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A difference method for a non-linear parabolic differential equation without mixed derivatives

by Z. KOWALSKI (Kraków)

§ 1. This paper deals with the mixed problem for the differential equation of the second order

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_p^2}\right), \tag{1.1}$$

where $t \in R^1$, $x = (x_1, \ldots, x_p) \in R^p$, $u \in R^1$.

We shall replace the derivatives at both sides of (1.1) by corresponding forward-difference and central-difference expressions. In particular,

$$\frac{\partial u}{\partial t} \text{ will be replaced by } \frac{1}{k}(v^{\omega(M)} - v^M),$$

$$\frac{\partial u}{\partial x_j} \text{ will be replaced by } \frac{1}{2h}(v^{j(M)} - v^{-j(M)}), \tag{1.2}$$

$$\frac{\partial^2 u}{\partial x_j^2} \text{ will be replaced by } \frac{1}{h^2}\left(v^{j(M)} - 2v^M + v^{-j(M)}\right)$$

($j = 1, \ldots, p$), v^M being the approximate value of the solution at the nodal point M with coordinates (t^μ, v^m), cf. fig. 1.

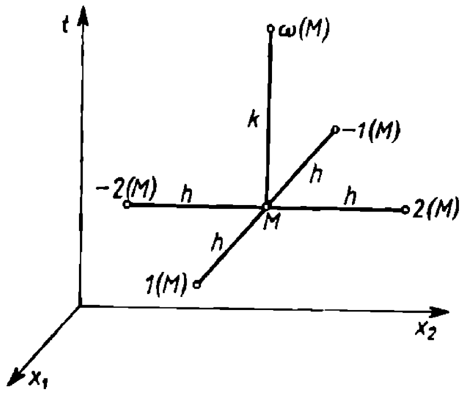

Fig. 1. Basic configuration of nodal points in the case $p = 2$

This means that we solve the differential equation (1.1) with the aid of an explicit type difference equation

$$v^{\omega(M)} = v^M + kf\left(t^\mu, x^m, v^M, \frac{v^{1(M)} - v^{-1(M)}}{2h}, \dots, \frac{v^{p(M)} - v^{-p(M)}}{2h}, \frac{v^{1(M)} - 2v^M + v^{-1(M)}}{h^2}, \dots, \frac{v^{p(M)} - 2v^M + v^{-p(M)}}{h^2}\right), \tag{1.3}$$

cf. equation (4.6).

The difference equation (1.3) can be used for digital computers, since (1.3) enables one to compute the value $v^{\omega(M)}$ with the aid of the preceding values $v^M, v^{1(M)}, v^{-1(M)}, \dots, v^{p(M)}, v^{-p(M)}$, only.

It turns out, cf. Theorem 1, that procedure (1.3) is convergent if the quantity

$$a = k/h^2, \tag{1.4}$$

satisfies the condition

$$0 < a \leqslant 1/2pG, \tag{1.5}$$

cf. (3.3), where

$$0 < g \leqslant \frac{\partial f}{\partial w_j} \leqslant G \quad (j = 1, \dots, p), \tag{1.6}$$

h and k being the space and time intervals, respectively.

We shall also give an error estimate of the method, cf. (7.1).

It is worth noting that if (1.1) reduces to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1.7}$$

then the condition of convergence (1.5) is of the form

$$0 < a \leqslant 1/2. \tag{1.8}$$

In fact, in this case we have $f = w$, $\partial f/\partial w = 1$, $g = G = 1$, $p = 1$ which means that (1.8) is the consequence of the condition (1.5), cf. [2] and [1].

The proofs are based, as in [3], on the method of difference inequalities.

§ 2. We shall denote by E the set of points of the real $(p+1)$-dimensional space R^{p+1}:

$$E: 0 \leqslant t \leqslant \tau, \ 0 \leqslant x_j \leqslant \tau, \ \tau > 0 \ (j = 1, \dots, p). \tag{2.1}$$

We shall introduce the nodal points with coordinates defined by

$$t^{\mu} = \mu k\,, \quad x_j^{\nu} = \nu h \tag{2.2}$$

$(\mu = 0, 1, \ldots, N_1;\ \nu = 0, 1, \ldots, N;\ j = 1, \ldots, p)$, $0 < k = \tau/N_1$, $0 < h = \tau/N$, N and N_1 being two natural numbers.

The sequence of indices

$$M = (\mu, m_1, \ldots, m_p), \quad 0 \leqslant m_j \leqslant N \quad (j = 1, \ldots, p)$$

of the nodal point (2.2) will be denoted by

$$M = (\mu, m)\,, \tag{2.3}$$

where

$$m = (m_1, \ldots, m_p)\,, \tag{2.4}$$

and the coordinates of nodal points (2.2) by

$$(t^{\mu}, x^{m})\,, \tag{2.5}$$

where

$$x^{m} = (x_1^{m_1}, x_2^{m_2}, \ldots, x_p^{m_p})\,. \tag{2.6}$$

We shall also consider the nodal points in the set E characterized by the following sequences of indices:

$$\omega(M) = (\mu+1, m)\,, \tag{2.7}$$

$$j(M) = \big(\mu, j(m)\big)\,, \quad -j(M) = \big(\mu, -j(m)\big)\,, \tag{2.8}$$

where

$$j(m) = (m_1, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_p)\,, \tag{2.9}$$

$$-j(m) = (m_1, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_p) \tag{2.10}$$

for $j = 1, \ldots, p$.

Relations (2.8), (2.9) and (2.10) imply that

$$j(\mu, m) = \big(\mu, j(m)\big)\,, \quad -j(\mu, m) = \big(\mu, -j(m)\big) \quad (j = 1, \ldots, p)\,. \tag{2.11}$$

Suppose that to each nodal point (2.2) defined by the sequence M, cf. (2.3), there corresponds a number

$$v^{M}\,. \tag{2.12}$$

We can then introduce the forward-difference expression

$$v^{M\sim} = \frac{1}{k}\big(v^{\omega(M)} - v^{M}\big)\,, \tag{2.13}$$

the central difference expressions

$$v^{Mj} = \frac{1}{2h}(v^{j(M)} - v^{-j(M)}) \qquad (j = 1, \ldots, p), \tag{2.14}$$

and the vector

$$v^{M\Delta} = (v^{M1}, \ldots, v^{Mp}). \tag{2.15}$$

We shall also use the central-difference expressions

$$v^{Mjj} = \frac{1}{h^2}(v^{j(M)} - 2v^M + v^{-j(M)}) \qquad (j = 1, \ldots, p), \tag{2.16}$$

and the vector

$$v^{M\Box} = (v^{M11}, v^{M22}, \ldots, v^{Mpp}). \tag{2.17}$$

(2.13) will replace the derivative with respect to the time variable t on the left-hand member of (1.1). The derivatives of the first and second order with respect to space variables x_j $(j = 1, \ldots, p)$ will be replaced by (2.14) and (2.16), respectively.

§ 3. Throughout this paper we shall use the following assumptions **H**:

ASSUMPTIONS H. 1) *Suppose that the scalar function* $f(t, x, u, q, w)$, $t \in R^1$, $x = (x_1, \ldots, x_p) \in R^p$, $u \in R^1$, $q = (q_1, \ldots, q_p) \in R^p$, $w = (w_1, \ldots, w_p) \in R^p$, *is of the class* C^1 *in the set* D:

$$D\colon\ 0 \leqslant t \leqslant \tau,\ 0 \leqslant x_j \leqslant \tau,\ -\infty < u < +\infty,\ -\infty < q_j < +\infty,\ -\infty < w_j < +\infty \tag{3.1}$$

$(j = 1, \ldots, p;\ \tau > 0)$.

2) *There exist positive constants* L, Γ, G, g *such that*

$$0 \leqslant \frac{\partial f}{\partial u} \leqslant L, \quad \left|\frac{\partial f}{\partial q_j}\right| \leqslant \Gamma, \quad 0 < g \leqslant \frac{\partial f}{\partial w_j} \leqslant G \qquad (j = 1, \ldots, p), \tag{3.2}$$

hold for $(t, x, u, q, w) \in D$.

3) *The time interval* k *and the space interval* h *are chosen so as to obtain*

$$\frac{g}{h} - \frac{\Gamma}{2} \geqslant 0, \quad \frac{2pG}{h^2} - \frac{1}{k} \leqslant 0. \tag{3.3}$$

4) *The scalar function* $u(t, x)$ *is of the class* C^2 *in the set* E, *cf.* (2.1), *satisfies the boundary conditions*

$$\begin{cases} u(0, x) = \varphi_0(x), \\ u(t, x) = \varphi_j(t, x) \quad \text{for} \quad (t, x) \in E,\ x_j = 0 \quad (j = 1, \ldots, p), \\ u(t, x) = \psi_j(t, x) \quad \text{for} \quad (t, x) \in E,\ x_j = \tau \quad (j = 1, \ldots, p), \end{cases} \tag{3.4}$$

and is the solution of the non-linear partial differential equation of the second order

$$\frac{\partial u}{\partial t}=f\left(t,x,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_p},\frac{\partial^2 u}{\partial x_1^2},\ldots,\frac{\partial^2 u}{\partial x_p^2}\right). \tag{3.5}$$

§ 4. Let us denote by u^M the value of the solution $u(t,x)$ of equation (3.5) at the nodal point $M=(\mu,m)$, cf. (2.2). The numbers u^M satisfy the following boundary conditions:

$$u^M=\begin{cases}\varphi_0(x^m) \quad \text{for } M=(0,m)\,,\\ \varphi_j(t^\mu,x_1^{m_1},\ldots,x_j^0,\ldots,x_p^{m_p})\\ \qquad\qquad \text{for } M=(\mu,m_1,\ldots,m_{j-1},0,m_{j+1},\ldots,m_p)\,,\\ \psi_j(t^\mu,x_1^{m_1},\ldots,x_j^N,\ldots,x_p^{m_p})\\ \qquad\qquad \text{for } M=(\mu,m_1,\ldots,m_{j-1},N,m_{j+1},\ldots,m_p)\,,\end{cases} \tag{4.1}$$

$(\mu=0,1,\ldots,N_1;\ j=1,\ldots,p)$.

It can be verified that the numbers u^M satisfy the equation

$$u^{M\sim}=f(t^\mu,x^m,u^M,u^{M\triangle},u^{M\square})+\eta^M\,, \tag{4.2}$$

and the condition

$$\varepsilon(h)\to 0\,,\quad \text{as}\quad h\to 0\,, \tag{4.3}$$

where

$$\varepsilon(h)=\max_M|\eta^M|\,, \tag{4.4}$$

at the nodal points $M=(\mu,m)$ of the set E, for $0\leqslant\mu\leqslant N_1-1$, $1\leqslant m_j\leqslant N-1$ $(j=1,\ldots,p)$, the quantities $u^{M\sim}$, $u^{M\triangle}$, $u^{M\square}$ being defined with the aid of u^M in the same way as $v^{M\sim}$, $v^{M\triangle}$, $v^{M\square}$ with the aid of v^M, cf. (2.13)-(2.17).

In fact, formulas (4.2) and (4.3) follow from (3.5), since $u(t,x)$ is the solution of equation (3.5) of the class C^2, and the right-hand member of (3.5) is of the class C^1.

We shall now define the approximate solution v^M of equation (3.5) at the nodal points M of the set E.

The boundary conditions for v^M are the same as for u^M, cf. (4.1):

(4.5)

$$v^M=\begin{cases}\varphi_0(x^m) \quad \text{for } M=(0,m)\,,\\ \varphi_j(t^\mu,x_1^{m_1},\ldots,x_j^0,\ldots,x_p^{m_p}) \quad \text{for } M=(\mu,m_1,\ldots,m_{j-1},0,m_{j+1},\ldots,m_p)\\ \psi_j(t^\nu,x_1^{m_1},\ldots,x_j^N,\ldots,x_p^{m_p}) \quad \text{for } M=(\mu,m_1,\ldots,m_{j-1},N,m_{j+1},\ldots,m_p)\end{cases}$$

$(\mu=0,1,\ldots,N_1;\ j=1,\ldots,p)$.

The values v^M at the remaining nodal points M of the set E are defined step by step starting from (4.5), with the aid of the difference equation

$$v^{M\sim} = f(t^\mu, x^m, v^M, v^{M\Delta}, v^{M\sqcap}) . \tag{4.6}$$

§ 5. We shall give without proof a lemma on linear difference inequalities.

LEMMA 1. *Suppose that the numbers* s^μ $(\mu = 0, 1, \ldots)$ *satisfy the non-homogeneous linear difference inequality*

$$s^{\mu\sim} \leqslant Ls^\mu + \varepsilon \quad (\mu = 0, 1, \ldots) , \tag{5.1}$$

and the initial condition $s^0 = 0$, *the difference* $s^{\mu\sim}$ *being defined by*

$$s^{\mu\sim} = \frac{1}{H}(s^{\mu+1} - s^\mu) \quad (\mu = 0, 1, \ldots) , \tag{5.2}$$

where $0 < H = \text{const}$, $0 < L = \text{const}$, $0 < \varepsilon = \text{const}$.

Under these assumptions

$$s^\mu \leqslant \frac{\varepsilon}{L}(e^{LH\mu} - 1) \quad (\mu = 0, 1, \ldots) . \tag{5.3}$$

This lemma can be proved with the aid of induction.

§ 6. LEMMA 2. *Suppose that the values* u^M *and the approximation* v^M *are defined with the aid of formulas* (4.1), (4.2) *and* (4.5), (4.6), *respectively.*

Suppose in addition that the assumptions H *are fulfilled, and let us write*

$$r^M = u^M - v^M , \tag{6.1}$$

$$s^\mu = \max_m r^{\mu,m} , \quad z^\mu = \min_m r^{\mu,m} \quad (\mu = 0, 1, \ldots, N_1) \tag{6.2}$$

at the nodal points M *of the set* E.

Under these assumptions the numbers s^μ *and* z^μ *satisfy the conditions*

$$s^\mu \geqslant 0 , \quad z^\mu \leqslant 0 \quad (\mu = 0, 1, \ldots, N_1) , \tag{6.3}$$

the initial conditions $s^0 = 0$, $z^0 = 0$, *and the non-homogeneous linear difference inequalities*

$$\begin{aligned} s^{\mu\sim} &\leqslant Ls^\mu + \varepsilon(h) , \\ z^{\mu\sim} &\geqslant Lz^\mu - \varepsilon(h) , \end{aligned} \tag{6.4}$$

$\mu = 0, 1, \ldots, N_1$, $\varepsilon(h)$ *being defined by* (4.4).

Proof. From (6.1) and the equality of boundary conditions for u^M and v^M, cf. (4.1) and (4.5), it follows that $r^M = 0$ at the boundary nodal points M of the set E. Therefore the greatest value s^μ must be non-negative, $s^\mu \geqslant 0$, and the smallest z^μ non-positive, $z^\mu \leqslant 0$, cf. (6.2).

This completes the proof of (6.3).

We shall now deal with (6.4). There exist nodal points $A = (\mu, a)$, $B = (\mu, b)$, $a = (a_1, ..., a_p)$, $b = (b_1, ..., b_p)$, such that

$$s^{\mu+1} = \max_m r^{\mu+1,m} = r^{\mu+1,a} = r^{\omega(A)}, \tag{6.5}$$
$$s^{\mu} = \max_m r^{\mu,m} = r^{\mu,b} = r^B.$$

Hence, the difference

$$s^{\mu\sim} = \frac{1}{k}(s^{\mu+1} - s^{\mu}) \tag{6.6}$$

can be written in the form

$$s^{\mu\sim} = \frac{1}{k}(r^{\omega(A)} - r^A) + \frac{1}{k}(r^A - r^B). \tag{6.7}$$

We shall consider the right-hand member of (6.7) in the case of $1 \leqslant a_j \leqslant N-1$ $(j = 1, ..., p)$ since, if for some j $(1 \leqslant j \leqslant p)$ we have $a_j = 0$ or $a_j = N$, then inequalities (6.4) are evident. From the definition (6.1) of r^M it follows that

$$\frac{1}{k}(r^{\omega(A)} - r^A) = \frac{1}{k}(u^{\omega(A)} - u^A) - \frac{1}{k}(v^{\omega(A)} - v^A) = u^{A\sim} - v^{A\sim}; \tag{6.8}$$

therefore from (6.8) and the difference equations (4.2), (4.6) we obtain

$$\frac{1}{k}(r^{\omega(A)} - r^A) \tag{6.9}$$
$$= \eta^A + f(t^\mu, x^a, u^A, u^{A\Delta}, u^{A\Box}) - f(t^\mu, x^a, v^A, v^{A\Delta}, v^{A\Box}).$$

We apply the mean-value theorem to the right-hand member of (6.9) and we get

$$\frac{1}{k}(r^{\omega(A)} - r^A) \tag{6.10}$$
$$= \eta^A + \frac{\partial f}{\partial u}(\sim) r^A + \sum_{j=1}^{p} \frac{\partial f}{\partial q_j}(\sim)(u^{Aj} - v^{Aj}) + \sum_{j=1}^{p} \frac{\partial f}{\partial w_j}(\sim)(u^{Ajj} - v^{Ajj}),$$

the derivatives being taken at a suitable point $(\sim)$. From the definitions (2.14) and (2.16) it follows that

$$u^{Aj} - v^{Aj} = \frac{1}{2h}(r^{j(A)} - r^{-j(A)}), \tag{6.11}$$
$$u^{Ajj} - v^{Ajj} = \frac{1}{h^2}(r^{j(A)} - 2r^A + r^{-j(A)}),$$

whence, combining (6.11), (6.10) and (6.7), we obtain the equality

$$(6.12)\quad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim) r^A + \sum_{j=1}^{p} \frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}(r^{j(A)} - r^{-j(A)}) + \sum_{j=1}^{p} \frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2}(r^{j(A)} - 2r^A + r^{-j(A)}) + \frac{1}{k}(r^A - r^B).$$

Now we write the right-hand member of (6.12) in an equivalent form, cf. fig. 2:

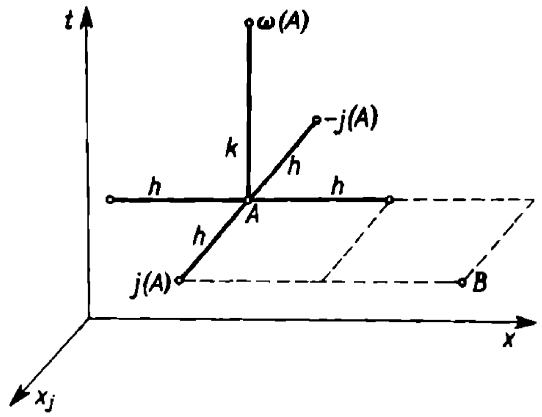

Fig. 2. The nodal points intervening in the formulae (6.12) and (6.13): $\omega(A)$—max for $r^{\mu+1,m}$, B—max for $r^{\mu,m}$

$$(6.13)\quad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim) r^A + \sum_{j=1}^{p} \frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}[(r^{j(A)} - r^B) - (r^{-j(A)} - r^B)] + \sum_{j=1}^{p} \frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2}[(r^{j(A)} - r^B) + 2(r^B - r^A) + (r^{-j(A)} - r^B)] + \frac{1}{k}(r^A - r^B),$$

collect terms, and arrive at the equation

$$(6.14)\quad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim) r^A + \sum_{j=1}^{p}\left[\frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2} + \frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}\right](r^{j(A)} - r^B) + \sum_{j=1}^{p}\left[\frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2} - \frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}\right](r^{-j(A)} - r^B) + \left(\sum_{j=1}^{p}\frac{\partial f}{\partial w_j}(\sim)\frac{2}{h^2} - \frac{1}{k}\right)(r^B - r^A).$$

With (6.14) at hand we can proceed with the majorization of the right-hand member of (6.14) so as to obtain (6.4). Let us observe that

$$(6.15)\qquad \frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2}+\frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}\geqslant -\Gamma\frac{1}{2h}+g\frac{1}{h^2}=\frac{1}{h}\left(\frac{g}{h}-\frac{\Gamma}{2}\right),$$

$$(6.16)\qquad \frac{\partial f}{\partial w_j}(\sim)\frac{1}{h^2}-\frac{\partial f}{\partial q_j}(\sim)\frac{1}{2h}\geqslant -\Gamma\frac{1}{2h}+g\frac{1}{h^2}=\frac{1}{h}\left(\frac{g}{h}-\frac{\Gamma}{2}\right),$$

because of (3.2) and (3.3).

We multiply both sides of (6.15) by the non-positive quantity $r^{j(A)}-r^B\leqslant 0$, cf. (6.5), and (6.16) by the non-positive quantity $r^{-j(A)}-r^B\leqslant 0$, and we get

$$(6.17)\qquad \begin{aligned} &\text{second line in (6.14)} \leqslant \sum_{j=1}^{p}\frac{1}{h}\left(\frac{g}{h}-\frac{\Gamma}{2}\right)(r^{j(A)}-r^B)\leqslant 0\,,\\ &\text{third line in (6.14)} \leqslant \sum_{j=1}^{p}\frac{1}{h}\left(\frac{g}{h}-\frac{\Gamma}{2}\right)(r^{-j(A)}-r^B)\leqslant 0\,. \end{aligned}$$

Hence we can drop the second and third line in (6.14).

In a similar way, from assumptions (3.2) and (3.3) it follows that

$$(6.18)\qquad \sum_{j=1}^{p}\frac{\partial f}{\partial w_j}(\sim)\frac{2}{h^2}-\frac{1}{k}\leqslant\frac{2pG}{h^2}-\frac{1}{k}.$$

We multiply both sides of (6.18) by the non-negative quantity $r^B-r^A\geqslant 0$, cf. (6.5), and we see that

$$(6.19)\qquad \begin{aligned} \text{fourth line in (6.14)} &= \left(\sum_{j=1}^{p}\frac{\partial f}{\partial w_j}(\sim)\frac{2}{h^2}-\frac{1}{k}\right)(r^B-r^A)\\ &\leqslant\left(\frac{2pG}{h^2}-\frac{1}{k}\right)(r^B-r^A)\leqslant 0\,, \end{aligned}$$

whence, we can drop the fourth line in (6.14).

From (6.14), (6.17) and (6.19) we obtain the desired inequality

$$(6.20)\qquad s^{\mu\sim}\leqslant Ls^{\mu}+\varepsilon(h)\qquad (\mu=0,1,\ldots,N_1)\,,$$

because of the definition (4.4) of $\varepsilon(h)$.

This completes the proof of the first inequality (6.4).

In a similar manner we can prove the second inequality (6.4). It is sufficient only to introduce the notation

$$(6.21)\qquad \begin{aligned} z^{\mu+1}&=\min_m r^{\mu+1,m}=r^{\mu+1,c}=r^{\omega(C)}\,,\\ z^{\mu}&=\min_m r^{\mu,m}=r^{\mu,d}=r^D\,, \end{aligned}$$

where $C = (\mu, c)$, $D = (\mu, d)$, $c = (c_1, \ldots, c_p)$, $d = (d_1, \ldots, d_p)$, and, with (6.21) at hand, to repeat the argument connected with formulas (6.5) up to (6.20), the sense of the corresponding inequalities being reversed.

In particular, from equality (6.13) for $s^{\mu\sim}$ we obtain the corresponding equality for $z^{\mu\sim}$, if s^μ, A, B in (6.13) are replaced by z, C, D, respectively.

It turns out that we can drop the second, third and fourth lines of the equality for $z^{\mu\sim}$ just obtained, since they are non-negative, and this leads us to the second inequality (6.4).

This completes the proof of Lemma 2.

§ 7. THEOREM 1. *Suppose that*

i) *the right-hand member $f(t, x, u, q, w)$ of the equation* (3.5) *fulfils assumptions* H,

ii) *the values of the solution u^M and the approximation v^M are defined at the nodal points of the set E by* (4.1), (4.2) *and* (4.5), (4.6), *respectively,*

iii) *the function $\varepsilon(h)$ is defined by* (4.4) *and the error r^M by* (6.1).

Under these assumptions

1) *we have error estimate of the form*

$$|r^M| \leqslant \frac{\varepsilon(h)}{L}(e^{Lk\mu}-1) \qquad (\mu = 0, 1, \ldots, N_1) \tag{7.1}$$

at the nodal points M of the set E,

2) *the difference method* (4.6) *is convergent, i.e.,*

$$\lim_{h\to 0} r^M = 0 \,. \tag{7.2}$$

Proof. (7.2) follows from (7.1), as $h \to 0$, cf. (4.3), therefore we shall deal with relation (7.1).

Let us observe first that

$$z^\mu \leqslant r^{\mu,m} \leqslant s^\mu \qquad (\mu = 0, 1, \ldots, N_1) \tag{7.3}$$

at the nodal points (μ, m) of the set E, because of the definition (6.2).

The values $s^\mu \geqslant 0$ satisfy the first difference inequality (6.4) and the initial condition $s^0 = 0$, whence from Lemma 1 we obtain

$$s^\mu \leqslant \frac{\varepsilon(h)}{L}(e^{Lk\mu}-1) \qquad (\mu = 0, 1, \ldots, N_1)\,. \tag{7.4}$$

For the same reason, the values $(-z^\mu) \geqslant 0$ satisfy the first difference inequality (6.4), i.e. the inequality

$$(-z^\mu)^\sim \leqslant L(-z^\mu) + \varepsilon(h) \qquad (\mu = 0, 1, \ldots, N_1)\,, \tag{7.5}$$

and the initial condition $(-z^0) = 0$, whence from (7.5) and Lemma 1 it follows that

$$-z^{\mu} \leqslant \frac{\varepsilon(h)}{L}(e^{Lk\mu}-1) \quad (\mu = 0, 1, \ldots, N_1), \tag{7.6}$$

i.e.,

$$z^{\mu} \geqslant -\frac{\varepsilon(h)}{L}(e^{Lk\mu}-1) \quad (\mu = 0, 1, \ldots, N_1). \tag{7.7}$$

Combining (7.3), (7.4) and (7.7) we obtain the desired inequality (7.1). This completes the proof of Theorem 1.

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