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On a system of functional equations occurring in the theory of geometric objects

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In the present paper we shall give the general solution (without any supposition whatever about the required functiond $F(X)$ and $g(X)$) of the system of functional equations

$$F(BA) = F(B)F(A)\,, \tag{1}$$

$$g(BA) = F(B)g(A)+g(B)\,, \tag{2}$$

where A, B and F are 2×2 matrices [1] and g is a 2×1 matrix. Moreover, we assume that equations (1) and (2) are satisfied (and the functions $F(X)$ and $g(X)$ are defined) only for non-singular matrices A and B.

Equations (1) and (2) occur in the theory of geometric objects (cf. [2], p. 152).

Equation (1) does not contain the function g and therefore may be considered independently of equation (2). The general solution of equation (1) has been given in our paper [4] (cf. also [5]). Namely, a function $F(X)$, satisfying (for non-singular matrices A and B) equation (1), must have one of the following four forms:

$$F(X) = C\begin{Vmatrix}\varphi(\Delta) & 0\\ 0 & \varphi(\Delta)\end{Vmatrix} XC^{-1}, \tag{3}$$

$$F(X) = C\begin{Vmatrix}\varphi_1(\Delta) & 0\\ 0 & \varphi_2(\Delta)\end{Vmatrix} C^{-1}, \tag{4}$$

$$F(X) = C\begin{Vmatrix}\varphi(\Delta) & \varphi(\Delta)\alpha(\Delta)\\ 0 & \varphi(\Delta)\end{Vmatrix} C^{-1}, \tag{5}$$

$$F(X) = C\begin{Vmatrix}\varkappa(\Delta) & -\sigma(\Delta)\\ \sigma(\Delta) & \varkappa(\Delta)\end{Vmatrix} C^{-1}. \tag{6}$$

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[1] In the present paper we use capital Latin letters to denote 2×2 matrices (which in the sequel will be shortly called matrices), small Latin letters to denote 2×1 matrices (which will be shortly called vectors), and Greek letters (small as well as capital) to denote scalars. Moreover, the determinants of matrices X, A, B will be denoted by $\Delta, \Delta_A, \Delta_B$, respectively.

In formulae (3)-(6) Δ denotes the determinant of the matrix X, C is an arbitrary constant non-singular matrix, the functions $\varphi(\xi), \varphi_1(\xi), \varphi_2(\xi)$ are arbitrary solutions of the functional equation

$$\varphi(\xi\eta) = \varphi(\xi)\varphi(\eta) \tag{7}$$

(in the sequel such functions will be called multiplicative), with the restriction

$$\varphi(\xi) \not\equiv 0 , \tag{8}$$

(but we admit the case $\varphi_1(\xi) \equiv 0$, the case $\varphi_2(\xi) \equiv 0$, or both), $a(\xi)$ is an arbitrary function satisfying the equation

$$a(\xi\eta) = a(\xi) + a(\eta) \tag{9}$$

and the condition

$$a(\xi) \not\equiv 0 , \tag{10}$$

and the functions $\varkappa(\xi)$ and $\sigma(\xi)$ are a solution of the system of functional equations

$$\begin{aligned} \varkappa(\xi\eta) &= \varkappa(\xi)\varkappa(\eta) - \sigma(\xi)\sigma(\eta) , \\ \sigma(\xi\eta) &= \varkappa(\xi)\sigma(\eta) + \sigma(\xi)\varkappa(\eta) , \end{aligned} \tag{11}$$

fulfilling the condition

$$\sigma(\xi) \not\equiv 0 . \tag{12}$$

Restrictions (8), (10) and (12) are not essential, but it is convenient to make them. For if any of the inequalities (8), (10), (12) were not fulfilled, then the corresponding cases of (3), (5), (6) would be reduced to case (4).

Equations (7), (9) and (11) as well as their solutions are well known (cf. [1], [5]). However, we shall call the reader's attention to the following facts: Every solution of equation (7) is either an even or an odd function (this can easily be verified if we set in (7) successively: $\xi = \eta = 1$, $\xi = \eta = -1$, $\eta = -1$). Furthermore, it follows from (8) that $\varphi(\xi) \neq 0$ for $\xi \neq 0$ and in particular

$$\varphi(1) = 1 . \tag{13}$$

Every solution of equation (9) is an even function and we have

$$a(1) = a(-1) = 0 . \tag{14}$$

Finally, functions $\varkappa(\xi)$ and $\sigma(\xi)$, satisfying the system of equations (11), must be either both even or both odd.

From the above-mentioned properties of the solutions of equations (7), (9) and (11) it follows in particular that, if inequalities (8), (10), (12) are fulfilled, then they are also valid if we confine ourselves to $\xi > 0$ only.

Now we shall prove the following

THEOREM. *Functions $F(X)$ and $g(X)$, satisfying (for non-singular matrices A and B) equations* (1) *and* (2), *must have one of the following forms:*

(15) $F(X)$ *arbitrary of functions* (3)-(6), $g(X) = [F(X)-E]c$,

(16) $$F(X) = C\begin{Vmatrix}1 & 0\\ 0 & \varphi_2(\Delta)\end{Vmatrix}C^{-1}, \qquad g(X) = C\begin{Vmatrix}\ln|\varphi(\Delta)|\\ \lambda(\varphi_2(\Delta)-1)\end{Vmatrix},$$

(17) $$F(X) = E, \qquad g(X) = \begin{Vmatrix}\ln|\varphi(\Delta)|\\ \ln|\psi(\Delta)|\end{Vmatrix},$$

(18) $$F(X) = C\begin{Vmatrix}1 & a(\Delta)\\ 0 & 1\end{Vmatrix}C^{-1}, \qquad g(X) = C\begin{Vmatrix}\ln|\varphi(\Delta)|+\omega a^2(\Delta)\\ 2\omega a(\Delta)\end{Vmatrix}.$$

In formulae (15)-(18) λ, ω *are arbitrary constants, C is an arbitrary constant non-singular matrix, c is an arbitrary constant vector,* $E = \begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}$ *is the unit matrix, Δ (as usual) denotes the determinant of the matrix X, $\varphi(\xi)$, $\psi(\xi)$ and $\varphi_2(\xi)$ are arbitrary multiplicative functions, where $\varphi(\xi)\not\equiv 0$ and $\psi(\xi)\not\equiv 0$ (but $\varphi_2(\xi)$ may vanish identically).*

On the other hand, it may easily be verified that each of the pairs of functions (15)-(18) *actually satisfies the system of equations* (1) *and* (2).

Proof. We shall distinguish 4 cases, according to the form of the function $F(X)$.

I. The function $F(X)$ has forms (3), (6), or (5), in the last case with the additional restriction

(19) $$\varphi(\xi)\not\equiv 1 \quad \text{for} \quad \xi > 0.$$

Let $F_0(X)$ denote the corresponding matrix (according to the occurring case):

(20) $$F_0(X) = \begin{Vmatrix}\varphi(\Delta) & 0\\ 0 & \varphi(\Delta)\end{Vmatrix}X,$$

(21) $$F_0(X) = \begin{Vmatrix}\varkappa(\Delta) & -\sigma(\Delta)\\ \sigma(\Delta) & \varkappa(\Delta)\end{Vmatrix},$$

(22) $$F_0(X) = \begin{Vmatrix}\varphi(\Delta) & \varphi(\Delta)a(\Delta)\\ 0 & \varphi(\Delta)\end{Vmatrix} \quad (\varphi(\xi)\not\equiv 1 \text{ for } \xi>0),$$

so that $F(X) = CF_0(X)C^{-1}$. Thus, after setting this last equality into (2), we get

$$g(BA) = CF_0(B)C^{-1}g(A)+g(B),$$

i.e.

(23) $$C^{-1}g(BA) = F_0(B)C^{-1}g(A)+C^{-1}g(B).$$

Now let us write

(24) $$g_0(X) \stackrel{\text{df}}{=} C^{-1}g(X).$$

From (23) it follows that the function $g_0(X)$ satisfies the equation

$$g_0(BA) = F_0(B)g_0(A) + g_0(B)\,. \tag{25}$$

Now we shall prove that there exists a number β such that for $B_0 \overset{\mathrm{df}}{=} \begin{Vmatrix} \beta & 0 \\ 0 & \beta \end{Vmatrix}$ the matrix $F_0(B_0) - E$ is non-singular, i.e. that

$$\det[F_0(B_0) - E] \neq 0\,. \tag{26}$$

If case (20) occurs, then it is sufficient to assume $\beta = -1$. Then we have $\det B_0 = \beta^2 = 1$ and (by (13)) $\varphi(\det B_0) = 1$, and consequently $F_0(B_0) = B_0$ and $F_0(B_0) - E = B_0 - E = \begin{Vmatrix} -2 & 0 \\ 0 & -2 \end{Vmatrix}$. It is a non-singular matrix.

If (21) is the case, then we have for $B = \begin{Vmatrix} \xi & 0 \\ 0 & \xi \end{Vmatrix}$

$$\det[F_0(B) - E] = \big(\varkappa(\xi^2) - 1\big)^2 + \sigma^2(\xi^2)\,. \tag{27}$$

But it follows from condition (12) that there exists a $\xi = \beta$ such that expression (27) does not equal zero.

Finally, in case (22) we have for $B = \begin{Vmatrix} \xi & 0 \\ 0 & \xi \end{Vmatrix}$

$$\det[F_0(B) - E] = \big(\varphi(\xi^2) - 1\big)^2,$$

which again, according to (19), is not equal to zero for a certain $\xi = \beta$.

Now let B_0 be a fixed matrix such that (26) holds and let us put in (25) first $B = B_0$, $A = X$, and then $B = X$, $A = B_0$. We obtain

$$g_0(B_0X) = F_0(B_0)g_0(X) + g_0(B_0)\,, \tag{28}$$

$$g_0(XB_0) = F_0(X)g_0(B_0) + g_0(X)\,, \tag{29}$$

respectively. The matrix B_0 evidently commutes with every matrix: $B_0X = XB_0$. Thus $g_0(B_0X) = g_0(XB_0)$ and we obtain by (28) and (29)

$$F_0(B_0)g_0(X) + g_0(B_0) = F_0(X)g_0(B_0) + g_0(X)\,,$$

whence

$$[F_0(B_0) - E]g_0(X) = [F_0(X) - E]g_0(B_0)$$

and by (26)

$$g_0(X) = [F_0(B_0) - E]^{-1}[F_0(X) - E]g_0(B_0)\,. \tag{30}$$

But the matrices $F_0(B_0) - E$ and $F_0(X) - E$ commute, since

$$\begin{aligned}
[F_0(B_0) - E][F_0(X) - E] &= F_0(B_0)F_0(X) - F_0(B_0) - F_0(X) + E \\
&= F_0(B_0X) - F_0(X) - F_0(B_0) + E = F_0(XB_0) - F_0(X) - F_0(B_0) + E \\
&= F_0(X)F_0(B_0) - F_0(X) - F_0(B_0) + E = [F_0(X) - E][F_0(B_0) - E]\,.
\end{aligned}$$

Consequently the matrices $[F_0(B_0)-E]^{-1}$ and $[F_0(X)-E]$ also commute. Thus, putting $k \overset{\mathrm{df}}{=} [F_0(B_0)-E]^{-1}g_0(B_0)$, we can write (30) in the form

$$g_0(X) = [F_0(X)-E]k\,. \tag{31}$$

Going back to the functions $F(X)$ and $g(X)$, we have by (31)

$$\begin{aligned} g(X) = Cg_0(X) &= C[C^{-1}F(X)C-E]k = CC^{-1}[F(X)-E]Ck \\ &= [F(X)-E]c\,, \end{aligned}$$

where we have put $c \overset{\mathrm{df}}{=} Ck$. Thus in this case we have obtained functions (15) as the solution of equations (1) and (2).

II. The function $F(X)$ has form (4) (the possibilities $\varphi_1(\xi) \equiv 0$, $\varphi_2(\xi) \equiv 0$, or both, are also admitted).

Writing

$$F_0(X) = \begin{Vmatrix} \varphi_1(\Delta) & 0 \\ 0 & \varphi_2(\Delta) \end{Vmatrix},$$

we have $F(X) = CF_0(X)C^{-1}$, and introducing the function $g_0(X)$, defined by (24), we obtain, as in the preceding case, equation (25) for the function $g_0(X)$.

Let $g_0(X) = \begin{Vmatrix} \gamma_1(X) \\ \gamma_2(X) \end{Vmatrix}$. Writing equation (25) in coordinates we obtain for the functions $\gamma_\iota(X)$ the equations

$$\gamma_\iota(BA) = \varphi_\iota(\Delta_B)\gamma_\iota(A) + \gamma_\iota(B)\,, \qquad \iota = 1, 2\,. \tag{32}$$

The general solution of an equation of form (32) has been given in [6]:

$$\begin{aligned} \gamma_\iota(X) &= \lambda_\iota(\varphi_\iota(\Delta)-1) \quad &&\text{if} \quad \varphi_\iota(\xi) \not\equiv 1\,, \\ \gamma_\iota(X) &= \ln|\varphi(\Delta)| \quad &&\text{if} \quad \varphi_\iota(\xi) \equiv 1 \end{aligned}$$

($\varphi(\xi)$ denotes here an arbitrary multiplicative function, not vanishing identically). Thus there are 4 possible subcases:

1. $\varphi_1(\xi) \not\equiv 1$ and $\varphi_2(\xi) \not\equiv 1$. Then

$$g_0(X) = \begin{Vmatrix} \lambda_1(\varphi_1(\Delta)-1) \\ \lambda_2(\varphi_2(\Delta)-1) \end{Vmatrix} = [F_0(X)-E]k\,,$$

where $k = \begin{Vmatrix} \lambda_1 \\ \lambda_2 \end{Vmatrix}$, and then, as in case I (compare (31)), we obtain functions (15) as the solution of equations (1) and (2).

2. $\varphi_1(\xi) \equiv 1$, $\varphi_2(\xi) \not\equiv 1$. Then

$$g_0(X) = \begin{Vmatrix} \ln|\varphi(\Delta)| \\ \lambda(\varphi_2(\Delta)-1) \end{Vmatrix}$$

and taking into account (24) we obtain functions (16) as the solution of equations (1) and (2).

3. $\varphi_1(\xi) \not\equiv 1$, $\varphi_2(\xi) \equiv 1$. On account of the relation

$$J \begin{Vmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{Vmatrix} J^{-1} = \begin{Vmatrix} \varphi_2 & 0 \\ 0 & \varphi_1 \end{Vmatrix},$$

where

(33)
$$J = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

this case can easily be reduced to the former by a suitable choice of matrix C.

4. $\varphi_1(\xi) \equiv 1$ and $\varphi_2(\xi) \equiv 1$. Then

$$g_0(X) = \begin{Vmatrix} \ln|\varphi(\Delta)| \\ \ln|\psi(\Delta)| \end{Vmatrix}$$

(φ, ψ — multiplicative functions, not vanishing identically), and

$$F(X) = F_0(X) = E .$$

Thus in this case we obtain functions (17) as the solution of equations (1) and (2).

It remains to consider the case where the function $F(X)$ has form (5) and condition (19) is not fulfilled. Since the function $\varphi(\xi)$ is either even or odd, we have two possibilities: either $\varphi(\xi) \equiv 1$, or $\varphi(\xi) = \operatorname{sgn}\xi$. We shall discuss these two cases separately.

III. The function $F(X)$ has the form $F(X) = CF_0(X)C^{-1}$, where

$$F_0(X) = \begin{Vmatrix} 1 & a(\Delta) \\ 0 & 1 \end{Vmatrix}.$$

Let us introduce the function $g_0(X)$ defined by formula (24). This function satisfies equation (25).

Let $g_0(X) = \begin{Vmatrix} \gamma_1(X) \\ \gamma_2(X) \end{Vmatrix}$. Writing (25) in coordinates we get

(34)
$$\gamma_1(BA) = \gamma_1(A) + a(\Delta_B)\gamma_2(A) + \gamma_1(B) ,$$

(35)
$$\gamma_2(BA) = \gamma_2(A) + \gamma_2(B) .$$

Equation (34) alone is sufficient to determine the two functions $\gamma_1(X)$ and $\gamma_2(X)$. We need not use equation (35).

It follows from (10) that there exists a β such that $a(\beta^2) \neq 0$. Writing $B_0 \overset{\mathrm{df}}{=} \begin{Vmatrix} \beta & 0 \\ 0 & \beta \end{Vmatrix}$ we have $B_0 X = X B_0$ for every matrix X. Thus we obtain from (34), putting first $B = B_0$, $A = X$, and next $A = B_0$, $B = X$

$$\gamma_1(X) + a(\beta^2)\gamma_2(X) + \gamma_1(B_0) = \gamma_1(B_0) + a(\Delta)\gamma_2(B_0) + \gamma_1(X) ,$$

i.e.

$$a(\beta^2)\gamma_2(X) = a(\Delta)\gamma_2(B_0).$$

Hence, writing $\dfrac{\gamma_2(B_0)}{a(\beta^2)} = 2\omega$, we have

$$\gamma_2(X) = 2\omega a(\Delta) \tag{36}$$

and

$$\gamma_1(BA) = \gamma_1(A)+\gamma_1(B)+2\omega a(\Delta_A)a(\Delta_B). \tag{37}$$

Now let us put

$$\gamma_0(X) \stackrel{\mathrm{df}}{=} \gamma_1(X)-\omega a^2(\Delta). \tag{38}$$

It follows from (37) that the function $\gamma_0(X)$ satisfies the equation

$$\gamma_0(BA) = \gamma_0(A)+\gamma_0(B). \tag{39}$$

Putting further

$$\mu(X) \stackrel{\mathrm{df}}{=} e^{\gamma_0(X)}, \tag{40}$$

we have according to (39)

$$\mu(BA) = \mu(A)\mu(B). \tag{41}$$

As has been proved in [3], a function $\mu(X)$ satisfying (41) must be of the form $\mu(X) = \varphi(\Delta)$, where $\varphi(\xi)$ is an arbitrary multiplicative function. It follows from (40) that $\varphi(\xi) \not\equiv 0$ and $\varphi(\xi) = |\varphi(\xi)|$ for all $\xi \neq 0$. Thus finally we obtain

$$\gamma_0(X) = \ln|\varphi(\Delta)|$$

and by (38) and (36)

$$g_0(X) = \begin{Vmatrix} \ln|\varphi(\Delta)|+\omega a^2(\Delta) \\ 2\omega a(\Delta) \end{Vmatrix}.$$

Consequently, taking into account (24), we obtain functions (18) as the solution of equations (1) and (2).

IV. The function $F(X)$ has the form $F(X) = CF_0(X)C^{-1}$, where

$$F_0(X) = \begin{Vmatrix} \operatorname{sgn}\Delta & (\operatorname{sgn}\Delta)a(\Delta) \\ 0 & \operatorname{sgn}\Delta \end{Vmatrix}.$$

Let us introduce the function $g_0(X)$ defined by formula (24). This function satisfies equation (25).

Let $g_0(X) = \begin{Vmatrix} \gamma_1(X) \\ \gamma_2(X) \end{Vmatrix}$. Writing equation (25) in coordinates we obtain

$$\gamma_1(BA) = \operatorname{sgn}\Delta_B\gamma_1(A)+\operatorname{sgn}\Delta_B a(\Delta_B)\gamma_2(A)+\gamma_1(B), \tag{42}$$

$$\gamma_2(BA) = \operatorname{sgn}\Delta_B\gamma_2(A)+\gamma_2(B). \tag{43}$$

It follows from (43) (cf. [6]) that

$$\gamma_2(X) = \gamma(\operatorname{sgn}\Delta-1) \tag{44}$$

($\gamma = \mathrm{const}$). Let us write

(45) $$\delta \stackrel{\mathrm{df}}{=} \tfrac{1}{2}\gamma_1(J)\ (^2)$$

(cf. (33)) and

(46) $$\gamma_0(X) \stackrel{\mathrm{df}}{=} \gamma a(\varDelta)\operatorname{sgn}\varDelta - \delta(\operatorname{sgn}\varDelta - 1).$$

The functions $\gamma_0(X)$, $\gamma_2(X)$ satisfy equation (42). In fact, we have by (46), (9) and (44)

$$\begin{aligned}\gamma_0(BA) &= \gamma a(\varDelta_A)\operatorname{sgn}\varDelta_A\operatorname{sgn}\varDelta_B + \gamma a(\varDelta_B)\operatorname{sgn}\varDelta_A\operatorname{sgn}\varDelta_B - \delta(\operatorname{sgn}\varDelta_A\operatorname{sgn}\varDelta_B - 1)\\ &= \operatorname{sgn}\varDelta_B[\gamma a(\varDelta_A)\operatorname{sgn}\varDelta_A - \delta(\operatorname{sgn}\varDelta_A - 1)] - \delta(\operatorname{sgn}\varDelta_B - 1) +\\ &\quad + \gamma a(\varDelta_B)\operatorname{sgn}\varDelta_B - \gamma a(\varDelta_B)\operatorname{sgn}\varDelta_B + \gamma a(\varDelta_B)\operatorname{sgn}\varDelta_A\operatorname{sgn}\varDelta_B\\ &= \operatorname{sgn}\varDelta_B\gamma_0(A) + \gamma_0(B) + \gamma a(\varDelta_B)\operatorname{sgn}\varDelta_B(\operatorname{sgn}\varDelta_A - 1)\\ &= \operatorname{sgn}\varDelta_B\gamma_0(A) + \gamma_0(B) + \operatorname{sgn}\varDelta_B a(\varDelta_B)\gamma_2(A).\end{aligned}$$

Now let us put

$$\varrho(X) \stackrel{\mathrm{df}}{=} \gamma_1(X) - \gamma_0(X).$$

The function $\varrho(X)$ evidently satisfies the equation

(47) $$\varrho(BA) = \operatorname{sgn}\varDelta_B\,\varrho(A) + \varrho(B)$$

and, according to (14) and (45), $\varrho(J) = 0$. From (47) it follows (cf. [6]) that

(48) $$\varrho(X) = \lambda(\operatorname{sgn}\varDelta - 1).$$

Setting in (47) $X = J$, we obtain $\lambda = 0$. Thus finally $\varrho(X) \equiv 0$, $\gamma_1(X) \equiv \gamma_0(X)$ and by (46) and (44)

$$g_0(X) = \begin{Vmatrix}\gamma a(\varDelta)\operatorname{sgn}\varDelta - \delta(\operatorname{sgn}\varDelta - 1)\\ \gamma(\operatorname{sgn}\varDelta - 1)\end{Vmatrix},$$

which can be written in the form

$$g_0(X) = \begin{Vmatrix}\operatorname{sgn}\varDelta - 1 & \operatorname{sgn}\varDelta\, a(\varDelta)\\ 0 & \operatorname{sgn}\varDelta - 1\end{Vmatrix}\begin{Vmatrix}-\delta\\ \gamma\end{Vmatrix} = [F_0(X) - E]k,$$

where we have put $k = \begin{Vmatrix}-\delta\\ \gamma\end{Vmatrix}$. Thus, as in case I (cf. (31)), also in the present case we obtain functions (15) as the solution of equations (1) and (2).

Thus we have considered all the possible cases and so the proof of the theorem has been completed.

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(2) Instead of J we might take any other matrix with the determinant equal -1.

References

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