continuously extended for $u \leqslant 0$.

We shall also require the Bernoulli equation associated with (2.2), obtained by substituting $z = x'^2$ on intervals of constant sign for x'

$$\frac{dz}{dx}+2f(x)h(z)+2g(x) = 0. \tag{2.3}$$

Utz considered (2.2) when h is linear and gave the following result:

THEOREM [9]. *Let the zeros of g be isolated. Then* (2.2) *has infinitely many periodic solutions if and only if there is a real number c and a deleted neighbourhood I of c such that $(x-c)g(x) > 0$, for $x \in I$.*

We shall prove the following generalizations of this theorem:

THEOREM 1. *Let $\int_{+0}^{1} \frac{du}{h(u)} = +\infty$ and assume that g has countably many zeros and that* (2.3) *has uniquely solvable initial value problems. Then* (2.2) *has infinitely many periodic solutions if and only if*

(P) *there exist a, b with $a < b$ such that $\int_0^a g(u)du = \int_0^b g(u)du > \int_0^x g(u)du$ for all $x \in (a, b)$.*

With a slightly greater restriction placed on h, we may dispense completely with conditions on the zero set of g to obtain

THEOREM 2. *Let h be locally Lipschitz in a neighbourhood of zero. Then condition* (P) *is necessary and sufficient for there to be infinitely many periodic solutions of* (2.2).

For the case $h(u) = u^n$ $(n > 1)$, Theorems 1 and 2 are sharper than the result given in Theorem 1 of [10].

3. In this section, we shall prove Theorems 1 and 2. It will be convenient to denote $\int_0^x g(u)du$ by $G(x)$.

Proof of Theorem 1. To demonstrate the necessity of condition (P), let $x(t)$ be a periodic solution of (2.2). We can find a, β, t_0, t_1 with $a < \beta$, $t_0 < t_1$, such that $a = x(t_0) < x(t) < x(t_1) = \beta$, and $x'(t) > 0 = x'(t_0) = x'(t_1)$ for $t \in (t_0, t_1)$. Our next step is to show that for any $\delta > 0$, there exists $x \in (a, a+\delta)$ with $g(x) < 0$. For otherwise $g(x) \geqslant 0$ in $(a, a+\delta_0)$ for some $\delta_0 > 0$, and we may suppose that $\delta_0 < \beta - a$. Then (2.3) implies that

$$\frac{dz}{dx} + 2f(x)h(z) \leqslant 0, \qquad \text{on } (a, a+\delta_0)$$

which gives

$$\int_{z(x)}^{z(a+\delta_0)} \frac{du}{h(u)} \leqslant -2\int_x^{a+\delta_0} f(u)du.$$

Letting $x \to a+$, we obtain

$$+\infty = \int_{+0}^{z(a+\delta_0)} \frac{du}{h(u)} \leqslant -2\int_0^{a+\delta_0} f(u)du,$$

clearly a contradiction.

In a similar fashion, we may show that for any $\delta > 0$, there exists $x \in (\beta-\delta, \beta)$ such that $g(x) > 0$. Combining these conclusions with the continuity of $g(x)$, we find x_1, x_2, x_3, x_4 with $a < x_1 < x_2 < x_3 < x_4 < \beta$ such that $G(x_1) > G(x_2)$, $G(x_3) < G(x_4)$.

A simple argument on the continuity of G now yields the existence of numbers a, b satisfying condition (P).

Now let us assume that condition (P) is satisfied by two numbers a, b. We have $G(a) = G(b) > G(x)$ for $a < x < b$. Let c_0 be the least of those numbers x in (a, b) for which $G(x) = \min_{a<u<b} G(u)$. Then $g(c_0) = 0$. Let $M = \int_a^b |f(u)|du$ and choose $K > 0$, but less than $\min\left(\frac{1}{3}, G(b) - G(c_0)\right)$, such that $\int_{3K}^1 \frac{du}{h(u)} > 2M$.

Define a_0 to be $\sup\{x\colon x < c_0 \text{ and } G(x) = G(c_0)+K\}$ and b_0 to be $\inf\{x\colon x > c_0 \text{ and } G(x) = G(c_0)+K\}$. If $z(x)$ solves the initial value problem comprising (2.3) and $z(x_0) = z_0$, where $a_0 < x_0 < b_0$ and $|z_0| < K$, we have, on integrating (2.3) over any subinterval (x_0, x) of (a_0, b_0) for which $z(x)$

is extendable,

$$z(x) \leqslant z(x_0) + 2 \int_{x_0}^{x} |f(u)||h(z(u))|\,du - 2\big(G(x) - G(x_0)\big). \tag{3.1}$$

Thus

$$|z(x)| \leqslant 3K + 2 \int_{x_0}^{x} |f(u)|\,h(|z(u)|)\,du.$$

Bihari's inequality now gives

$$\int_{3K}^{|z(x)|} \frac{du}{h(u)} \leqslant 2 \int_{x_0}^{x} |f(u)|\,du \leqslant 2M$$

which implies that $|z(x)| < 1$. It follows that z and dz/dx are uniformly bounded on all subintervals of (a_0, b_0) on which z is extendable. Thus z is extendable to the whole of (a_0, b_0).

Now choose ε with $0 < \varepsilon < \frac{1}{2}K$ so that for $0 \leqslant u \leqslant \varepsilon$, we have $h(u) < K/2M$, and then choose θ with $0 < \theta < \varepsilon$ so that $\int_{\theta}^{\varepsilon} \frac{du}{h(u)} > 2M$. Finally, choose r_0 so that $a_0 < r_0 < c_0$ and $2\big(G(x) - G(x_0)\big) > -\theta$ for all x_0 in (r_0, c_0) and all x in (c_0, b_0). Let $x_0 \epsilon (r_0, c_0)$ such that $g(x_0) < 0$. The existence of a subinterval J of such values of x_0 contained in (r_0, c_0) is guaranteed by the definition of c_0. Consider the solution $z(x)$ of (2.3) which vanishes at x_0. We shall show that it vanishes again for some value of $x > x_0$ and is positive between these two zeros. That the solution is positive locally to the right of x_0 follows since $\left.\frac{dz}{dz}\right|_{x=x_0} = -2g(x_0) > 0$. Integrating (2.3) and using Bihari's inequality as above gives $z(x) \leqslant \varepsilon$ on the maximal subinterval of (x_0, b_0) for which $z > 0$. Using this bound in (3.1), we have

$$\begin{aligned} z(x) &\leqslant \frac{K}{2M}\cdot 2M + 2\big(G(x_0) - G(x)\big) \\ &= K + 2\big(G(x_0) - G(c_0) + G(c_0) - G(x)\big) \\ &< K + 2\theta + 2\big(G(c_0) - G(x)\big) \\ &< 2K + 2\big(G(c_0) - G(x)\big). \end{aligned}$$

As x tends to $b_0 -$ the right-hand side of the above inequality tends to zero so that $z(x)$ must vanish again before $x = b_0$. Denoting the first zero in (x_0, b_0) of $z(x)$ by $y(x_0)$, we note that the uniqueness condition on (2.3) implies that $y(x_0)$ is a (one-to-one) strictly decreasing map of J into the real line. Since $g(x)$ has by hypothesis countably many zeros, there exist (uncountably many) x_0 in J such that $0 = z(x_0) = z\big(y(x_0)\big)$

$< z(x)$ for $x \in (x_0, y(x_0))$ and $\frac{dz}{dx}(x_0) \cdot \frac{dz}{dx}(y(x_0)) \neq 0$. A standard argument now shows the existence of a periodic solution $x(t)$ of (2.2) with $x(0) = x_0$, $x'(0) = 0$, and the proof of Theorem 1 is complete.

Proof of Theorem 2. The necessity of condition (P) follows by an identical proof to that given in Theorem 1. As regards the sufficiency, the proof follows that given in Theorem 1 up to and including the definition of $y(x_0)$ for each $x_0 \in J \subset (r_0, c_0)$. The one modification required up to this point is as follows: $h(u)$ is uniformly Lipschitz in some interval $|u| < p$. Choose (a_0, b_0) with the additional requirement that there exists $\varepsilon_0 > 0$ such that all solutions $z(x)$ of (2.3) with $|z(x)| < \varepsilon_0$, for some $x \in \in (a_0, b_0)$, are uniformly bounded by p in (a_0, b_0).

The problem that remains is to exhibit the existence of $x_0 \in J$ for which $g(y(x_0)) \neq 0$, since we no longer have a restriction on the zero set of g to enable us to proceed with the argument given in Theorem 1.

It will be convenient at this point to introduce the notation $z(u, v; x)$ for the solution $z(x)$ of (2.3) with $z(u) = v$. Without loss of generality, we may assume that $g(x) < -m < 0$ for some $m > 0$ and for all $x \in J$. We have for $u \in J$ and $u < x < y(u)$

$$0 = z(u, 0; u) = z(u, 0; y(u)) < z(u, 0; x).$$

Suppose it were the case that $g(y(u)) = 0$ for all $u \in J$. By the uniqueness of solutions of initial value problems for (2.3), it follows that $y(u)$ is strictly decreasing for $u \in J$. Thus $y(u)$ is differentiable for almost all $u \in J$. If $u \in J$ and $\delta > 0$ and sufficiently small, we have

$$(3.2) \qquad 0 = \frac{z(u+\delta, 0; y(u+\delta)) - z(u, 0; y(u))}{\delta}$$
$$= \left[\frac{z(u, 0; y(u+\delta)) - z(u, 0; y(u))}{y(u+\delta) - y(u)}\right]\left[\frac{y(u+\delta) - y(u)}{\delta}\right] + $$
$$+ \left[\frac{z(u+\delta, 0; y(u+\delta)) - z(u, 0; y(u+\delta))}{\delta}\right].$$

For almost all $u \in J$,

$$\lim_{\delta \to 0} \frac{y(u+\delta) - y(u)}{\delta}$$

is finite and

$$\lim_{\delta \to 0} \frac{z(u, 0; y(u+\delta)) - z(u, 0; y(u))}{y(u+\delta) - y(u)} = \frac{\partial}{\partial x} z(u, 0; x)\Big|_{x=y(u)}$$

which is

$$-f(y(u))h(z(y(u))) - g(y(u)) = 0.$$

Thus for almost all $u \in J$, we have

$$\lim_{\delta \to 0} \frac{z(u+\delta, 0; y(u+\delta)) - z(u, 0; y(u+\delta))}{\delta} = 0, \tag{3.3}$$

and so for $\delta > 0$ and sufficiently small we may write $z(u, 0; y(u+\delta)) = \delta E(u, \delta)$, where $\lim_{\delta \to 0} E(u, \delta) = 0$.

Choose $u \in J$ for which (3.3) holds. Denote $z(y(u+\delta), \delta E(u, \delta); x)$ by $z_1(x)$, $z(y(u+\delta), 0; x)$ by $z_2(x)$, and let $\zeta(x) = z_1(x) - z_2(x)$. Note that

$$z(y(u+\delta), \delta E(u, \delta); y(u+\delta)) = \delta E(u, \delta) = z(u, 0; y(u+\delta)),$$

so that uniqueness of solutions of initial value problems of (2.3) implies that

$$z_1(u) = 0. \tag{3.4}$$

We have

$$\zeta(x) = \delta E(u, \delta) + \int_{y(u+\delta)}^{x} f(s)\{h(z_1(s)) - h(z_2(s))\} ds$$

so that

$$|\zeta(x)| \leqslant \delta E(u, \delta) + A \int_{y(u+\delta)}^{x} |f(s)||\zeta(s)| ds,$$

where A is the Lispschitz constant associated with h, and then Gronwall's inequality gives

$$|\zeta(x)| = o(\delta) \quad \text{uniformly for } x \in (a_0, b_0). \tag{3.5}$$

Similarly, we may show that

$$|z_1(x)| = o(\delta) \quad \text{uniformly for } x \in (a_0, b_0). \tag{3.6}$$

However, on integrating (2.3) and using (3.4), (3.6) we have

$$\zeta(u+\delta) = z_1(u+\delta) = -\int_{u}^{u+\delta} g(s) ds - \int_{u}^{n+\delta} f(s) h(z_1(s)) ds \geqslant m\delta + o(\delta)$$

in contradiction to (3.5).

We may therefore conclude, as in Theorem 1, that there are infinitely many $x_0 \in J$ with $g(y(x_0)) \neq 0$ and so Theorem 2 is proved.

Remarks 1. As an illustration of a function $g(x)$ which has an uncountable zero set with no isolated zeros and satisfies condition (P), let S be the ternary cantor set on $[0, 1)$ repeated periodically and let $\gamma(x)$ be a continuous function on $[0, 1]$ with $0 = \gamma(0) = \gamma(1) < \gamma(x)$ for

$0 < x < 1$. Then define

$$g(x) = \begin{cases} 0, & \text{if } x \in S, \\ 3^{-n}\gamma\left(\dfrac{x-a_n}{b_n-a_n}\right)\operatorname{sgn} x & \text{if } x \in (a_n, b_n), \\ \quad \text{where } (a_n, b_n) \text{ is a "deleted interval" of length } 3^{-n}. \end{cases}$$

2. Some condition similar to $\int\limits_{0+} \frac{du}{h(u)} = +\infty$ is to be expected, as is demonstrated by the equation $x'' - 2|x'| + x = 0$, which has no periodic solutions, and yet satisfies all the hypotheses of Theorem 1 other than the integral condition on h.

3. For the equation $x'' + g(x) = 0$, H. I. Freedman and the author [2] have shown that (P) is a necessary and sufficient condition for the existence (in the Carathéodory sense) of a periodic solution assuming only that $g(x)$ is locally integrable

4. Our investigation of the existence of periodic solutions of (2.2) with arbitrarily large period begins with the following result of Heidel:

THEOREM ([4], see also [7], [5]). *Assume that*

(i) $xg(x) > 0$ *for* $x \neq 0$, $f(x) > 0$ *for all* x,

(ii) *initial value problems for* (2.2) *are unique, and*

(iii) *there exists a function* $\psi(x) \in C^1(-\infty, 0]$ *such that* $\psi(x) > 0$,

$$\frac{d\psi}{dx} > -2f(x)h(\psi) - 2g(x) \quad \text{for } x \leqslant 0, \quad \text{and} \quad \int\limits_{-\infty}^{0} \frac{du}{\sqrt{\psi(u)}} = \infty.$$

Then (2.2) *has periodic solutions with arbitrarily large periods.*

The method of proof is essentially in three parts: (a) the existence of a periodic solution $x_1(t)$ is shown; (b) Ważewski's topological method (see [3]) is used to demonstrate the existence of an extendable, non-constant, non-periodic solution $x_2(t)$ whose graph intersects that of $x_1(t)$; (c) the use of continuous dependence of solutions of (2.2) on initial conditions enables the proof of the Theorem to be completed. The last part of the argument may be used in the presence of solutions $x_1(t)$, $x_2(t)$ of the appropriate type for the more general equation that we are dealing with and so we state the following

LEMMA 4.1. *Assume that the hypotheses, either of Theorem* 1 *or of Theorem* 2 *hold and that initial value problems for* (2.2) *are unique. Then the existence of a periodic solution* $x_1(t)$ *of* (2.2) *and of a non-constant, non-periodic, extendable solution* $x_2(t)$ *of* (2.2) *whose graph intersects that of* $x_1(t)$, *will guarantee that* (2.2) *has periodic solutions of arbitrarily large period.*

Proof. We refer the reader to the last part of the proof of Theorem 1 in [4].

Without a sign restriction on g, however, we are unable to employ the topological method used in the paper referred to above to demonstrate the existence of $x_2(t)$ and need therefore to impose alternative conditions.

We shall prove the following

THEOREM 3. *Assume that* $\lim_{u\to\infty} h(u) = \infty$ *and*

(i) *the conditions of either Theorem* 1 *or of Theorem* 2 *hold, and that* $f(x) > 0$ *for all* x.

(ii) *initial value problems for* (2.2) *are unique, and*

(iii) *there exist* a, β, γ, δ *with* $\gamma < a < \beta < \delta$ *such that* $G(a) = G(\beta) > G(x)$ *for* $x \in (a, \beta)$, *and* $g(\gamma) = g(\delta) = 0$.

Then (2.2) *has periodic solutions with arbitrary large period.*

Proof. We shall adopt the notation $x(\xi, t)$ for the solution $x(t)$ of (2.2) satisfying $x(0) = \xi$, $x'(0) = 0$, and shall use $z(\xi, x)$ for the corresponding function $z(x)$.

By Theorems 1 and 2, there exist numbers a, b with $a \not< a < b < \beta$ and a periodic solution $x_1(t)$ of (2.2) with $\min_{t\in(-\infty,\infty)} x_1(t) = a$, $\max_{t\in(-\infty,\infty)} x_1(t) = b$. From the uniqueness of initial value problems, we have $g(a) < 0$, $g(b) > 0$, and so we can find $\delta_1 \leqslant \delta$ such that $g(x) > 0$ for $b < x < \delta_1$ and $g(\delta_1) = 0$. Define ξ to be $\sup\{B: b \leqslant B < \delta_1$, and $x(B, t)$ is periodic$\}$. We have $\xi \leqslant \delta_1$. If $\xi = \delta_1$, then $x(\xi, t) \equiv \xi$, a rest point in the phase plane for (2.2); if $\xi < \delta_1$, and $x(\xi, t)$ were periodic, then from the continuous dependence of solutions on initial conditions, it would follow that $x(\xi+\varepsilon, t)$ is periodic for sufficiently small $|\varepsilon|$, contradicting the definition of ξ. Thus $x(\xi, t)$ is a non-periodic solution of (2.2). A similar argument with $\eta = \inf\{A: A \leqslant a$, and $x(A, t)$ is periodic$\}$ reveals that $x(\eta, t)$ is a non-periodic solution of (2.2). We note $\eta < \xi$. Suppose that both $x(\xi, t)$ and $x(\eta, t)$ were non-extendable, and denote their maximal intervals of existence by $(\omega_-(\xi), \omega_+(\xi))$, $(\omega_-(\eta), \omega_+(\eta))$, respectively. If it were also the case that there exist ξ_1, η_1 such that

$$\lim_{t\to\omega_-(\xi)} x(\xi, t) = \xi_1, \qquad \lim_{t\to\omega_+(\eta)} x(\eta, t) = \eta_1,$$

we would have

$$\lim_{x\to\xi_1+} z(\xi, x) = \infty, \qquad \lim_{x\to\eta_1-} z(\eta, x) = \infty.$$

By the uniqueness of initial value problems, any solution $x(\lambda, t)$ with $\eta < \lambda < \xi$ must satisfy

$$\eta \leqslant \inf_{t\in(-\infty,\infty)} x(\lambda, t) \leqslant \sup_{t\in(-\infty,\infty)} x(\lambda, t) \leqslant \xi.$$

Now, from the definition of ξ, it follows that there exists a sequence

$\xi_n \uparrow \xi$ such that $x(\xi_n, t)$ is periodic. By the continuous dependence of solutions on initial conditions, it follows that $z(\xi_n, x_n) \to \infty$ as $n \to \infty$, where x_n is such that $z(\xi_n, x)$ attains its maximum value at $x = x_n$. It follows that

$$0 = \frac{dz}{dx}(\xi_n, x_n) = -2f(x_n)h\big(z(\xi_n, x_n)\big) - 2g(x_n);$$

since $h\big(z(\xi_n, x_n)\big) \to \infty$ as $n \to \infty$, and $g(x)$ is bounded on $[\lambda, \xi]$, we deduce that $f(x_n) \to 0$ as $n \to \infty$, and so there exists $x_0 \in [\lambda, \xi]$ with $f(x_0) = 0$, contradicting hypothesis (i). It follows that $\lim_{t\to\omega_-(\xi)} x(\xi, t) = -\infty$, or $\lim_{t\to\omega_+(\xi)} x(\eta, t) = +\infty$. In either case, an argument in the phase plane shows that a contradiction of the uniqueness of initial value problems of (2.2) is obtained. Thus we finally conclude that $x(\xi, t)$ or $x(\eta, t)$ is extendable. It now follows from Lemma 4.1, or by direct appeal to continuous dependence on initial conditions, that (2.2) has periodic solutions of arbitrarily large period, and the Theorem is proved.

Remarks 1. It is quite easy to find examples which show that the existence of just one rest point of the phase plane for (2.2) lying outside a periodic orbit is not sufficient to ensure the existence of solutions with arbitrarily large period.

2. It is apparent from the proof of the Theorem that we used the non-vanishing of f, rather than its positivity; if this assumption is removed, it would seem that some kind of growth condition on f is necessary to obtain the required result.

The following is an immediate Corollary to Theorem 3 (see Lemma 1.1).

COROLLARY. *Let the hypotheses of Theorem* 3 *hold. Then* (2.1) *has periodic solutions of period* ω *if* $|\mu|$ *is sufficiently small.*

To conclude, we indicate one special case of (2.2) for which quite precise information concerning the existence of periodic solutions may be obtained.

THEOREM 4. *Let* $f(x), g(x)$ *be continuous, with* $xg(x) > 0$ *for* $x \neq 0$. *Define* $F(x)$ *to be* $\int_0^x f(u)du$ *and* $G(x)$ *to be* $\int_0^x g(u)du$. *Let* $\int_0^\infty e^{F(u)}du$ *and* $\int_0^{-\infty} e^{F(u)}du$ *be denoted by* a^+ *and* a^-, *respectively, and let* φ, Φ *be defined on* $I = (a^-, a^+)$ *by*

$$\varphi\Big(\int_0^x e^{F(u)}du\Big) = e^{F(x)}g(x), \qquad \Phi(x) = \int_0^x \varphi(u)du.$$

Let E_0 *be* $\min\big(\Phi(a^-), \Phi(a^+)\big)$ *and let* ε_0 *be* $\sqrt{2E_0}$. *Finally, for* $x \in I$, *define* $X(x)$ *to be* $\sqrt{2\Phi(x)}\,\operatorname{sgn} x$, *with range* J, *say, and for* $X \in J$, *define* $k(X)$ *to be* $\varphi\big(x(X)\big)$, *where* $x(X)$ *is the inverse function of* $X(x)$.

Then a necessary and sufficient condition for

$$x''+f(x)x'^2+g(x) = 0 \tag{4.1}$$

to have periodic solutions with different periods is that for no choice of positive λ *is the function* $\frac{\lambda X}{k(X)}-1$ *an odd integrable function of* X *on* $(0, E_0)$.

Proof. If $r(x) = \int_0^x e^{F(u)}du$, the substitution $y = r(x)$ transforms (4.1) into

$$y''+\varphi(y) = 0, \quad y\in I. \tag{4.2}$$

Periodic solutions of (4.1) with the same period are transformed into periodic solutions of (4.2) with the same period and whose energy functions $E = \frac{1}{2}y'^2+\Phi(y)$ are bounded by E_0, and the converse occurs under the inverse transformation $x = r^{-1}(y)$.

The Theorem now follows because the condition stated is precisely the contrapositive of that obtained by Levin and Shatz [6] for (4.2) to have all solutions, whose energy functions are bounded by E_0, periodic with the same period.

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Reçu par la Rédaction le 18. 1. 1973