

Generalized n -dimensional Gronwall's inequality and its applications to non-selfadjoint and non-linear hyperbolic equations

by SUDHANSHU K. GHOSHAL, ABHA GHOSHAL, and M. ABU MASOOD
(Calcutta, India)

Abstract. The Gronwall's inequality for differential equations has been generalized in this paper in order to study the properties of a class of non-selfadjoint, in general non-linear, vector hyperbolic differential equations. It has been shown that the existing Gronwall inequalities for selfadjoint partial differential equations in two independent variables, are particular cases of our present generalized inequality. This inequality has been used to prove uniqueness theorems, comparison theorems, and continuous dependence theorems for non-linear and non-selfadjoint vector differential equations. It has many other important applications in stability problems and numerical solutions of partial differential equations.

1. Introduction. The Gronwall's inequality in the original form can be found in [5] and [2]. Owing to its important applications in the study of differential equations, it has received considerable attention of both pure and applied mathematicians. Bellman [1] gave a more general form of the inequality which was extended further by Snow [6], [7] to the study of selfadjoint partial differential equations and by Ghoshal and Masood [3] to the investigation of non-selfadjoint cases in two dimensions. In the present paper, we have presented a generalized n -dimensional form of the inequality and applied to the study of uniqueness theorems, comparison theorems and continuous dependence theorems of non-selfadjoint vector partial differential equations of hyperbolic type.

2. The generalized inequality and particular cases.

THEOREM. *If $f(x, y)$, $g(x, y)$ are all continuous n -vector functions on a domain D , and $p(x, y)$, $q(x, y)$, $H(x, y)$ are symmetric non-negative matrix functions (matrix with non-negative elements) on D . Let $X_0(x_0, y_0)$ and $X(x, y)$ be two points in D such that $(x - x_0) \cdot (y - y_0) \geq 0$ and let R be the rectangular region whose diagonal is the line joining the points X_0 and X and let $V(s, t; x, y)$ be the $n \times n$ matrix function satisfying the matrix initial value problem*

$$(1) \quad M(V) = 0,$$

where M is the adjoint of the operator L given by

$$(2) \quad L(u) = u_{st} + au_s + bu_t + cu$$

and $a = -Hq$, $b = -Hp$, $c = -H$, with the boundary conditions:

$$(3) \quad V(x, y; x, y) \equiv V(x, y) \equiv V(X) = I,$$

$$(4) \quad V(x, t) = \text{Exp} \left(\int_y^t a(x, \varrho) d\varrho \right),$$

$$V(s, y) = \text{Exp} \left(\int_x^s b(\varrho, y) d\varrho \right);$$

I is the identity matrix and $V(s, t; x, y) \equiv V(s, t)$ is the matrix generalization of Riemann's function relative to the point $X(x, y)$ associated with the operator L .

Let G be the connected sub-domain of D which contains X and on which $V \geq 0$. If $R \subset G$ and $f(x, y)$ satisfies

$$(5) \quad f(x, y) \leq g(x, y) + p(x, y) \int_{x_0}^x H(s, y) f(s, y) ds +$$

$$+ q(x, y) \int_{y_0}^y H(x, t) f(x, t) dt + \int_{x_0}^x \int_{y_0}^y H(s, t) f(s, t) ds dt,$$

where the inequality holds component-wise; then $f(x, y)$ also satisfies

$$(6) \quad f(x, y) \leq g(x, y) + p(x, y) \int_{x_0}^x H(s, y) f(s, y) ds + q(x, y) \int_{y_0}^y H(x, t) f(x, t) dt +$$

$$+ \int_{x_0}^x \int_{y_0}^y V^T(s, t; x, y) H(s, t) g(s, t) ds dt.$$

Further, if $q(x, y) = 0$, then

$$(7) \quad f(x, y) \leq g(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t) H(s, t) g(s, t) ds dt +$$

$$+ p(x, y) \int_{x_0}^x H(s, y) \left\{ g(s, y) + \int_{x_0}^s \int_{y_0}^y V^T(\theta, t) H(\theta, t) g(\theta, t) d\theta dt \right\} \times$$

$$\times \left[\exp \int_s^x H(\xi, y) p(\xi, y) d\xi \right] ds.$$

Also if $p(x, y) = 0$, then

$$(8) \quad f(x, y) \leq g(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t) H(s, t) g(s, t) ds dt + \\ + q(x, y) \int_{y_0}^y H(x, t) \left\{ g(x, t) + \int_{x_0}^x \int_{y_0}^t V^T(s, \Phi) H(s, \Phi) g(s, \Phi) ds d\Phi \right\} \times \\ \times \left[\exp \int_t^y H(x, \varphi) q(x, \varphi) d\varphi \right].$$

LEMMA 1. Let $a(s, t)$, $b(s, t)$ and $H(s, t)$ be continuous matrix functions. Then the matrix characteristic initial value problem (1) under (3) and (4) has a unique solution $V(s, t; x, y)$ for s and t near to $X(x, y)$ and satisfying $(s-x) \cdot (t-y) \geq 0$. The solution is continuous, and if a, b, H are non-negative, so is $V(s, t)$.

Proof. Now equation (1) together with conditions (3) and (4), is equivalent to the Volterra integral equation

$$(9) \quad V(s, t) = I + \int_y^t a(x, \eta) V(x, \eta) d\eta + \int_x^s b(\xi, y) V(\xi, y) d\xi + \\ + \int_x^s \int_y^t H(\xi, \eta) V(\xi, \eta) d\xi d\eta$$

[since $M(V) = V_{st} - (aV)_s - (bV)_t + cV = 0$, $H = -c$].

The proof of the theorem is based on the successive approximation argument. Let T represent the transformation

$$(10) \quad TV = \int_y^t aV d\eta + \int_x^s bV d\xi + \int_x^s \int_y^t HV d\xi d\eta,$$

so that the integral equation (9) can be written as

$$(11) \quad V = I + TV.$$

Let us write $V_0(s, t) = I$, and define $V_{n+1} = I + TV_n$. When V is continuous, TV is also continuous under the assumptions stated in the main body of the lemma, and so by induction V_n is defined and continuous for all 'n'. Let $\|\cdot\|$ be a matrix norm. As $(s-x) \cdot (t-y) \geq 0$,

$$(12) \quad \|TV\| \leq \int_y^t \|a\| \cdot \|V\| d\eta + \int_x^s \|b\| \|V\| d\xi + \int_x^s \int_y^t \|H\| \cdot \|V(\xi, \eta)\| d\xi d\eta \\ \leq \left[\int_y^t \|a\| d\eta + \int_x^s \|b\| d\xi + \int_x^s \int_y^t \|H\| d\xi d\eta \right] \cdot \max \|V(\xi, \eta)\| \\ \leq \alpha \cdot \max \|V(\xi, \eta)\|,$$

where $0 < \alpha < I$ if s and t are close enough to (x, y) . Then

$$\|V_{n+1} - V_n\| = \|T(V_n - V_{n-1})\| \leq \alpha \cdot \max \|V_n - V_{n-1}\| \leq \dots \leq \alpha^n \cdot \|V_1 - V_0\| \max.$$

Now $V_{n+1} = V_0 + \sum_{r=0}^n (V_{r+1} - V_r)$ is the n -th partial sum of a matrix series, which is majorized by the following convergent geometric series (in matrix norm)

$$\max \|V_1 - V_0\| \sum_{r=0}^{\infty} \alpha^r.$$

It is evident that the matrix sequence $\{V_n\}$ converges uniformly in the domain, where (12) holds. Each V_n is continuous and so is the limit function $V(s, t)$.

Now

$$I + TV = I + T\{\lim V_n\} = I + T\lim V_n$$

owing to the continuity of the operator T)

$$= \lim(I + TV_n) = \lim(V_{n+1}) = V.$$

So that $V = I + TV$. This implies that V is a solution of (11).

If possible, let W be any other solution; so $V - W = T(V - W)$. Hence

$$\|V - W\| = \|T(V - W)\| \leq \alpha \cdot \max \|V - W\|.$$

Thus

$$V = W,$$

which proves the uniqueness of the solution V .

Now if a, b, H are all positive, then $V \geq 0$, implies $TV \geq 0$ (since $(s-x)(t-y) \geq 0$). As $V_0 = I > 0$, it follows by induction that $V_n \geq 0$ for all n , so that the limit function $V(s, t; x, y) \geq 0$.

Proof of the main theorem. Let

$$u(x, y) = \int_{x_0}^x \int_{y_0}^y H(s, t) f(s, t) ds dt \quad [\text{since } u(x_0, y) = u(x, y_0) = 0].$$

Then we obtain

$$u_{xy} = H(x, y) f(x, y) \leq H(x, y) \{g(x, y) + pu_y + qu_x + u\},$$

$$L(u) \equiv u_{xy} + au_x + bu_y + cu \leq Hg,$$

where

$$a = -Hq, \quad b = -Hp, \quad c = -H.$$

This is a hyperbolic partial differential inequality for u ; L is a non-selfadjoint operator.

Now for any two functions $u, V \in C^2$, we have

$$V^T L(u) - u^T M(V) = V^T [u_{xy} + au_x + bu_y + cu] - u^T [V_{xy} - aV_x - bV_y + (c - a_x - b_y)V].$$

Here the relations are scalar and hold true for each column of V . If a, b, c are symmetric matrices, then we can show that the last expression is

$$\left(u^T a V + V^T \frac{u_y}{2} - \frac{u^T V_y}{2} \right)_x + \left(V^T b u + V^T \frac{u_x}{2} - u^T \frac{V_x}{2} \right)_y.$$

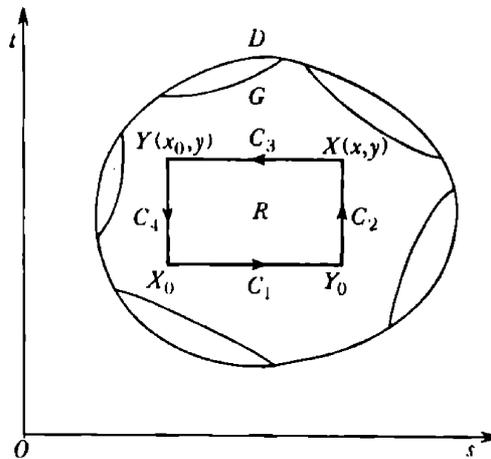


Fig. 1

Taking the region R referred to in the main theorem in the form of the rectangle [4] of our previous lemma and applying Green's theorem, we find (see Fig. 1)

$$\begin{aligned} & \iint_R [V^T L(u) - u^T M(V)] ds dt \\ &= \int_{c=c_1+c_2+c_3+c_4} \left\{ \left(u^T a V + V^T \frac{u_t}{2} - \frac{u^T V_t}{2} \right) dt - \left(V^T b u + V^T \frac{u_s}{2} - u^T \frac{V_s}{2} \right) ds \right\} \\ &= \int_{c_2} \left[u^T a V + V^T \frac{u_t}{2} - u^T \frac{V_t}{2} \right] dt - \int_{c_3} \left[V^T b u + V^T \frac{u_s}{2} - u^T \frac{V_s}{2} \right] ds \end{aligned}$$

[thus u is zero on C_1 and C_4 ; also ds does not vary on C_2 and dt on C_3 ; $u_t = 0$ on C_4 , $u_s = 0$ on C_1]

$$= \int_{c_2} [u^T (aV - V_t) + \frac{1}{2}(V^T u)_t] dt - \int_{c_3} [u^T (bV - V_s) + \frac{1}{2}(V^T u)_s] ds.$$

(The relations are scalar relations and hold for each column of V .) If V is the Riemann function with the initial conditions

$$\begin{aligned} aV - V_t &= 0 & \text{on } C_2, \\ bV - V_s &= 0 & \text{on } C_3, \end{aligned}$$

and

$$V(x, y) = I = V^T(X)$$

(which are conditions (3) and (4)) and satisfies the equation $M(V) = 0$, then we obtain

$$\int_R \int V^T L(u) ds dt = V^T(x, y)u(x, y) - \frac{1}{2}u^T(x, y_0)V(x, y_0) - \frac{1}{2}u(x_0, y)V(x_0, y_0),$$

so that

$$u(X) \equiv u(x, y) = \int_{x_0}^x \int_{y_0}^y V^T(s, t)L(u) ds dt$$

since

$$u(x, y) \leq \int_{x_0}^x \int_{y_0}^y V^T(s, t)\{H(s, t)g(s, t)\} ds dt;$$

hence

$$\begin{aligned} f(x, y) &\leq g(x, y) + p(x, y) \int_{x_0}^x H(s, y)f(s, y) ds + \\ &\quad + q(x, y) \int_{y_0}^y H(x, t)f(x, t) dt + \int_{x_0}^x \int_{y_0}^y V^T(s, t)g(s, t) ds dt. \end{aligned}$$

Now, let $q(x, y) = 0$ and suppose

$$\omega(x, y) = g(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t)H(s, t)g(s, t) ds dt$$

thus

$$f(x, y) \leq \omega(x, y) + p(x, y) \int_{x_0}^x H(s, y)f(s, y) ds.$$

This inequality may be treated as a one-dimensional Gronwall's inequality for any fixed "y" between y_0 to y .

For a fixed y , let

$$\psi(x, y) = \int_{x_0}^x H(s, y)f(s, y) ds;$$

therefore

$$\psi(x_0, y) = 0,$$

and

$$\psi_s(s, y) = H(s, y)f(s, y) \leq H(s, y)\{\omega(s, y) + p(s, y)\psi(s, y)\}$$

[since $H(s, y) \geq 0$], so that we have

$$\psi_s(s, y) - H(s, y) p \psi(s, y) \leq H(s, y) \omega(s, y).$$

Hence we obtain

$$\psi(x, y) \leq \int_{x_0}^x H(s, y) \omega(s, y) e^{\int_s^x H(\xi, y) p(\xi, y) d\xi} ds,$$

so that

$$f(x, y) \leq g(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t) H(s, t) g(s, t) ds dt + p(x, y) \times \\ \times \int_{x_0}^x H(s, y) \left\{ g(s, y) + \int_{x_0}^s \int_{y_0}^y V^T(\theta, t) H(\theta, t) g(\theta, t) d\theta dt \right\} e^{\int_s^x H(\xi, y) p(\xi, y) d\xi} ds.$$

Similarly, if $p(x, y) = 0$, we obtain

$$f(x, y) \leq g(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t) H(s, t) g(s, t) ds dt + q(x, y) \int_{y_0}^y H(x, t) \times \\ \times \left\{ g(x, t) + \int_{x_0}^x \int_{y_0}^t V^T(s, \Phi) H(s, \Phi) g(s, \Phi) ds d\Phi \right\} e^{\int_t^y H(x, \varphi) q(x, \varphi) d\varphi} dt.$$

COROLLARY 1. Putting $p(x, y) \equiv 0 \equiv q(x, y)$ in (5), we obtain

$$f(x, y) \leq g(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t) H(s, t) g(s, t) ds dt,$$

which was obtained by Snow [7]. The treatment given in [6] follows from here as a particular case.

COROLLARY 2. If $f(x, y) \leq \int_{x_0}^x \int_{y_0}^y H(s, t) f(s, t) ds dt$ and $H(s, t) \geq 0$ and $(x - x_0) \cdot (y - y_0) \geq 0$, then $f(x, y) \leq 0$.

COROLLARY 3. If inequality (5) is reversed, then so is inequality (6) [(7) and (8)].

COROLLARY 4. If $q(x, y) = 0$ and $g(x, y) = 0$, then (5) reduces to

$$f(x, y) \leq p(x, y) \int_{x_0}^x H(s, y) f(s, y) ds + \int_{x_0}^x \int_{y_0}^y H(s, t) f(s, t) ds dt;$$

then by (7)

$$f(x, y) \leq 0.$$

COROLLARY 5. Similarly, if $p(x, y) = 0 = g(x, y)$, (5) reduces to

$$f(x, y) \leq q(x, y) \int_{y_0}^y H(x, t) dt + \int_{x_0}^x \int_{y_0}^y H(s, t) f(s, t) ds dt,$$

then (8) gives

$$f(x, y) \leq 0.$$

Applications. The applications are analogous to those which have already been considered by the present authors in their treatment for the case of a single variable.

EXAMPLE 1. Let us discuss the uniqueness of the solution of the non-linear, non-selfadjoint, vector hyperbolic partial differential equation

$$u_{xy} = \{a(x, y)u(x, y)\}_y + a(x, y)\Phi(x, y, u)$$

with the conditions prescribed on $x = x_0$, and $y = y_0$. Suppose that $a(x, y)$, $\Phi(x, y, u)$ are continuous functions of their arguments, $a(x, y)$ is an $n \times n$ symmetric matrix, u and Φ are $n \times 1$ matrices, Φ satisfies a matrix Lipschitz condition, viz.,

$$|\Phi(x, y, u) - \Phi(x, y, u^*)| \leq K|u - u^*|$$

for any two vectors u and u^* , where the absolute values are taken componentwise.

Let the boundary conditions be such that the given partial differential equation is equivalent to the vector Volterra integral equation given by

$$u(x, y) = g(x, y) + \int_{x_0}^x a(s, y)u(s, y)ds + \int_{x_0}^x \int_{y_0}^y a(s, t)\Phi(s, t, u)dsdt,$$

where $g(x, y)$ is a continuous vector function depending on boundary conditions. Then for any two solutions u and u^* of the integral equation we have

$$u - u^* = \int_{x_0}^x a(s, y)\{u(s, y) - u^*(s, y)\}ds + \\ + \int_{x_0}^x \int_{y_0}^y a(s, t)\{\Phi(s, t, u) - \Phi(s, t, u^*)\}dsdt.$$

Now if $(x - x_0) \cdot (y - y_0) \geq 0$, we have

$$|u - u^*| \leq K' \int_{x_0}^x |a| \cdot K \cdot |u - u^*| ds + \int_{x_0}^x \int_{y_0}^y |a| \cdot K \cdot |u - u^*| ds dt,$$

where $K'|a|K|u - u^*| = |a||u - u^*|$.

With the help of the Corollary 4, we obtain $|u - u^*| \leq 0$, componentwise, which implies $u = u^*$.

So there is at most one solution of the differential equation.

EXAMPLE 2. Let us consider the vector characteristic initial value problem

$$u_{xy} - \{a(x, y)u(x, y)\}_y - a(x, y)u(x, y) = f(x, y),$$

where all the functions involved are continuous, and $a(x, y)$ is a non-negative matrix, and $u(x, y)$ is prescribed on $x = x_0, y = y_0$. This problem with the given conditions is equivalent to the vector Volterra integral equation

$$u(x, y) = h(x, y) + \int_{x_0}^x a(s, y)u(s, y)ds + \int_{x_0}^x \int_{y_0}^y a(s, t)u(s, t)dsdt,$$

where $h(x, y)$ is computed from $f(x, y)$ and the conditions at $x = x_0, y = y_0$.

Let the vectors $\bar{u}(x, y)$ and $\bar{\bar{u}}(x, y)$, satisfy

$$\bar{u}(x, y) \leq h(x, y) + \int_{x_0}^x a(s, y)\bar{u}(s, y)ds + \int_{x_0}^x \int_{y_0}^y a(s, t)\bar{u}(s, t)dsdt$$

and

$$\bar{\bar{u}}(x, y) \geq h(x, y) + \int_{x_0}^x a(s, y)\bar{\bar{u}}(s, y)ds + \int_{x_0}^x \int_{y_0}^y a(s, t)\bar{\bar{u}}(s, t)dsdt.$$

Now by (7) and Corollary 3, we find that for any solution vector u to the boundary valued problem we have

$$\bar{u} \leq u \leq \bar{\bar{u}}.$$

This is a componentwise comparison theorem for the solution vector.

EXAMPLE 3. Continuous dependence test: Let us consider the following pair of vector boundary value problems:

$$u_{xy} = \{a(x, y)\Phi(x, y, u)\}_y + a(x, y)u(x, y)$$

with

$$u(x_0, y) = g(y), \quad u(x, y_0) = h(x), \quad g(y_0) = h(x_0), \\ \Phi(x, y_0, h(x)) = f(x) \quad (\text{say})$$

and

$$U_{xy} = \{a(x, y)\psi(x, y, U)\}_y + a(x, y)U(x, y)$$

with

$$U(x_0, y) = G(y), \quad U(x, y_0) = H(x), \quad G(y_0) = H(x_0), \\ \psi(x, y_0, H(x)) = F(x) \quad (\text{say}),$$

where all the functions involved are continuous and Φ satisfies the Lipschitz condition, viz.,

$$|\Phi(x, y, u) - \Phi(x, y, \bar{u})| \leq K \cdot |u - \bar{u}|$$

and K is the non-negative Lipschitz constant matrix for Φ for two vectors u and \bar{u} . Here we form equivalent vector integral equations and subtract

$$\begin{aligned} u - U &= (g - G) + (h - H) - [g(y_0) - G(y_0)] - \int_{x_0}^x a(s, y_0)(f - F) ds + \\ &+ \int_{x_0}^x a(s, y) [\Phi(s, y, u(s, y)) - \psi(s, y, U(s, y))] ds + \\ &+ \int_{x_0}^x \int_{y_0}^y a(s, t) [u(s, t) - U(s, t)] ds dt. \end{aligned}$$

Adding and subtracting $\Phi(U)$ in the integrand and taking absolute values componentwise, we obtain, for $(x - x_0) \cdot (y - y_0) \geq 0$,

$$\begin{aligned} |u - U| &\leq |g - G| + |h - H| + |g(y_0) - G(y_0)| + \int_{x_0}^x |a| \cdot |f - F| ds + \\ &+ \int_{x_0}^x |a| \cdot |\Phi(u) - \Phi(U)| ds + \int_{x_0}^x |a| \cdot |\Phi(U) - \psi(U)| ds + \\ &+ \int_{x_0}^x \int_{y_0}^y |a| \cdot |u - U| ds dt. \end{aligned}$$

Now if $|g - G| \leq \varepsilon$, $|h - H| \leq \varepsilon$, $|\Phi(s, t, U) - \psi(s, t, U)| \leq \varepsilon$, and $|f - F| \leq \varepsilon$, where ε is a non-negative vector, then

$$|u - U| \leq \varepsilon q(x, y) + K' \int_{x_0}^x A \cdot |u - U| ds + \int_{x_0}^x \int_{y_0}^y A \cdot |u - U| ds dt,$$

where

$$q(x, y) = 3 + 2A(x - x_0), \quad K' A |u - U| = AK |u - U|$$

and

$$A = \{a_{ij}\}.$$

so that by (7), we obtain

$$\begin{aligned} |u - U| &\leq \varepsilon \left[q(x, y) + \int_{x_0}^x \int_{y_0}^y V_{\Sigma}^T(s, t) A q(s, t) ds dt + \right. \\ &+ K \int_{x_0}^x A \cdot \left\{ q(s, y) + \int_{x_0}^s \int_{y_0}^y V^T(\theta, t) A(\theta, t) q(\theta, t) d\theta dt \right\} \cdot e^{\int_s^x A(\xi, y) d\xi} ds \left. \right] \\ &\leq [M(x, y)] \varepsilon, \end{aligned}$$

where M is a continuous matrix function and obviously bounded. If $\varepsilon \rightarrow 0$, then $u \rightarrow U$ in the domain. This means that the solution of the characteristic initial value problem depends continuously on the initial data.

There are quite a few more useful applications of the main theorem, but for the sake of brevity they are not presented here.

Note. Similar results hold in the case of the non-selfadjoint linear hyperbolic vector partial differential equations of the form

$$u_{xy} = \{b(x, y) \Phi(x, y, u)\}_y + b(x, y) \Phi(x, y, u).$$

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DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
CALCUTTA, INDIA

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