ANNALES
POLONICI MATHEMATICI
XXVI (1972)

The radius of univalence and starlikeness of a certain class of analytic functions

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1. Introduction and statement of results. Suppose that $f(z) = z+a_2z^2+ +\dots$ and $g(z) = z+b_2z^2+\dots$ are analytic for $|z|<1$.

Recently, Ratti [4] proved, *inter alia*, the following theorems:

A. *If* $\operatorname{Re}\{g(z)/z\}>0$ *and* $\operatorname{Re}\{f(z)/g(z)\}>0$ *for* $|z|<1$, *then* f *is starlike and univalent for* $|z|<\sqrt{5}-2$.

B. *If* $\operatorname{Re}\{g(z)/z\}>0$ *and* $|f(z)/g(z)-1|<1$ *for* $|z|<1$, *then* f *is univalent and starlike for* $|z|<\frac{1}{4}(\sqrt{17}-3)$.

C. *If* $\operatorname{Re}\{g(z)/z\}>\frac{1}{2}$ *and* $\operatorname{Re}\{f(z)/g(z)\}>0$ *for* $|z|<1$, *then* f *is starlike and univalent for* $|z|<\frac{1}{3}$.

D. *If* $\operatorname{Re}\{g(z)/z\}>\frac{1}{2}$ *and* $|f(z)/g(z)-1|<1$ *for* $|z|<1$, *then* f *is univalent and starlike for* $|z|<r_0$, *where* r_0 *is the smallest positive root of the equation* $4-4r-13r^2-2r^3-r^4=0$ $(0.4<r_0<\sqrt{2}-1)$.

All the results are sharp.

The object of this note is to generalize the results stated above and to prove the following theorems:

THEOREM 1. *If* $\operatorname{Re}\{g(z)/z\}>0$ *and* $|f(z)/g(z)-a|<a$ *for* $|z|<1$, *where* a *is a fixed real number* $>\frac{1}{2}$, *then* f *is univalent and starlike for* $|z|<r_0$, *where* r_0 *is the smallest positive root of the equation* $r^3(1-a)+r^2(3-5a)- -3ra+a=0$, *and the bound* r_0 *is sharp.*

Putting $a=1$ one obtains Theorem B as a special case of Theorem 1, and letting a tend to infinity, Theorem A can be deduced from Theorem 1 as a corollary.

THEOREM 2. *If* $\operatorname{Re}\{g(z)/z\}>\frac{1}{2}$ *and* $|f(z)/g(z)-a|<a$ *for* $|z|<1$, *where* a *is a fixed real number* $\geqslant 1$, *then* f *is univalent and starlike for* $|z|<r_0$, *where* r_0 *is the smallest positive root of the equation* $4a^2-4ar-(24a^2-12a+1)r^2- -(32a^2-36a+6)r^3-(12a^2-20a+9)r^4=0$. *The bound* r_0 *is sharp.*

Theorem D is special case of the above theorem with $a=1$, and Theorem C is then deduced by letting a tend to infinity.

2. Proofs of theorems. We need the following lemmas for our discussion.

LEMMA 1. *Let $F(z)$ be analytic for $|z| < 1$ and satisfy* $\mathrm{Re}\{F(z)\} > a$, $0 \leqslant a < 1$ *for* $|z| < 1$ *and let* $F(0) = 1$. *Then we have*

$$F(z) = \{1+(2a-1)z\psi(z)\}/\{1+z\psi(z)\}, \tag{1}$$

where $\psi(z)$ is analytic for $|z| < 1$ and satisfies $|\psi(z)| \leqslant 1$ for $|z| < 1$; conversely, any function $F(z)$ given by the above formula is analytic for $|z| < 1$ and satisfies $\mathrm{Re}\{F(z)\} > a$ *for* $|z| < 1$.

The above lemma was proved by the author in [3].

LEMMA 2. *Suppose $G(z)$ is analytic for $|z| < 1$ and $G(0) = 1$. If $|G(z)-a$ $< a$, where a is any real number greater than $\frac{1}{2}$, then*

$$G(z) = \{1+z\varphi(z)\}/\left\{1+\left(\frac{1-a}{a}\right)z\varphi(z)\right\}, \tag{2}$$

where $\varphi(z)$ is analytic for $|z| < 1$ and $|\varphi(z)| \leqslant 1$ for $|z| < 1$. Conversely, any function $G(z)$ given by the above formula is analytic in the unit disc and satisfies the condition $|G(z)-a| < a$, $a > \frac{1}{2}$.

Proof. Setting $h(z) = (G(z)-a)/a$, we note that $h(z)$ is analytic in the unit disc and $|h(z)| < 1$ for $|z| < 1$, $h(0) = (1-a)/a$. Putting

$$\psi(z) = \frac{h(z)-h(0)}{1-h(0)h(z)},$$

we observe that $\psi(z)$ is analytic in the unit disc, $\psi(0) = 0$, $|\psi(z)| < 1$ for $|z| < 1$ since $|h(z)| < 1$ for $a > \frac{1}{2}$ and $|z| < 1$. Therefore, Schwarz's lemma applied to $\psi(z)$ yields $|\psi(z)| \leqslant |z|$ for $|z| < 1$. Hence we can write $\psi(z) = z\varphi(z)$, where $\varphi(z)$ is analytic in the unit disc and satisfies $|\varphi(z)| \leqslant 1$ there.

Expressing $h(z)$ in terms of $\varphi(z)$, we have

$$h(z) = \{(1-a)/a+z\varphi(z)\}/\{1+z\varphi(z)(1-a)/a\}.$$

Thus we get

$$G(z) = a+ah(z) = \{1+z\varphi(z)\}/\{1+z\varphi(z)(1-a)/a\}.$$

Conversely, if $G(z)$ is given by the above formula, where $\varphi(z)$ is analytic for $|z| < 1$ and $|\varphi(z)| \leqslant 1$, then clearly $G(z)$ is analytic for $|z| < 1$, since

$$|(1-a)z\varphi(z)/a| \leqslant |1-a|\,|z|/a < 1 \quad \text{for } |z| < 1,$$

since $a > \frac{1}{2}$. Moreover,

$$|(G(z)-a)/a| = \left|\frac{1-a+az\varphi(z)}{a+(1-a)z\varphi(z)}\right| < 1,$$

provided

$$|1-a+az\varphi(z)|^2 \leqslant |a+(1-a)z\varphi(z)|^2.$$

The above inequality is equivalent to the following one:

$$\{a^2-(1-a)^2\}|z\varphi(z)|^2 \leqslant \{a^2-(1-a)^2\},$$

which is true for $|z| < 1$, since $|\varphi(z)| \leqslant 1$ and $a > \frac{1}{2}$.

Thus $G(z)$ given by formula (2) represents an analytic function satisfying $|G(z)-a| < a$, $a > \frac{1}{2}$ for $|z| < 1$.

The proof of the lemma is complete.

LEMMA 3. *Let* $g(z) = z+b_2z^2+\dots$ *be analytic for* $|z| < 1$ *and satisfy* $\mathrm{Re}\{g(z)/z\} > 0$ *for* $|z| < 1$. *Then for* $|z| < 1$, *we have*

$$\mathrm{Re}\{zg'(z)/g(z)\} > \frac{(1-2|z|-|z|^2)}{(1-|z|^2)}.$$

The proof of the above lemma is implicitly contained in Theorem 2, [2], but the independent proof which we give below also is of some interest.

Proof. Since $\mathrm{Re}\{g(z)/z\} > 0$ for $|z| < 1$, we can apply Lemma 1 to $g(z)/z$ with $a = 0$ and write

$$g(z)/z = \big(1-z\varphi(z)\big)/\big(1+z\varphi(z)\big), \tag{3}$$

where $\varphi(z)$ is analytic and satisfies $|\varphi(z)| \leqslant 1$ for $|z| < 1$. Differentiating (3) yields

$$zg'(z)/g(z) = 1-2\left\{\frac{z\varphi(z)+z^2\varphi'(z)}{1-\big(z\varphi(z)\big)^2}\right\}. \tag{4}$$

For a function $\varphi(z)$ with the above properties we have ([1], p. 18)

$$|\varphi'(z)| \leqslant \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{5}$$

Denoting $|\varphi(z)|$ by x, $|z|$ by a and using (5), we obtain

$$\left|\frac{z\varphi(z)+z^2\varphi'(z)}{1-\big(z\varphi(z)\big)^2}\right| \leqslant \frac{ax+a^2(1-x^2)/(1-a^2)}{(1-a^2x^2)} = \frac{a(x+a)}{(1-a^2)(1+ax)}.$$

The expression $(a+x)/(1+ax)$ increases with x for a fixed a and its maximal value 1 is attained for $x = 1$. Hence, by (4), we get

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant 1-2\left|\frac{z\varphi(z)+z^2\varphi'(z)}{1-\big(z\varphi(z)\big)^2}\right| \geqslant 1-\frac{2a}{1-a^2} = \frac{1-2a-a^2}{1-a^2}.$$

The proof of the lemma is complete.

We now are in a position to prove the theorems.

Proof of Theorem 1. Clearly $f(z)/g(z)$ is analytic for $|z| < 1$, since the condition $\operatorname{Re}\{g(z)/z\} > 0$, $|z| < 1$ ensures that $g(z) \neq 0$ for $z \neq 0$ in the unit disc. $f(z)/g(z)$ satisfies the hypotheses of Lemma 2 and hence we have

$$f(z)/g(z) = \frac{1+z\varphi(z)}{1+(1-a)z\varphi(z)/a},$$

where $\varphi(z)$ is analytic for $|z| < 1$ and $|\varphi(z)| \leqslant 1$ for $|z| < 1$. Differentiation of the above formula gives

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{zg'(z)}{g(z)} + \frac{z\varphi(z)+z^2\varphi'(z)}{1+z\varphi(z)} - \frac{(1-a)}{a}\,\frac{\big(z\varphi(z)+z^2\varphi'(z)\big)}{\big(1+(1-a)z\varphi(z)/a\big)}\\ &= \frac{zg'(z)}{g(z)} + \frac{(2a-1)\big(z\varphi(z)+z^2\varphi'(z)\big)}{\big(1+z\varphi(z)\big)\big(a+(1-a)z\varphi(z)\big)}.\end{aligned}$$

Thus we get

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} - (2a-1)\left|\frac{z\varphi(z)+z^2\varphi'(z)}{\big(1+z\varphi(z)\big)\big(a+(1-a)z\varphi(z)\big)}\right|. \tag{6}$$

We have the following estimates

$$|z\varphi(z)+z^2\varphi'(z)| \leqslant |z\varphi(z)| + \frac{|z|^2\big(1-|\varphi(z)|^2\big)}{(1-|z|^2)} \tag{7}$$

and, for $1 \geqslant a > \frac{1}{2}$,

$$\begin{aligned}\Big|\big(1+z\varphi(z)\big)\big(a+(1-a)z\varphi(z)\big)\Big| &\geqslant \big(1-|z\varphi(z)|\big)\big(a-(1-a)\,|z\varphi(z)|\big)\\ &= a-|z\varphi(z)|+(1-a)\,|z\varphi(z)|^2.\end{aligned}$$

On the other hand, for $a > 1$,

$$\begin{aligned}\Big|\big(1+z\varphi(z)\big)\big(a+(1-a)z\varphi(z)\big)\Big| &= \Big|a+z\varphi(z)+(1-a)\big(z\varphi(z)\big)^2\Big|\\ &\geqslant a-|z\varphi(z)|-(a-1)\,|z\varphi(z)|^2\\ &= \big(a-(1-a)\,|z\varphi(z)|\big)\big(1-|z\varphi(z)|\big).\end{aligned}$$

Thus, for all $a > \frac{1}{2}$, we have

$$\Big|\big(1+z\varphi(z)\big)\big(a+(1-a)z\varphi(z)\big)\Big| \geqslant a-|z\varphi(z)|+(1-a)\,|z\varphi(z)|^2. \tag{8}$$

Using the above estimates, the estimate for $\operatorname{Re}\{zg'(z)/g(z)\}$ from Lemma 3, and writing $|z| = a$, $t = |z\varphi(z)|$, we obtain from (6)

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1-2a-a^2}{1-a^2} - \frac{(2a-1)\{t(1-a^2)+a^2-t^2\}}{(1-a^2)\{a-t+(1-a)t^2\}}. \tag{9}$$

Therefore $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, provided

$$(10)\qquad t^2\{(1-a)(1-2a-a^2)+(2a-1)\}+t\{(1-2a)(1-a^2)-1+2a+a^2\}+ \\ +a(1-2a-a^2)-(2a-1)a^2>0.$$

We note that $0 \leqslant a < 1$ and $0 \leqslant t \leqslant a$.

Denoting the left-hand member of inequality (10) by $E(t)$, we note that $E'(t)$ vanishes for

$$(11)\qquad t = t_1 = \frac{a(1-a^2)-a}{a-(1-a)(a^2+2a)}.$$

For $0 \leqslant a \leqslant \sqrt{2}-1$ and $a > \frac{1}{2}$, it is easily seen that the expressions $a(1-a^2)-a$ and $a-(1-a)(a^2+2a)$ both are positive. Thus t_1 is positive. Also $E''(t)$ is positive for $a > \frac{1}{2}$ and $a < \sqrt{2}-1$.

Now, $t_1 \gtreqless a$, respectively, when

$$a^3(1-a)+a^2(2-3a)-a(1+a)+a \gtreqless 0.$$

Let $P(a)$ denote the left-hand side of the above inequality. An analysis of the equation $P(a) = 0$ shows that for any $a > \frac{1}{2}$ there is only one root of $P(a) = 0$ lying between 0 and 1, namely $\sqrt{2}-1$. Thus for $0 \leqslant a < \sqrt{2}-1$, t_1 exceeds a and $E(t)$ attains its minimum at $t = a$ for $0 \leqslant t \leqslant a$. Hence $E(a) > 0$ would imply that $E(t) > 0$ for $0 \leqslant t \leqslant a$. This condition, after a simplification reduces to

$$(12)\qquad a^3(1-a)+a^2(3-5a)-3aa+a>0.$$

Denote by $Q(a)$ the left-hand side of the above inequality. An analysis of the equation $Q(a) = 0$ shows that for any $a > \frac{1}{2}$, it has only one positive root lying between 0 and 1. Let the root be called r_0. Then $Q(a) > 0$ for $0 \leqslant a < r_0$; we shall show that $r_0 < \sqrt{2}-1$. In fact, a direct computation shows that $Q(\sqrt{2}-1) = (2-\sqrt{2})(1-2a) < 0$ for $a > \frac{1}{2}$. Since $Q(a)$ vanishes only once for $0 \leqslant a < 1$, it immediately follows that $r_0 < \sqrt{2}-1$. Thus the condition $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ is satisfied for $|z| < r_0$ and it follows directly that $f(z)$ is univalent and starlike for $|z| < r_0$. To see that the bound r_0 is sharp, we choose $g(z) = z(1+z)/(1-z)$ and $f(z)$ so that $f(z)/g(z) = (1+z)/\big(1+z(1-a)/a\big)$. Evidently $\mathrm{Re}\{g(z)/z\} = \mathrm{Re}\left\{\dfrac{1+z}{1-z}\right\} > 0$ for $|z| < 1$ and

$$|f(z)/g(z)-a| = \left|\frac{1-a+az}{1+z(1-a)/a}\right| = a\left|\big(z+(1-a)/a\big)/\big(1+z(1-a)/a\big)\right| < a$$

for $|z| < 1$, since $|(1-a)/a| < 1$; consequently $\left\{\dfrac{z+(1-a)/a}{1+z(1-a)/a}\right\}$ defines a bilinear transformation which maps the disc $|z| < 1$ onto itself. For our choice of f and g, we have

$$zf'(z)/f(z) = (1+2z-z^2)/(1-z^2)+z(2a-1)/\{(1+z)(a+z(1-a))\} = 0,$$

whenever

$$a+3az+(3-5a)z^2-(1-a)z^3 = 0.$$

$z = -r_0$ satisfies the above equation and, consequently, the function $f(z)$ is not univalent in any disc $|z| < R$ if R exceeds r_0. The proof of Theorem 1 is complete.

Proof of Theorem 2. Arguing, as in Theorem 1, we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \text{Re}\left\{\frac{zg'(z)}{g(z)}\right\} - \frac{(2a-1)|z\varphi(z)+z^2\varphi'(z)|}{|(1+z\varphi(z))(a+(1-a)z\varphi(z))|}. \tag{13}$$

Estimates (7) and (8) yield

$$\left|\frac{\{z\varphi(z)+z^2\varphi'(z)\}(2a-1)}{(1+z\varphi(z))(a+(1-a)z\varphi(z))}\right| \leqslant \frac{(2a-1)\{t(1-a^2)+(a^2-t^2)\}}{(1-a^2)\{a-t+(1-a)t^2\}}, \tag{14}$$

where a is written for $|z|$ and t for $|z\varphi(z)|$. Denoting the right-hand member of inequality (14) by $F(t)$, we observe that $F(t)$ increases with t for a fixed a, and hence attains its maximum at $t = a$ for $0 \leqslant t \leqslant a$. This maximal value $F(a) = (2a-1)a/\{a-a+(1-a)a^2\}$. Hence we can replace the right-hand side of (14) by $F(a)$ and obtain

$$\left|\frac{\{z\varphi(z)+z^2\varphi'(z)\}(2a-1)}{(1+z\varphi(z))(a+(1-a)z\varphi(z))}\right| \leqslant \frac{(2a-1)a}{\{a+a(a-1)\}\{1-a\}}. \tag{15}$$

Again, since $\text{Re}\{g(z)/z\} > \frac{1}{2}$ for $|z| < 1$, we have, by Lemma 1 with $a = \frac{1}{2}$,

$$g(z)/z = 1/\{1+z\varphi(z)\}, \tag{16}$$

where $\psi(z)$ is analytic in the unit disc and $|\psi(z)| \leqslant 1$ for $|z| < 1$. Differentiating (15) we obtain

$$\frac{zg'(z)}{g(z)} = 1 - \frac{(z\psi(z)+z^2\psi'(tz))}{(1+z\psi(z))}. \tag{17}$$

Substituting (15) and (17) into (13) we get

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant 1 - \frac{(2a-1)a}{(a+a(a-1))(1-a)} - \text{Re}\left\{\frac{z\psi(z)+z^2\psi'(z)}{1+z\psi(z)}\right\} \tag{18}$$

$$\geqslant \frac{a-2aa+(1-a)a^2}{a-a+(1-a)a^2} - \left|\frac{z\psi(z)+z^2\psi'(z)}{1+z\psi(z)}\right|.$$

Writing a for $|z|$, x for $|\psi(z)|$ and using the estimate $|\psi'(z)| \leqslant (1-|\psi(z)|^2)/(1-|z|^2)$ for $|z|<1$ we obtain

$$\left|\frac{z\psi(z)+z^2\psi'(z)}{1+z\psi(z)}\right| \leqslant \frac{a(x+a)}{1-a^2} \leqslant \frac{a}{1-a}. \tag{19}$$

Setting (19) in (18) we get

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{a-2\alpha a+(1-\alpha)a^2}{a-\alpha+(1-\alpha)a^2} - \frac{a}{1-a}. \tag{20}$$

Therefore $\operatorname{Re}\{zf'(z)/f(z)\} > 0$, provided

$$2a^2(1-\alpha)-3a\alpha+\alpha > 0. \tag{21}$$

Let us denote the left-hand side of inequality (21) by $P_1(a)$ and observe that the equation $P_1(a) = 0$ has a unique positive root r_1 lying between 0 and 1 so that, for $0 \leqslant a < r_1$, $P_1(a) > 0$. It follows that $f(z)$ is starlike and univalent for $|z| < r_1$. We are now going to investigate whether $f(z)$ must be starlike in a larger disc. To this end we sharpen the first of the estimates (18) for $|z| \geqslant r_1$ as follows:

Since

$$\operatorname{Re}\left\{\frac{z\psi(z)+z^2\psi'(z)}{1+z\psi(z)}\right\} = \operatorname{Re}\frac{\big(z\psi(z)+z^2\psi'(z)\big)\big(1+\overline{z\psi(z)}\big)}{|1+z\psi(z)|^2},$$

it follows from the first of inequalities (18) that $\operatorname{Re}\{zf'(z)|f(z)\} > 0$ provided

$$\operatorname{Re}[\{a-2\alpha a+(1-\alpha)a^2\}|1+z\psi(z)|^2 - \\ -\{(a-\alpha)(1-\alpha)a^2\}\{z\psi(z)+z^2\psi'(z)\}\{1+\overline{z\psi(z)}\}] > 0,$$

that is, provided

$$a-2a\alpha+(1-\alpha)a^2+|z\psi(z)|^2a(1-2\alpha)+ \\ +\operatorname{Re}\big[\big(z\psi(z)\big)\big(\alpha-a(4\alpha-1)+a^2(1-\alpha)\big)- \\ -\{a-\alpha+(1-\alpha)a^2\}z^2\psi'(z)\big(1+\overline{z\psi(z)}\big)\big] > 0.$$

Since the real part of a complex number is equal to that of its conjugate we are free to replace, in the left-hand member of the above inequality, a complex quantity by its conjugate. Doing this wherever necessary, and re-arranging we note that the above inequality holds, provided

$$\operatorname{Re}\big[\big\{\big(a-\alpha+(1-\alpha)a^2\big)z^2\psi'(z)+\big((\alpha-1)a^2+a(4\alpha-1)-\alpha\big)\big\}\overline{\{1+z\psi(z)\}}\big] \\ < a(2\alpha-1)\big(1-|z\psi(z)|^2\big). \tag{22}$$

Now we have the following estimates:

$$(23)\qquad \operatorname{Re}\{z^2\psi'(z)(1+\overline{z\psi(z)})\} \leqslant |z|^2|\psi'(z)|(1+|z\psi(z)|) \leqslant |z|^2(1-|\psi(z)|^2)(1+|z\psi(z)|)/(1-|z|^2) = \{a^2(1-x^2)/(1-a^2)\}\{1+|z\psi(z)|\},$$

where $a = |z|$, $x = |\psi(z)|$.

Again, for $1 > a \geqslant r_1$, $\{-\alpha+a(4\alpha-1)+(\alpha-1)a^2\} \geqslant 0$ provided r_1 is not less than the only positive root a_0 of the equation $(\alpha-1)a^2 + a(4\alpha-1)-\alpha = 0$. To see this, let us set $P_2(a) = (\alpha-1)a^2+a(4\alpha-1)-\alpha$ and compare it with $P_1(a)$, the left-hand side of formula (21). Let a_0 denote the positive root of $P_2(a) = 0$. Since $\alpha \geqslant 1$, we have $P_1(a_0) = P_1(a_0) + P_2(a_0) = a_0^2(1-\alpha)+a_0(\alpha-1) = (\alpha-1)(a_0-a_0^2) \geqslant 0$. Since r_1 is the only positive root of the equation $P_1(a) = 0$ and $P_1(a) > 0$ for $0 \leqslant a < r_1$, it follows that $a_0 \leqslant r_1$. Thus $(\alpha-1)a^2+a(4\alpha-1)-\alpha > 0$ and

$$(\alpha-a)+(1-\alpha)a^2 = (\alpha+(\alpha-1)a)(1-a) > 0 \qquad \text{for } r_1 < a < 1.$$

So, we see, by (23), that the left-hand member of inequality (22) does not exceed, for $a \geqslant r_1$, the value of the expression

$$\{(\alpha+(\alpha-1)a)a^2(1-x^2)/(1+a)+(\alpha-1)a^2+a(4\alpha-1)-\alpha\}\{1+ax\}.$$

Hence inequality (22) holds provided $a \geqslant r_1$ and

$$(\alpha+(\alpha-1)a)a^2(1-x^2)+(1+a)\{(\alpha-1)a^2+a(4\alpha-1)-\alpha\} < a(1+a)(2\alpha-1)(1-ax),$$

that is, provided

$$(24)\qquad a^2x^2(\alpha+(\alpha-1)a)-ax(2\alpha-1)(a+a^2)+(1+a)\{\alpha-2\alpha a+a^2(1-\alpha)\} - -a^2\alpha-a^3(\alpha-1) > 0.$$

Denoting the left-hand side of (24) by $p(x)$, we see that $p'(x) = 0$ for $x = x_1 = (2\alpha-1)(1+a)/\{2\alpha+2(\alpha-1)a\}$ and $p''(x) > 0$. Thus, for a fixed a, $p(x)$ attains its minimum at $x = x_1$. Also $x_1 < 1$ for $a < 1$. Hence $p(x_1) > 0$ would imply that $p(x) > 0$ for any fixed a under consideration. This condition reduces after a simplification to the following inequality:

$$(25)\qquad -a^4(12\alpha^2-20\alpha+9)-a^3(32\alpha^2-36\alpha+6)-a^2(24\alpha^2-12\alpha+1) - -4\alpha a+4\alpha^2 > 0.$$

Let $T(a)$ denote the left-hand member of the above inequality. For $\alpha \geqslant 1$ the equation $T(a) = 0$ has only one positive root which lies in the interval $(0,1)$, which we call r_0. For $a < r_0$ inequality (25) holds and consequently inequality (22) holds for $a \geqslant r_1$ and $a < r_0$. Thus $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $|z| \geqslant r_1$ and $|z| < r_0$. In fact, one could verify directly that $r_0 > r_1$ for $\alpha \geqslant 1$. However, we shall produce an example of a function $f(z)$ such that $f'(z) = 0$ for $z = r_0$. This would imply not only that r_0 cannot be less than r_1 (since we have already proved that all functions $f(z)$ of the class under consideration are univalent and starlike in $|z| < r_1$), but also prove that the bound r_0 we have obtained is sharp. To this end we consider the function $f(z) = g(z)(1-z)/\big(1-z(1-a)/a\big)$, where $g(z) = z/\{1+z\psi(z)\}$, $\psi(z) = (z-b)/(1-bz)$ and b is defined by

$$\frac{r_0-b}{1-br_0} = \frac{(2a-1)(1+r_0)}{2\{a+(a-1)r_0\}}. \tag{26}$$

Simplifying (26) we get

$$b = \frac{1+r_0}{2a+(2a-1)r_0} - 1.$$

Evidently $0 > b > -1$ and $\psi(z)$ is a bilinear transformation mapping the unit disc onto itselt. Thus $\operatorname{Re}\{g(z)/z\} > \frac{1}{2}$ for $|z| < 1$ and

$$\left|\frac{f(z)}{g(z)} - a\right| = a\left|\frac{z+(a-1)/a}{1+z(a-1)/a}\right| < a \quad \text{for } |z| < 1.$$

Also we have

$$\frac{zf'(z)}{f(z)} = \frac{(1-z)\big(a+(a-1)z\big)-(2a-1)z}{(1-z)\big(a+(a-1)z\big)} - \frac{z\psi(z)+z^2\psi'(z)}{1+z\psi(z)}. \tag{27}$$

An actual computation yields

$$\psi'(z) = \frac{1-\big(\psi(z)\big)^2}{1-z^2}. \tag{28}$$

Substituting (28) into (27) and simplifying we see that $f'(z) = 0$ whenever

$$\begin{aligned}\big(z\psi(z)\big)^2\big(a+(a-1)z\big)+z\psi(z)\big\{(1+z)\big(a-2az-(a-1)z^2\big)&-\\ -(1-z^2)\big(a+(a-1)z\big)\big\}&+\\ +(1+z)\big(a-2az-(a-1)z^2\big)-az^2-(a-1)z^3 &= 0.\end{aligned}$$

Replacing $\psi(z)$ and b by their defining expressions we verify easily that $z = r_0$ satisfies the above equation. This shows that our function $f(z)$ is not univalent in any disc $|z| < R$ if R exceeds r_0.

The proof of the theorem is complete.

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Reçu par la Rédaction le 2. 1. 1971