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A characteristic property of orthogonal pencils of coaxal circles from the standpoint of conformal mapping

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Abstract. This paper gives a characteristic property of orthogonal pencils of coaxal circles from the standpoint of conformal mapping in analytic function theory.

1. The following theorem of circle geometry plays an important role in the present note:

THEOREM A. *For orthogonal parabolic or elliptic and hyperbolic pencils of coaxal circles* (see [4], p. 64 and 67), *the four vertices of a curvilinear rectangle formed by any four members arbitrarily chosen are concyclic.*

In Section 2 we shall state a proof of Theorem A from the standpoint of conformal mapping in analytic function theory. Conversely, in Section 3, we shall prove that the property in Theorem A characterizes the orthogonal pencils of coaxal circles.

The well-known principle of circle-transformation of a linear rational function (see [2]) says:

Suppose that $f = f(z)$ ($\not\equiv$ const) is meromorphic in $|z| < +\infty$. Then $w = f(z)$ transforms circles on the z-plane into circles on the w-plane, including straight lines among circles, if and only if f is a linear rational function.

The following theorem, which is a generalization of the above principle, was proved in [1]:

THEOREM B. *Suppose that* $f = f(z)$ ($\not\equiv$ const) *is meromorphic in* $|z| < +\infty$. *Then* $w = f(z)$ *transforms straight lines* $\operatorname{Im}(z) = t$ *on the z-plane, where t is a real variable, into circles on the w-plane, including straight lines among circles, if and only if*

$$f(z) = (az+b)/(cz+d) \quad \text{or} \quad f(z) = \big(a\exp(kz)+b\big)/\big(c\exp(kz)+d\big),$$

where a, b, c, d are arbitrary complex constants and k is an arbitrary real or purely imaginary constant with $(ad-bc)k \neq 0$.

In Section 4 we shall prove Theorem B by using the main theorem in Section 3.

2. We shall give a proof of Theorem A. We discuss two cases.

Case 1. Consider orthogonal parabolic pencils of coaxal circles.

We may assume that the two pencils lie on the w-plane. Furthermore, we may assume that the limiting point of the two pencils is at the origin on the w-plane and that the two common radical axes coincide with the real and imaginary axes on the w-plane. Let C be a curvilinear rectangle formed by any four members arbitrarily chosen from the two pencils. Consider the function $w=f(z)=1/z$. Then, there exist a non-empty simply connected domain D and four points A_1, A_2, A_3, A_4 on the z-plane satisfying the following three conditions:

(i) f is regular (and univalent) in D.

(ii) The four points A_1, A_2, A_3, A_4 form the four vertices of a rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane.

(iii) The four points $f(A_1), f(A_2), f(A_3), f(A_4)$ coincide with the four vertices of the curvilinear rectangle C on the w-plane.

The above facts result from the following mapping property of $f(z)=1/z$:

The horizontal and vertical lines $\mathrm{Im}(z)=\mathrm{const}$ and $\mathrm{Re}(z)=\mathrm{const}$ on the z-plane are transformed by the function $w=f(z)=1/z$ into orthogonal parabolic pencils of coaxal circles on the w-plane whose limiting point is at the origin and whose two common radical axes coincide with the real and imaginary axes.

By (ii) the four points A_1, A_2, A_3, A_4 are concyclic. Since $f(z)=1/z$ is a linear rational function, by the "if" part of the principle of circle-transformation the four points $f(A_1), f(A_2), f(A_3), f(A_4)$ are also concyclic. Hence, by (iii) Theorem A is proved in this case.

Case 2. Consider orthogonal elliptic and hyperbolic pencils of coaxal circles.

We may assume that the two pencils lie on the w-plane. Furthermore, we may assume that the two limiting points of the two pencils are at 1 and -1. Let C be a curvilinear rectangle formed by any four members arbitrarily chosen from the two pencils. Consider the function $w=f(z)=\tanh z$. Then, there exist a non-empty simply connected domain D and four points A_1, A_2, A_3, A_4 on the z-plane satisfying the following three conditions:

(i) f is regular and univalent in D.

(ii) The four points A_1, A_2, A_3, A_4 form the four vertices of a rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane.

(iii) The four points $f(A_1), f(A_2), f(A_3), f(A_4)$ coincide with the four vertices of the curvilinear rectangle C on the w-plane.

The above facts result from the following mapping property of $f(z) = \tanh z$:

The horizontal and vertical lines $\mathrm{Im}(z) = \mathrm{const}$ and $\mathrm{Re}(z) = \mathrm{const}$ on the z-plane are transformed by the function $w = f(z) = \tanh z$ into orthogonal elliptic and hyperbolic pencils of coaxal circles on the w-plane whose two limiting points are at 1 and -1.

The transformation

$$w = \tanh z = \left(\exp(2z)-1\right) / \left(\exp(2z)+1\right)$$

may be obtained by performing successively in the order given in the following transformations

(1) $$z' = 2z,$$

(2) $$z'' = \exp(z'),$$

(3) $$w = (z''-1)/(z''+1).$$

Under transformations (1) and (2) the four points A_1, A_2, A_3, A_4 are mapped on the four vertices $A_1'', A_2'', A_3'', A_4''$ of an isosceles trapezoid on the z''-plane. Hence the four points $A_1'', A_2'', A_3'', A_4''$ are concyclic. Since by (3) w is a linear rational function of z'', by the "if" part of the principle of circle-transformation the four points $f(A_1), f(A_2), f(A_3), f(A_4)$ are also concyclic. Hence, by (iii) Theorem A is proved in this case.

3. We shall state the main theorem and prove it.

Let $f = f(z)$ be a non-constant meromorphic function of a complex variable z in $|z| < +\infty$ and let D be a non-empty simply connected domain, where f is regular and univalent. Let $A_1A_2A_3A_4$ be an arbitrary rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane. We denote the set of all domains D satisfying the above conditions by S. The purpose of the present note is, as stated in Section 1, to prove the following

THEOREM. *Let D be an arbitrary domain belonging to S. Suppose that $w = f(z)$ ($\not\equiv$ const) is meromorphic in $|z| < +\infty$. Then the four points $f(A_1), f(A_2), f(A_3), f(A_4)$ are concyclic on the w-plane, including straight lines among circles, if and only if*

$$f(z) = (az+b)/(cz+d) \quad \text{or} \quad f(z) = \left(a\exp(kz)+b\right)/\left(c\exp(kz)+d\right),$$

where a, b, c, d are arbitrary complex constants and k is an arbitrary real or purely imaginary constant with $(ad-bc)k \neq 0$.

Remark. By the following two facts we see that the property in Theorem A characterizes the orthogonal pencils of coaxal circles.

(i) The horizontal and vertical lines $\mathrm{Im}(z) = \mathrm{const}$ and $\mathrm{Re}(z) = \mathrm{const}$ on the z-plane are transformed by the function $w = f(z) = (az+b)/(cz+d)$

$(ad-bc \neq 0)$ into orthogonal parabolic pencils of coaxal circles on the w-plane, including degenerate cases.

(ii) The horizontal and vertical lines $\mathrm{Im}(z) = \mathrm{const}$ and $\mathrm{Re}(z) = \mathrm{const}$ on the z-plane are transformed by the function $w = f(z) = \big(a\exp(kz)+ + b\big)/\big(c\exp(kz)+d\big)$ $((ad-bc)k \neq 0)$ into orthogonal elliptic and hyperbolic pencils of coaxal circles on the w-plane, including degenerate cases. Here k is real or purely imaginary.

Proof of the Theorem. By the above remark and by Theorem A we have only to prove the "only if" part of the theorem.

Let A_1, A_2, A_3, A_4 represent the complex numbers $x+t\exp(i\varphi)$, $x-t\exp(-i\varphi)$, $x-t\exp(i\varphi)$, $x+t\exp(-i\varphi)$, respectively, x denoting the center of the rectangle $A_1A_2A_3A_4$ whose sides are parallel to the real and imaginary axes on the z-plane and t, φ being real variables. By hypothesis the four points

$$w_1 = f(A_1) = f\big(x+t\exp(i\varphi)\big), \tag{4}$$

$$w_2 = f(A_2) = f\big(x-t\exp(-i\varphi)\big), \tag{5}$$

$$w_3 = f(A_3) = f\big(x-t\exp(i\varphi)\big), \tag{6}$$

$$w_4 = f(A_4) = f\big(x+t\exp(-i\varphi)\big) \tag{7}$$

are concyclic, including straight lines among circles. Hence, by a well-known theorem (see [4], p. 40) of circle geometry we have

$$\begin{vmatrix} w_1\overline{w}_1 & w_1 & \overline{w}_1 & 1 \\ w_2\overline{w}_2 & w_2 & \overline{w}_2 & 1 \\ w_3\overline{w}_3 & w_3 & \overline{w}_3 & 1 \\ w_4\overline{w}_4 & w_4 & \overline{w}_4 & 1 \end{vmatrix} = 0. \tag{8}$$

Developing the left-hand side of (8) according to its first column, we obtain

$$w_1\overline{w}_1C_1(x,t,\varphi)+w_2\overline{w}_2C_2(x,t,\varphi)+w_3\overline{w}_3C_3(x,t,\varphi)+ \\ +w_4\overline{w}_4C_4(x,t,\varphi) = 0, \tag{9}$$

where $C_1(x,t,\varphi)$, $C_2(x,t,\varphi)$, $C_3(x,t,\varphi)$, $C_4(x,t,\varphi)$ denote the cofactors of the elements of its first column, respectively.

Differentiating both sides of (9) six times with respect to t, putting $t = 0$ in the resulting equality and writing

$$P_j(x,\varphi) = \Big((\partial^6/\partial t^6)\big(w_j\overline{w}_jC_j(x,t,\varphi)\big)\Big)_{t=0} \quad (j = 1,2,3,4), \tag{10}$$

we obtain

$$P_1(x,\varphi)+P_2(x,\varphi)+P_3(x,\varphi)+P_4(x,\varphi) = 0, \tag{11}$$

where x is an arbitrary point in D and φ is a real variable. Here $P_1(x, \varphi)$, $P_2(x, \varphi), P_3(x, \varphi), P_4(x, \varphi)$ are trigonometric polynomials in φ whose coefficients are functions of x in D.

Let the absolute term, i.e., the term which does not contain φ, of each of $P_1(x, \varphi), P_2(x, \varphi), P_3(x, \varphi), P_4(x, \varphi)$ be $a_1(x), a_2(x), a_3(x), a_4(x)$, respectively.

If we put

$$Q_j(x, t, \varphi) = w_j\overline{w}_jC_j(x, t, \varphi) \quad (j = 1, 2, 3, 4), \tag{12}$$

then by the definitions of $C_1(x, t, \varphi), C_2(x, t, \varphi), C_3(x, t, \varphi), C_4(x, t, \varphi)$ and by (4), (5), (6), (7), (12), after a simple computation, we see that

$$Q_2(x, t, \varphi) = Q_1(x, -t, -\varphi), \tag{13}$$

$$Q_3(x, t, \varphi) = Q_1(x, -t, \varphi), \tag{14}$$

$$Q_4(x, t, \varphi) = Q_1(x, t, -\varphi). \tag{15}$$

By (10), (12) we have

$$P_j(x, \varphi) = \big((\partial^6/\partial t^6)Q_j(x, t, \varphi)\big)_{t=0} \quad (j = 1, 2, 3, 4). \tag{16}$$

Differentiating both sides of (13), (14), (15) six times with respect to t, putting $t = 0$ and using (16), we obtain in turn

$$P_2(x, \varphi) = P_1(x, -\varphi), \tag{17}$$

$$P_3(x, \varphi) = P_1(x, \varphi), \tag{18}$$

$$P_4(x, \varphi) = P_1(x, -\varphi). \tag{19}$$

Since $a_j(x) = \big(1/(2\pi)\big)\int_{-\pi}^{\pi} P_j(x, \varphi)\,d\varphi$ $(j = 1, 2, 3, 4)$, by (17), (18), (19) we see that

$$a_1(x) = a_2(x) = a_3(x) = a_4(x). \tag{20}$$

Since the representation of a trigonometric polynomial is unique, the absolute term of the left-hand side of (11), i.e., $a_1(x) + a_2(x) + a_3(x) + a_4(x)$ must be 0. Hence, by (20) we have in D

$$a_1(x) = 0. \tag{21}$$

By a direct calculation of $a_1(x)$, taking (4), (5), (6), (7) into account, we have

$$a_1(x) = 240\big(f'''(x)f'(x)\overline{f'(x)}^2 - \overline{f'''(x)}\,\overline{f'(x)}f'(x)^2\big) - \\ - 360\big(f''(x)^2\overline{f'(x)}^2 - \overline{f''(x)}^2 f'(x)^2\big). \tag{22}$$

By (21), (22) we have in D

$$(23)\quad f'''(x)f'(x)\overline{f'(x)}^2-\tfrac{3}{2}f''(x)^2\overline{f'(x)}^2=\overline{f'''(x)}\,\overline{f'(x)}f'(x)^2-\tfrac{3}{2}\overline{f''(x)}^2f'(x)^2.$$

Since f is regular and univalent in D, we have in D

$$(24)\qquad f'(x)^2\overline{f'(x)}^2\neq 0.$$

By (23), (24) we have in D

$$(25)\quad f'''(x)/f'(x)-\tfrac{3}{2}\big(f''(x)/f'(x)\big)^2=\overline{\big(f'''(x)/f'(x)-\tfrac{3}{2}(f''(x)/f'(x))^2\big)}.$$

By (25) we infer that $f'''(x)/f'(x)-\frac{3}{2}\big(f''(x)/f'(x)\big)^2$ is real in D. Hence, by a well-known theorem in analytic function theory we have in D

$$(26)\qquad f'''(x)/f'(x)-\tfrac{3}{2}\big(f''(x)/f'(x)\big)^2=K,$$

where K is a real constant.

By the Identity Theorem, (26) holds at every point x, where f is regular with $f'(x)\neq 0$.

The left-hand side of (26) is the Schwarzian derivative of f. Solving (26), we have, according as $K=0$ or $K<0$ or $K>0$, $f(z)=(az+b)/(cz+d)$ or $f(z)=\big(a\exp(kz)+b\big)/\big(c\exp(kz)+d\big)$ (k is a real constant), or $f(z)=\big(a\exp(kz)+b\big)/\big(c\exp(kz)+d\big)$ (k is a purely imaginary constant), where a, b, c, d are complex constants with $(ad-bc)k\neq 0$. Thus the theorem is proved.

4. We shall give another proof of Theorem B by using the main theorem in Section 3.

By the remark preceding the proof of the theorem in Section 3 we have only to prove the "only if" part of Theorem B. Let D be a non-empty simply connected domain, where f is regular and univalent and let $A_1A_2A_3A_4$ be an arbitrary rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes, x denoting the centre of the rectangle $A_1A_2A_3A_4$. We may assume that the two points A_1, A_4 and the two points A_2, A_3 are symmetric, respectively, with respect to the straight line L: $\mathrm{Im}(z)=\mathrm{Im}(x)$ on the z-plane. By hypothesis $f(L)$ is a circle, including straight lines among circles. Hence, by the Reflection Principle of Analytic Functions (see [3]) with respect to circles, including straight lines among circles, the two points $f(A_1)$, $f(A_4)$ and the two points $f(A_2)$, $f(A_3)$ are symmetric with respect to the circle $f(L)$ on the w-plane. Hence, by a simple theorem of inversive geometry the four points $f(A_1)$, $f(A_2)$, $f(A_3)$, $f(A_4)$ are concyclic on the w-plane, including straight lines among circles. Hence, by the main theorem we have $f(z)=(az+b)/(cz+d)$ or $f(z)=\big(a\exp(kz)+b\big)/\big(c\exp(kz)+d\big)$, where a, b, c, d are complex constants and k is a real or purely imaginary constant with $(ad-bc)k\neq 0$. Thus the theorem is proved.

References

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