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On the growth of entire functions of n complex variables

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Abstract. For entire functions of n complex variables with gap series of homogeneous polynomials the rate of growth in the whole space is determined by the rate of growth on a real cone K such that $C\bar{K}$ is non-pluripolar.

1. The main result. Let f be an entire function of n complex variables defined by a series of homogeneous polynomials

$$f(z) = \sum_{\nu=1}^{\infty} f_\nu(z), \quad z \in C^n \quad (\deg f_\nu = \nu).$$

Suppose that $f_\nu = 0$ for $\nu \neq \lambda_k$ with $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$. Let H: $[0, \infty) \to (0, \infty)$ be a function with continuous, positive and increasing first derivative. Let K denote a real cone in C^n such that $C\bar{K} = \{\zeta z\colon \zeta \in C,\ z \in \bar{K}\}$ is non-pluripolar. We write $\theta = \max\{r\colon \{\|z\| \leqslant r\} \subset \tilde{K}\}$, $\tilde{K}$ being the convex hull of $\{\|z\| = 1\} \cap C\bar{K}$ with respect to the family of all absolutely homogeneous plurisubharmonic functions on C^n. Since $C\bar{K}$ is non-pluripolar, θ is a positive number ([4], Corollary 11.2).

With these denotations we have

1.1. THEOREM. (i) *If* $|f(z)| \leqslant H(\|z\|)$, $z \in K$ *and* $H(r)r^{-\nu} \to \infty$, $r \to \infty$ *for every positive integer* ν, *then for each* $\sigma > 1$ *there is a constant* A *such that*

$$M(r) = \sup\{|f(z)\colon \|z\| \leqslant r\} \leqslant AH\left(\frac{\sigma}{\theta} r\right), \quad r \in [0, \infty).$$

(ii) *If* $|f(z)| \leqslant H(\|z\|)$, $z \in K$ *and* $H(r)r^{-\mu} \to 0$, $r \to \infty$ *for some positive integer* μ, *then* f *is a polynomial.*

Let us note that for $n = 1$ Theorem 1.1 was proved by Anderson and Binmore ([1], Theorem 1, Theorem A).

2. Remarks and auxiliary theorems.

2.1. Remark. For a suitable $r_0 > 0$ a function

$$[r_0, \infty) \ni t \to H^2(t) t^{-1} \in (0, \infty)$$

is increasing and so

$$\int_{r_0}^{r} H^2(t) t^{-1} \, dt \leqslant H^2(r), \qquad r_0 \leqslant r < \infty.$$

2.2. Remark. $\int_0^r |f(tz)|^2 t^{-1} \, dt \leqslant M(r)^2$, $r > 0$, $\|z\| \leqslant 1$. Indeed, let $r > 0$ and $z \in C^n$, $\|z\| \leqslant 1$. Since $f(0) = 0$ a function

$$\psi_z \colon C \ni \zeta \to \zeta^{-1} f(\zeta z) \in C$$

is analytic and $|\psi_z(\zeta)| \leqslant r^{-1} M(r)$ for $|\zeta| = r$. By the maximum principle this inequality holds for $|\zeta| \leqslant r$. Hence

$$|f(tz)|^2 t^{-1} \leqslant M(r)^2 r^{-1}, \qquad t \in [0, r],$$

from where the required inequality follows.

To prove Theorem 1.1 we need the following results due to Anderson, Binmore and Gaier.

2.3. Theorem ([2], Theorem 1). *Let* $\varphi \colon C \ni \zeta \to \varphi(\zeta) = \sum_{\nu=1}^{\infty} a_\nu \zeta^\nu \in C$ *be an entire function with* $a_\nu = 0$ *for* $\nu \neq \lambda_k$, *where* $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$. *Then for every* $r > 0$

$$|a_\nu r^\nu| \leqslant \sqrt{2\nu}\, \Pi(\nu) \Big[\int_0^r |\varphi(t)|^2 t^{-1} \, dt \Big]^{1/2}, \qquad \nu = 1, 2, \ldots,$$

where

$$\Pi(\nu) = \begin{cases} \prod\limits_{j \neq k} \dfrac{\lambda_k + \lambda_j}{|\lambda_k - \lambda_j|} & \text{if } \nu = \lambda_k \ (k = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$

2.4. Theorem ([3], Theorem 1). *If* $\{\lambda_k\}_{k \geqslant 1}$ *is an increasing sequence of positive integers and* $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$, *then* $\log \Pi(\lambda_k) = o(\lambda_k)$ $(k \to \infty)$.

3. Proof of Theorem 1.1. Given $z \in C^n$, we write $\varphi_z(\zeta) = f(\zeta z) = \sum_{\nu=1}^{\infty} f_\nu(z) \zeta^\nu$. Then the function φ_z satisfies the hypothesis of Theorem 2.3

and so

$$3(1) \qquad |f_\nu(z)|\, r^\nu \leqslant \sqrt{2\nu}\, \Pi(\nu) \Big[\int_0^r |f(tz)|^2 t^{-1}\, dt\Big]^{1/2}, \qquad \nu \geqslant 1,\ r \geqslant 0.$$

Put $K_1 = K \cap \{\|z\| = 1\}$ and take a number r_0 according to Remark 2.1. We claim that for every $\alpha > 1$ there exists a positive integer N such that

$$3(2) \qquad |f_\nu(z)|\, r^\nu \leqslant \alpha^\nu H(r), \qquad z \in K_1,\ r \geqslant r_0,\ \nu \geqslant N.$$

Indeed, by hypothesis we have

$$\int_0^r |f(tz)|^2 t^{-1}\, dt \leqslant \int_0^{r_0} |f(tz)|^2 t^{-1}\, dt + \int_{r_0}^r H^2(t) t^{-1}\, dt, \qquad z \in K_1,\ r \geqslant r_0.$$

So, by Remarks 2.2 and 2.1,

$$\int_0^r |f(tz)|^2 t^{-1}\, dt \leqslant M(r_0)^2 + H^2(r), \qquad z \in K_1,\ r \geqslant r_0.$$

Using Theorem 2.4, we can find a positive integer N such that $\sqrt{2\nu}\, \Pi(\nu) \leqslant \alpha^{\nu/2}$, $\nu \geqslant N$ and since H is increasing, the integer N can be enlarged so that

$$[M(r_0)^2 + H^2(r)]^{1/2} \leqslant \alpha^{\nu/2} H(r), \qquad \nu \geqslant N,\ r \geqslant r_0.$$

Hence by 3(1) the required inequality 3(2) holds.

Now, given $\sigma > 1$ take any $\alpha \in (1, \sigma)$ and choose N for which 3(2) is satisfied. Let $r \geqslant r_1 = \max\{r_0, 1\}$. Then for $z \in K_1$ and $|\zeta| \leqslant r$

$$|f(\zeta z)| \leqslant \sum_{\nu=1}^{N-1} |f_\nu(z)|\, r^N + \sum_{\nu=N}^{\infty} |f_\nu(z)|\, r^\nu \leqslant Br^N + \sum_{\nu=N}^{\infty} (|f_\nu(z)|\, r^\nu \sigma^\nu)\sigma^{-\nu}$$

$$\leqslant Br^N + H(r\sigma) \sum_{\nu=N}^{\infty} \alpha^\nu \sigma^{-\nu} \leqslant C(r^N + H(r\sigma)),$$

where B, C are appropriate constants.

Finally,

$$|f(z)| \leqslant C(r^N + H(r\sigma)), \qquad z \in C\bar{K} \cap \{\|z\| \leqslant r\}.$$

Since $C\bar{K}$ is non-pluripolar, by the Sibony–Wong inequality ([4], Corollary 11.2)

$$\sup\{|f(z)|:\ \|z\| \leqslant r\} \leqslant \sup\{|f(z)|:\ \|z\| \leqslant r/\theta,\ z \in C\bar{K}\}.$$

Hence

$$M(r) \leqslant C[(r/\theta)^N + H(\sigma r/\theta)], \qquad r \geqslant r_1.$$

Now, if $H(r)r^{-N} \to \infty$, $r \to \infty$, we can find a number $r_2 \geqslant r_1$ and a constant A so that

$$M(r) \leqslant AH(\sigma r/\theta), \qquad r \geqslant r_2.$$

Taking A greater than $M(r_2)/H(0)$, we obtain (i) of Theorem 1.1.

On the other hand, if $H(r)r^{-\mu} \to 0$, $r \to \infty$ for some positive integer μ, then for $N \geqslant \mu$ we have

$$M(r) \leqslant Dr^N, \qquad r \geqslant r_1$$

with a suitable constant D. Hence f is a polynomial.

References

[1] J. M. Anderson, K. G. Binmore, *Closure theorems with applications to entire functions with gaps*, Trans. Amer. Math. Soc. 161 (1971), 381–400.

[2] –, *Coefficient estimates for lacunary power series and Dirichlet series*. I, Proc. London Math. Soc. (3) 18 (1968), 36–48.

[3] D. Gaier, *On the coefficients and the growth of gap power series*, SIAM J. Numer. Anal. 3 (1966), 248–265.

[4] J. Siciak, *Extremal plurisubharmonic functions and capacities in C^n*, Lecture Notes of the Sophia Univ. Tokyo 1982, 1–97.

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