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On a system of functional equations occurring in the theory of geometric objects

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Abstract. In the present paper all solutions of system of functional equations (0.1), (0.2) are given, where x, y are non-singular 2×2 real matrices, i.e., $x, y \in \mathrm{GL}(2, R)$, F and g are unknown functions defined for every $x, y \in \mathrm{GL}(2, R)$. The values of the function F are 3×3 real matrices and, moreover, $F\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ is a non-singular matrix, i.e., F is a non-singular solution of the functional equation (0.1) (cf. [3] for the properties of the solutions of (0.1)). The values of the function g are 3×1 real matrices. The dot denotes the matrix multiplications.

We do not make any assumptions concerning the regularity of the functions F and g. Moreover, we assume that equations (0.1) and (0.2) are satisfied by the functions F and g only for all $x, y \in \mathrm{GL}(2, R)$.

The system of functional equations (0.1), (0.2) occurs in the theory of geometric objects. Namely, the solutions of this system of functional equations (0.1) and (0.2) have been applied to determine all purely differential geometric objects of first class with three components in the two-dimensional space, i.e., according to the terminology of J. Aczél and S. Gołąb, objects of type [3, 2, 1] with linear non-homogeneous transformation rule.

The method used in the present paper (except Section 5) is analogous to that used by M. Kucharzewski and M. Kuczma in [10]; however, the present problem is more complicated.

Equation (0.1) does not contain the function g and therefore it can be considered independently of equation (0.2). The general non-singular solution of (0.1) for all $x, y \in \mathrm{GL}(2, R)$ has been given in [3]. In the present paper all non-singular solutions F of the functional equation (0.1) are given by formulae (1.1)–(1.9).

The main result of the paper is Theorem 3.1 and Theorem 5.3.

Introduction. In the present paper all solutions of the system of functional equations

$$F(x\cdot y) = F(x)\cdot F(y), \tag{0.1}$$

$$g(x\cdot y) = F(x)\cdot g(y)+g(x), \tag{0.2}$$

are given, where x, y are non-singular 2×2 real matrices, i.e., $x, y \in \mathrm{GL}(2, R)$, and F and g are unknown functions defined on $\mathrm{GL}(2, R)$.

The values of the function F are 3×3 real matrices and, moreover, $F(e)$ (where $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$) is a non-singular matrix, i.e., F is a non-singular solution of the functional equation (0.1). The values of the function g are 3×1 real matrices. The dot "$\cdot$" denotes multiplication of matrices.

We do not make any assumptions concerning the regularity of the functions F and g.

The system of functional equations (0.1) and (0.2) occurs in the theory of geometric objects (cf. [2], p. 152). Namely, the solutions of this system have been applied to determine all purely differential geometric objects of first class with three components in a two-dimensional space, i.e., according to the terminology of J. Aczél and S. Gołąb, all objects of type $[3, 2, 1]$ (cf. [2], p. 15) with linear non-homogeneous transformation rule.

The method used in the present paper (except Section 5) is analogous to that used by M. Kucharzewski and M. Kuczma in [10]; however, the present problem is more complicated.

1. All non-singular solutions F of functional equation (0.1). Equation (0.1) does not contain the function g and therefore it can be considered independently of equation (0.2). The general non-singular solution of functional equation (0.1) for all $x, y \in \mathrm{GL}(2, R)$ has been given in my paper [3] (cf. p. 4–6).

Namely, all non-singular solutions F of the functional equation (0.1) for every $x, y \in \mathrm{GL}(2, R)$ are given by the formula

$$F(x) = C\cdot F_0(x)\cdot C^{-1}, \tag{1.1}$$

where $F_0(x)$ has one of the following forms:

$$F_0(x) = \begin{bmatrix} \varphi(\Delta) & 0 & 0 \\ 0 & \varphi(\Delta) & 0 \\ 0 & 0 & \varphi_3(\Delta) \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.2}$$

$$F_0(x) = \begin{bmatrix} \varphi_1(\Delta) & 0 & 0 \\ 0 & \varphi_2(\Delta) & 0 \\ 0 & 0 & \varphi_3(\Delta) \end{bmatrix}, \tag{1.3}$$

$$F_0(x) = \begin{bmatrix} \varphi(\Delta) & \varphi(\Delta)a(\Delta) & 0 \\ 0 & \varphi(\Delta) & 0 \\ 0 & 0 & \varphi_3(\Delta) \end{bmatrix}, \tag{1.4}$$

(1.5)
$$F_0(x) = \begin{bmatrix} \varkappa(\Delta) & -\sigma(\Delta) & 0 \\ \sigma(\Delta) & \varkappa(\Delta) & 0 \\ 0 & 0 & \varphi(\Delta) \end{bmatrix},$$

(1.6)
$$F_0(x) = \varphi(\Delta)\begin{bmatrix} 1 & 0 & a_1(\Delta) \\ 0 & 1 & a_2(\Delta) \\ 0 & 0 & 1 \end{bmatrix},$$

(1.7)
$$F_0(x) = \varphi(\Delta)\begin{bmatrix} 1 & a_1(\Delta) & a_2(\Delta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(1.8)
$$F_0(x) = \varphi(\Delta)\begin{bmatrix} 1 & a_1(\Delta) & \frac{1}{2}a_1^2(\Delta)+a_2(\Delta) \\ 0 & 1 & a_1(\Delta) \\ 0 & 0 & 1 \end{bmatrix},$$

(1.9)
$$F_0(x) = \varphi(\Delta)\begin{bmatrix} x_{11}^2 & 2x_{11}x_{12} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22}+x_{12}x_{21} & x_{12}x_{22} \\ x_{21}^2 & 2x_{21}x_{22} & x_{22}^2 \end{bmatrix}.$$

In formulae (1.2)–(1.9):

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in \mathrm{GL}(2, R),$$

$\Delta = \det x$ is the determinant of x; C is an arbitrary constant non-singular 3×3 real matrix playing in formula (1.1) the role of a parameter; φ, φ_i are arbitrary multiplicative functions not vanishing identically, i.e., they are solutions of the functional equation

(1.10)
$$\varphi(\xi\eta) = \varphi(\xi)\varphi(\eta), \quad \xi\eta \neq 0,$$

with the restriction

(1.11)
$$\varphi \not\equiv 0;$$

a is an arbitrary function satisfying the equation

(1.12)
$$a(\xi\eta) = a(\xi)+a(\eta), \quad \xi\eta \neq 0,$$

and the condition

(1.13)
$$a \not\equiv 0.$$

In formulae (1.6) and (1.7) a_1 and a_2 are arbitrary solutions of functional equation (1.12) fulfilling the condition

(1.14) the functions a_1 and a_2 are lineary independent for $\xi \neq 0$.

In formula (1.8), a_1 and a_2 are arbitrary solutions of (1.12) with the restriction

$$a_1 \not\equiv 0. \tag{1.15}$$

The functions $\varkappa$ and σ are a solution of the system of functional equations

$$\begin{aligned} \varkappa(\xi\eta) &= \varkappa(\xi)\varkappa(\eta) - \sigma(\xi)\sigma(\eta), \\ \sigma(\xi\eta) &= \varkappa(\xi)\sigma(\eta) + \varkappa(\eta)\sigma(\xi), \end{aligned} \qquad \xi\eta \neq 0, \tag{1.16}$$

fulfilling the condition

$$\sigma \not\equiv 0. \tag{1.17}$$

Remark 1.1. Restrictions (1.13), (1.14), (1.15) and (1.17) are not essential. For if any of inequalities (1.13), (1.14), (1.15) and (1.17) were not fulfilled, then the corresponding cases of (1.4)–(1.8) would be reduced to case (1.3) or (1.4).

Remark 1.2. Equations (1.10), (1.12), and (1.16), as well as their solutions, are well known (cf. [1]). By (1.11) and the properties of the solutions of equation (1.10) we get

$$\varphi(\xi) \neq 0 \quad \text{and} \quad \varphi_i(\xi) \neq 0 \quad \text{for every } \xi \neq 0,\ i = 1, 2, 3. \tag{1.18}$$

In particular

$$\varphi(1) = 1 \quad \text{and} \quad \varphi_i(1) = 1 \quad \text{for } i = 1, 2, 3. \tag{1.19}$$

Furthermore, from (1.12) we get

$$\begin{gathered} a(1) = a(-1) = 0 \quad \text{and} \quad a_i(1) = a_i(-1) = 0, \\ a(-\xi) = a(\xi), \quad a_i(-\xi) = a_i(\xi) \quad \text{for } i = 1, 2. \end{gathered} \tag{1.20}$$

Using (1.16), we obtain

$$\varkappa(1) = 1 \quad \text{and} \quad \sigma(\pm 1) = 0. \tag{1.21}$$

The general non-singular solution of equation (0.1) represents all homomorphisms F: $\mathrm{GL}(2, R) \to \mathrm{GL}(3, R)$, that means, all real linear 3×3 representations of the group $\mathrm{GL}(2, R)$.

From the properties of the solutions of equations (1.10), (1.12), (1.16) it follows, in particular, that if inequalities (1.11), (1.13), (1.14), and (1.15) are fulfilled, then they are also valid if we confine ourselves to $\xi > 0$ only.

Then for every $\xi > 0$ we have

$$\varphi(\xi) \neq 0, \quad \varphi_i(\xi) \neq 0 \quad (i = 1, 2, 3), \tag{1.22}$$

$$a(\xi) \not\equiv 0; \quad a_1(\xi) \not\equiv 0 \quad \text{in formula (1.8)}, \tag{1.23}$$

(1.24) the functions a_1 and a_2 in formulae (1.6) and (1.7) are linearly independent,

$$\sigma(\xi) \not\equiv 0. \tag{1.25}$$

Let us notice that for every non-singular solution $F(x)$ of equation (0.1) and for every $g(x)$ satisfying equation (0.2) for $x, y \in \mathrm{GL}(2, R)$, we have

$$g(e) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{1.26}$$

where e is the unit matrix belonging to $\mathrm{GL}(2, R)$.

Now, $g(x) = g(e \cdot x) = F(e) \cdot g(x) + g(e) = E \cdot g(x) + g(e) = g(x) + g(e)$. Thus we obtain $g(x) = g(x) + g(e)$, and hence follows (1.26).

2. Auxiliary lemmas. In the sequel of the paper we shall repeatedly apply the following lemmas (Lemmas 2.1–2.5).

LEMMA 2.1 (cf. [11]). *The general solution of the functional equation*

$$\gamma(x \cdot y) = \gamma(y)\varphi(\Delta) + \gamma(x) \tag{2.1}$$

for all $x, y \in \mathrm{GL}(2, R)$, *where* φ *is an arbitrary not vanishing identically solution of equation* (1.10), *is given by the formulae*

$$\gamma(x) = \lambda[\varphi(\Delta) - 1] \quad \text{if } \varphi \not\equiv 1, \tag{2.2}$$

$$\gamma(x) = \ln|\Phi_0(\Delta)| \quad \text{if } \varphi \equiv 1. \tag{2.3}$$

In formula (2.3), Φ_0 *is an arbitrary multiplicative function non-identically zero for* $\boldsymbol{R} - \{0\}$; *in* (2.2) λ *is a real parameter.*

The essential results obtained by M. Kucharzewski and M. Kuczma in their paper [10] will be referred to in the present paper as

LEMMA 2.2 (cf. [10], p. 61). *Every pair of functions* $f(x)$ *and* $\bar{g}(x)$ *satisfying for all* $x, y \in \mathrm{GL}(2, R)$ *the system of functional equations*

$$f(x \cdot y) = f(x) \cdot f(y), \tag{2.4}$$

$$\bar{g}(x \cdot y) = f(x) \cdot \bar{g}(y) + \bar{g}(x), \tag{2.5}$$

where f is a non-singular 2×2 matrix-function, $\bar{g}$ is a 2×1 matrix-function, must have one of the following forms:

$$f(x) = c\cdot\begin{bmatrix}\varphi(\Delta) & 0\\ 0 & \varphi(\Delta)\end{bmatrix}\cdot\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}\cdot c^{-1}, \tag{2.6}$$

$$\bar{g}(x) = [f(x)-e]\cdot\bar{q} = \begin{bmatrix}\gamma_1(x)\\ \gamma_2(x)\end{bmatrix},$$

when $\varphi \not\equiv 1$ *or* $\varphi \equiv 1$,

$$f(x) = c\cdot\begin{bmatrix}\varphi_1(\Delta) & 0\\ 0 & \varphi_2(\Delta)\end{bmatrix}\cdot c^{-1}, \quad \bar{g}(x) = [f(x)-e]\cdot\bar{q} \tag{2.7}$$

if $\varphi_1 \not\equiv 1$ *and* $\varphi_2 \not\equiv 1$,

$$f(x) = c\cdot\begin{bmatrix}\varphi(\Delta) & \varphi(\Delta)a(\Delta)\\ 0 & \varphi(\Delta)\end{bmatrix}\cdot c^{-1}, \quad \bar{g}(x) = [f(x)-e]\cdot\bar{q} \tag{2.8}$$

if $\varphi \not\equiv 1$,

$$f(x) = c\cdot\begin{bmatrix}\varkappa(\Delta) & -\sigma(\Delta)\\ \sigma(\Delta) & \varkappa(\Delta)\end{bmatrix}\cdot c^{-1}, \quad \bar{g}(x) = [f(x)-e]\cdot\bar{q}, \tag{2.9}$$

when $\sigma \not\equiv 0$,

$$f(x) = c\cdot\begin{bmatrix}1 & 0\\ 0 & \varphi_2(\Delta)\end{bmatrix}\cdot c^{-1}, \quad \bar{g}(x) = c\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \lambda_2[\varphi_2(\Delta)-1]\end{bmatrix} \tag{2.10}$$

if $\varphi_2 \not\equiv 1$,

$$f(x) = e, \quad \bar{g}(x) = \begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \ln|\Phi_2(\Delta)|\end{bmatrix}, \tag{2.11}$$

$$f(x) = c\cdot\begin{bmatrix}1 & a(\Delta)\\ 0 & 1\end{bmatrix}\cdot c^{-1}, \quad \bar{g}(x) = c\cdot\begin{bmatrix}\ln|\Phi(\Delta)|+\omega a^2(\Delta)\\ 2\omega a(\Delta)\end{bmatrix} \tag{2.12}$$

if $a \not\equiv 0$.

In formulae (2.6)–(2.12) λ_2, ω are constants, c is a non-singular constant 2×2 matrix; λ_1, λ_2 are constants;

$$q = \begin{bmatrix}\lambda_1\\ \lambda\end{bmatrix}; \quad e = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};$$

$\varphi, \varphi_i, \Phi, \Phi_i$ $(i = 1, 2)$ are non-trivial multiplicative functions.

On the other hand, it can easily be verified that each of the pairs of functions (2.6)–(2.12), where the above-mentioned constants are arbitrary constants and the functions $\varphi, \varphi_i, \Phi, \Phi_i$ $(i = 1, 2)$ are arbitrary multiplicative functions, actually satisfy the system of equations (2.4) and (2.5).

LEMMA 2.3. *If a pair of functions $F_1(x)$, $g_1(x)$ is a solution of the system* (0.1) *and* (0.2), *then the pair of functions $F_2(x) = Y\cdot F_1(x)\cdot Y^{-1}$ and $g_2(x) = Y\cdot g_1(x)$, where Y is an arbitrary non-singular 3×3 matrix, is also a solution of the system* (0.1) *and* (0.2).

In particular, if a pair of functions $F(x) = C\cdot F_0(x)\cdot C^{-1}$ and $g(x)$, where $F_0(x)$ takes any form of (1.2)–(1.9), *is a solution of the system* (0.1) *and* (0.2), *then we have*

$$C\cdot F_0(x\cdot y)\cdot C^{-1} = C\cdot F_0(x)\cdot C^{-1}\cdot C\cdot F_0(y)\cdot C^{-1}$$

and

$$g(x\cdot y) = C\cdot F_0(x)\cdot C^{-1}\cdot g(y)+g(x),$$

or

$$F_0(x\cdot y) = F_0(x)\cdot F_0(y),$$

$$C^{-1}\cdot g(x\cdot y) = F_0(x)\cdot C^{-1}\cdot g(y)+C^{-1}\cdot g(x).$$

Introducing the function $g_0(x)$ defined by

$$g_0(x) = C^{-1}\cdot g(x) = \begin{bmatrix}\gamma_1(x)\\ \gamma_2(x)\\ \gamma_3(x)\end{bmatrix} = [\gamma_i(x)], \quad i = 1, 2, 3, \tag{2.13}$$

we obtain the pair $F_0(x)$ and $g_0(x)$, which, as well, is a solution of the system (0.1), (0.2). Thus the function $g_0(x)$ is a solution of the equation

$$g_0(x\cdot y) = F_0(x)\cdot g_0(y)+g_0(x), \tag{2.14}$$

where $F_0(x)$ denotes the corresponding matrix-function (1.2)–(1.9) (according to the case considered).

It follows from (2.14) that, if $F_0(x)$ is given by formulae (1.2)–(1.9), then to find of the corresponding function $g_0(x)$, equation (2.14) has to be solved.

For in the sequel, notice that if a pair $F(x)$ and $g(x)$ is a solution of the system of functional equations (0.1) and (0.2), where

$$F(x) = C\cdot\begin{bmatrix} & & 0\\ & f_0(x) & 0\\ 0 & 0 & \varphi_3(\Delta)\end{bmatrix}\cdot C^{-1}, \tag{2.15}$$

and the function

$$f_0(x) = c^{-1}\cdot f(x)\cdot c \tag{2.16}$$

is defined by one of the following formulae:

$$f_0(x) = \begin{bmatrix} \varphi(\Delta) & 0 \\ 0 & \varphi(\Delta) \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \varphi(\Delta)x, \tag{2.17}$$

$$f_0(x) = \begin{bmatrix} \varphi_1(\Delta) & 0 \\ 0 & \varphi_2(\Delta) \end{bmatrix}, \tag{2.18}$$

$$f_0(x) = \begin{bmatrix} \varphi(\Delta) & \varphi(\Delta)\alpha(\Delta) \\ 0 & \varphi(\Delta) \end{bmatrix}, \tag{2.19}$$

$$f_0(x) = \begin{bmatrix} \varkappa(\Delta) & -\sigma(\Delta) \\ \sigma(\Delta) & \varkappa(\Delta) \end{bmatrix}, \tag{2.20}$$

then in order to determine the function $g(x)$,

$$g(x) = C\cdot g_0(x) = C\cdot \begin{bmatrix} \gamma_1(x) \\ \gamma_2(x) \\ \gamma_3(x) \end{bmatrix}, \tag{2.21}$$

the following system of the functional equations has to be solved:

$$\bar{g}_0(x\cdot y) = f_0(x)\cdot \bar{g}_0(y) + \bar{g}_0(x), \tag{2.22}$$

$$\gamma_3(x\cdot y) = \varphi_3(\Delta)\gamma_3(y) + \gamma_3(x), \tag{2.23}$$

where

$$\bar{g}_0(x) = c^{-1}\cdot \bar{g}(x) = \begin{bmatrix} \bar{\gamma}_1(x) \\ \bar{\gamma}_2(x) \end{bmatrix}. \tag{2.24}$$

Since equation (2.23) has been solved by M. Kuczma in [11], and equation (2.22) by M. Kucharzewski and M. Kuczma in [10] (p. 61, Theorem, Lemma 2.2), then, in the case considered, all functions $g_0(x)$ and $g(x) = C\cdot g_0(x)$ are obtained immediately, and this gives all solutions of the system (0.1), (0.2) (in this case).

Now, using (2.13), we get

$$\begin{cases} \left.\begin{matrix} \gamma_1(x) = \bar{\gamma}_1(x) \\ \gamma_2(x) = \bar{\gamma}_2(x) \end{matrix}\right\}, & \text{where } \bar{g}_0(x) = \begin{bmatrix} \bar{\gamma}_1(x) \\ \bar{\gamma}_2(x) \end{bmatrix} \text{ is the solution of equation (2.22),} \\ \gamma_3(x) & \text{is the solution of equation (2.23).} \end{cases} \tag{2.25}$$

The result of the above considerations can be stated in the form of the following

LEMMA 2.4. *If a pair of functions $F(x)$ and $g(x)$, with $F(x)$ defined by formula* (2.15) *and $f_0(x)$ by formulae* (2.17)–(2.20), *is a solution of the system of functional equations* (0.1), (0.2), *then*

$$g_0(x) = C^{-1}\cdot g(x) = \begin{bmatrix}\bar{\gamma}_1(x)\\ \bar{\gamma}_2(x)\\ \gamma_3(x)\end{bmatrix}, \quad \text{where } \bar{g}_0(x) = \begin{bmatrix}\bar{\gamma}_1(x)\\ \bar{\gamma}_2(x)\end{bmatrix}$$

is the corresponding solution of equation (2.22), *and $\gamma_3(x)$ is the solution of equation* (2.23).

Now, applying the method of proof analogous to that used by M. Kucharzewski and M. Kuczma in [10] (p. 61–63), we obtain

LEMMA 2.5. *If a pair of functions $F(x) = C\cdot F_0(x)\cdot C^{-1}$ and $g(x)$ is a solution of the system of equations* (0.1), (0.2) *and there exists a number $\beta \neq 0$ such that for*

$$x_0 = \begin{bmatrix}\beta & 0\\ 0 & \beta\end{bmatrix} \in \mathrm{GL}(2, R) \tag{2.26}$$

the matrix $F_0(x_0)-E$ is non-singular, i.e.,

$$\det[F_0(x_0)-E] \neq 0, \tag{2.27}$$

then

$$g(x) = [F(x)-E]\cdot q_1, \tag{2.28}$$

where

$$q_1 = C\cdot p = C\cdot [F_0(x_0)-E]^{-1}\cdot g_0(x_0) \tag{2.29}$$

is a 3×1 matrix.

A direct verification shows that such a pair of functions $F(x)$ and $g(x)$ is a solution of the system equations (0.1), (0.2).

3. Determination of the function $g(x)$ in the case where $F_0(x)$ is of the form (1.2)–(1.5). Applying the results obtained in papers [10] and [11] and taking into account Lemmas 2.1–2.4, we have immediately $g(x)$ in the case where $F(x) = C\cdot F_0(x)\cdot C^{-1}$ with a function $F_0(x)$ of one of the forms (1.2)–(1.5).

THEOREM 3.1. *Functions $F(x)$ and $g(x)$ satisfying the system of functional equations* (0.1) *and* (0.2) *for all $x, y \in \mathrm{GL}(2, R)$, where $F_0(x) = C^{-1}\cdot F(x)\cdot C$ is defined by formulae* (1.2)–(1.5), *must have one of the following forms:*

$$F(x) = C\cdot F_0(x)\cdot C^{-1}, \tag{3.1}$$

where $F_0(x)$ *is an arbitrary function of* (1.2)–(1.5) *with additional restrictions* $\varphi \not\equiv 1$, $\varphi_i \equiv 1$ $(i = 1, 2, 3)$, $\sigma \not\equiv 0$,

$$g(x) = [F(x) - E]\cdot q;$$

(3.2)
$$F(x) = C\cdot\begin{bmatrix}\varphi(\Delta) & 0 & 0\\ 0 & \varphi(\Delta) & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}x_{11} & x_{12} & 0\\ x_{21} & x_{22} & 0\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1},$$
$$g(x) = C\cdot\begin{bmatrix}\lambda_1[x_{11}\varphi(\Delta)-1]+\lambda_2 x_{12}\varphi(\Delta)\\ \lambda_1 x_{21}\varphi(\Delta)+\lambda_2[\varphi(\Delta)x_{22}-1]\\ \ln|\Phi_3(\Delta)|\end{bmatrix},$$

(*if* $\varphi \not\equiv 1$ *or* $\varphi \equiv 1$);

(3.3) $$F(x) = C\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & \varphi_2(\Delta) & 0\\ 0 & 0 & \varphi_3(\Delta)\end{bmatrix}\cdot C^{-1}, \quad g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \lambda_2[\varphi_2(\Delta)-1]\\ \lambda_3[\varphi_3(\Delta)-1]\end{bmatrix};$$

(3.4) $$F(x) = C\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \varphi_3(\Delta)\end{bmatrix}\cdot C^{-1}, \quad g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \ln|\Phi_2(\Delta)|\\ \lambda_3[\varphi_3(\Delta)-1]\end{bmatrix};$$

(3.5) $$F(x) = C\cdot E\cdot C^{-1} = E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \ln|\Phi_2(\Delta)|\\ \ln|\Phi_3(\Delta)|\end{bmatrix};$$

(3.6)
$$F(x) = C\cdot\begin{bmatrix}\varkappa(\Delta) & -\sigma(\Delta) & 0\\ \sigma(\Delta) & \varkappa(\Delta) & 0\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1},$$
$$g(x) = C\cdot\begin{bmatrix}\lambda_1[\varkappa(\Delta)-1]-\lambda_2\sigma(\Delta)\\ \lambda_1\sigma(\Delta)+\lambda_2[\varkappa(\Delta)-1]\\ \ln|\Phi_3(\Delta)|\end{bmatrix};$$

(3.7)
$$F(x) = C\cdot\begin{bmatrix}\varphi(\Delta) & \varphi(\Delta)a(\Delta) & 0\\ 0 & \varphi(\Delta) & 0\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1},$$
$$g(x) = C\cdot\begin{bmatrix}\lambda_1[\varphi(\Delta)-1]+\lambda_2\varphi(\Delta)a(\Delta)\\ \lambda_2[\varphi(\Delta)-1]\\ \ln|\Phi_3(\Delta)|\end{bmatrix};$$

$$F(x) = C\cdot\begin{bmatrix}1 & a(\Delta) & 0\\ 0 & 1 & 0\\ 0 & 0 & \varphi_3(\Delta)\end{bmatrix}\cdot C^{-1},$$

(3.8)

$$g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\omega a^2(\Delta)\\ 2\omega a(\Delta)\\ \lambda_3[\varphi_3(\Delta)-1]\end{bmatrix};$$

(3.9) $$F(x) = C\cdot\begin{bmatrix}1 & a(\Delta) & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1}, \quad g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\omega a^2(\Delta)\\ 2\omega a(\Delta)\\ \ln|\Phi_3(\Delta)|\end{bmatrix}.$$

In formulae (3.1)–(3.9) $\lambda_1, \lambda_2, \lambda_3, \omega$ are constants; Φ_i $(i = 1, 2, 3)$ are any non-trivial multiplicative functions;

$$q = C\cdot\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}.$$

Remark 3.1. The cases where

$$F(x) = C\cdot\begin{bmatrix}\varphi_1(\Delta) & 0 & 0\\ 0 & \varphi_2(\Delta) & 0\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1} = C\cdot\{\varphi_1, \varphi_2, 1\}\cdot C^{-1},$$

or

$$F(x) = C\cdot\begin{bmatrix}\varphi_1(\Delta) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \varphi_3(\Delta)\end{bmatrix}\cdot C^{-1} = C\cdot\{\varphi_1, 1, \varphi_3\}\cdot C^{-1}$$

can easily be reduced to case (3.3). Now, writting

$$J = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} = J^{-1}, \quad P_1 = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} = P_1^{-1},$$

we have

$$\begin{aligned}F(x) &= C\cdot\{\varphi_1, \varphi_2, 1\}\cdot C^{-1} = C\cdot J\cdot(J\cdot\{\varphi_1, \varphi_2, 1\}\cdot J)\cdot J\cdot C^{-1}\\ &= C^*\cdot\{1, \varphi_2, \varphi_1\}\cdot C^{*-1}, \quad \text{where } C^* = C\cdot J.\end{aligned}$$

However,

$$\begin{aligned} F(x) &= C\cdot\{\varphi_1, 1, \varphi_3\}\cdot C^{-1} \\ &= C\cdot P_1\cdot(P_1\cdot\{\varphi_1, 1, \varphi_3\}\cdot P_1)\cdot P_1\cdot C^{-1} \\ &= C^{**}\cdot\{1, \varphi_1, \varphi_3\}\cdot C^{**-1}, \quad \text{where } C^{**} = C\cdot P_1. \end{aligned}$$

Analogously, the cases where

$$F(x) = C\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varphi_2(\Delta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot C^{-1} = C\cdot\{1, \varphi_2, 1\}\cdot C^{-1}$$

and

$$F(x) = C\cdot\begin{bmatrix} \varphi_1(\Delta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot C^{-1} = C\cdot\{\varphi_1, 1, 1\}\cdot C^{-1}$$

are reduced to (3.4).

On the other hand, it can easily be verified that each of the pairs of functions (3.1)–(3.9) for arbitrary constants and arbitrary non-trivial multiplicative functions actually satisfy the system of (0.1) and (0.2).

Thus we obtain all solutions of (0.1) and (0.2) in the case where the function $F_0(x)$ is defined by formulae (1.2)–(1.5). The cases are still to be considered where the function $F_0(x)$ takes the form (1.6)–(1.9).

4. Determination of the function $g(x)$ in the case where $F_0(x)$ is of form (1.6), (1.7) **and** (1.8). In this section first of all we shall consider the case where the function $F_0(x)$ in the formula $F(x) = C\cdot F_0(x)\cdot C^{-1}$ has form (1.6), (1.7) or (1.8), with the additional restriction

$$\varphi(\xi) \not\equiv 1 \quad \text{for } \xi > 0. \tag{4.1}$$

Taking into account restriction (4.1), we see that there is a number $\xi_0 > 0$ such that

$$\varphi(\xi_0) \neq 1 \tag{4.2}$$

(of course, $\xi_0 \neq 1$, since $\varphi(1) = 1$). Writing

$$x_0 = \begin{bmatrix} \sqrt{\xi_0} & 0 \\ 0 & \sqrt{\xi_0} \end{bmatrix}, \tag{4.3}$$

we have $\det x_0 = \xi_0 > 0$ and $x_0 \in \mathrm{GL}(2, R)$. Moreover, in the present case, F_0 being of forms (1.6)–(1.8); we obtain

$$\det[F_0(x_0) - E] = [\varphi(\xi_0) - 1]^3 \neq 0 \tag{4.4}$$

in view of (4.2).

Applying Lemma 2.5, we obtain

THEOREM 4.1. *If a pair of functions $F(x)$ and $g(x)$ is a solution of the system of equations* (0.1) *and* (0.2), *where $F_0(x) = C^{-1}\cdot F(x)\cdot C$ is of one of forms* (1.6)–(1.8), *and if $\varphi(\xi) \not\equiv 1$ for $\xi > 0$* (*i.e.* (4.1) *is fulfilled*), *then*

$$g(x) = [F(x)-E]\cdot q_1 \tag{4.5}$$

(q_1 is in Lemma 2.5, (2.29)).

We can directly verify that the pair $F(x)$ and $g(x)$ satisfies the system of equations (0.1), (0.2). Let us now considerer the case where the function $F(x)$ has form (1.6)–(1.8) but condition (4.1) is not satisfied. Since the function $\varphi(\xi)$ is either even or odd, we have two possibilities: either

$$\varphi(\xi) \equiv 1 \tag{4.6}$$

or

$$\varphi(\xi) = \operatorname{sgn}\xi. \tag{4.7}$$

We shall discuss these two cases separately.

In order to determine the corresponding functions $g(x) = C\cdot g_0(x)$ in case (4.6) we solve equation (2.14) with a function $F_0(x)$ of forms (1.6), (1.7) and (1.8) with the function $\varphi(\Delta) \equiv 1$, i.e.,

$$g_0(x\cdot y) = F_0(x)\cdot g_0(y)+g_0(x),$$

where

$$F_0(x) = \begin{bmatrix} 1 & 0 & a_1(\Delta) \\ 0 & 1 & a_2(\Delta) \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.8}$$

$$F_0(x) = \begin{bmatrix} 1 & a_1(\Delta) & a_2(\Delta) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.9}$$

$$F_0(x) = \begin{bmatrix} 1 & a_1(\Delta) & \frac{1}{2}a_1^2(\Delta)+a_2(\Delta) \\ 0 & 1 & a_1(\Delta) \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.10}$$

By [7] (Theorem 0.1), [8] (Theorem 0.1), [9] (Theorem 0.1) we get the general solution of equation (2.14) in every corresponding case (4.8)–(4.10).

Thus we obtain two following theorems:

THEOREM 4.2. *If a pair of functions $F(x) = C\cdot F_0(x)\cdot C^{-1}$ and $g(x)$ is a solution of the system of equations* (0.1) *and* (0.2), *where $F_0(x)$ is defined by* (4.8), (4.9) *or* (4.10) (*i.e., $F_0(x)$ is of form* (1.6), (1.7) *or* (1.8) *with the* *additional restriction* $\varphi \equiv 1$), *then*

$$g(x) = C\cdot g_0(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \ln|\Phi_2(\Delta)|\\ 0\end{bmatrix} \tag{4.11}$$

in case, when $F_0(x)$ *is defined by formula* (4.8),

$$g(x) = C\cdot g_0(x) \tag{4.12}$$

$$= C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\frac{1}{2}\varepsilon_1 a_1^2(\Delta)+\frac{1}{2}\bar{\varepsilon}_2 a_2^2(\Delta)+\bar{\varepsilon}_1 a_1(\Delta)a_2(\Delta)\\ \varepsilon_1 a_1(\Delta)+\bar{\varepsilon}_1 a_2(\Delta)\\ \bar{\varepsilon}_1 a_1(\Delta)+\bar{\varepsilon}_2 a_2(\Delta)\end{bmatrix}$$

in case where $F_0(x)$ *is defined by formula* (4.9),

$$g(x) = C\cdot g_0(x) \tag{4.13}$$

$$= C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\tau a_1^2(\Delta)+\frac{1}{3}\omega a_1^3(\Delta)+2\omega a_1(\Delta)a_2(\Delta)\\ 2\tau a_1(\Delta)+2\omega a_2(\Delta)+\omega a_1^2(\Delta)\\ 2\omega a_1(\Delta)\end{bmatrix}$$

in case (4.10).

In the above formulae $\varepsilon_1, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \tau, \omega$ are constants, Φ_1, Φ_2 denote multiplicative functions not vanishing identically.

We can directly verify that every such pair $F(x)$ and $g(x)$ actually satisfy the system (0.1), (0.2).

In order to determine the corresponding functions $g(x) = C\cdot g_0(x)$ in case (4.7) we solve equation (2.14) with a function $F_0(x)$ of the form (1.6), (1.7) and (1.8) with the function $\varphi(\Delta) = \operatorname{sgn}\Delta$, i.e.,

$$g_0(x\cdot y) = F_0(x)\cdot g_0(y)+g_0(x),$$

where

$$F_0(x) = \operatorname{sgn}\Delta\begin{bmatrix}1 & 0 & a_1(\Delta)\\ 0 & 1 & a_2(\Delta)\\ 0 & 0 & 1\end{bmatrix}, \tag{4.14}$$

$$F_0(x) = \operatorname{sgn}\Delta\begin{bmatrix}1 & a_1(\Delta) & a_2(\Delta)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{4.15}$$

$$F_0(x) = \operatorname{sgn}\Delta\begin{bmatrix}1 & a_1(\Delta) & \frac{1}{2}a_1^2(\Delta)+a_2(\Delta)\\ 0 & 1 & a_1(\Delta)\\ 0 & 0 & 1\end{bmatrix}. \tag{4.16}$$

Applying the results of papers [7] (Theorem 0.2), [8] (Theorem 0.2), and [9] (Theorem 0.2), we know the general solution of equation (2.14) in every corresponding case. Now, in either of cases under consideration, $g(x) = [F(x)-E]\cdot q$. Thus we have

THEOREM 4.3. *If a pair of functions* $F(x) = C\cdot F_0(x)\cdot C^{-1}$ *and* $g(x)$ *is a solution of system* (0.1), (0.2), *where* $F_0(x)$ *is defined by* (4.14), (4.15) *or* (4.16) (*i.e.*, $F_0(x)$ *is the form* (1.6), (1.7), (1.8) *with the additional restriction* $\varphi(\xi) = \operatorname{sgn}\xi$), *then*

$$g(x) = [F(x)-E]\cdot q. \tag{4.17}$$

On the other hand, it can easily be verified that all such pairs of functions actually satisfy the system of equations (0.1) and (0.2).

Remark 4.1. If a_1, a_2 in formulae (1.6) and (1.7) are linearly dependent or if in formula (1.8) $a_1 \equiv 0$, then these cases reduce to (1.4), in which all solutions of equation (2.14) are known (cf. (3.7), (3.1)). For example, we demonstrate the consideration in case (4.8). We proceed in an analogous way in the remaining cases.

I. The function $F_0(x)$ has form (4.8) and the functions a_1, a_2 occurring in formula (4.8) are linearly dependent. We shall distinguish two subcases:

$$a_1(\xi) \equiv 0 \quad \text{for } \xi \neq 0, \tag{4.18}$$

$$a_1(\xi) \not\equiv 0 \quad \text{for } \xi \neq 0. \tag{4.19}$$

Now let $a_1 \equiv 0$. Then putting

$$P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{4.20}$$

we get

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P_3^{-1}\cdot F_0(x)\cdot P_3$$

$$= \begin{bmatrix} 1 & a_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_1(x).$$

Thus $F_0(x) = P_3\cdot F_1(x)\cdot P_3^{-1}$ and equation (2.14) becomes

$$g_0(x\cdot y) = P_3\cdot F_1(x)\cdot P_3^{-1}\cdot g_0(y)+g_0(x).$$

Consequently,

$$P_3^{-1}\cdot g_0(x\cdot y) = F_1(x)\cdot[P_3^{-1}\cdot g_0(y)]+P_3^{-1}\cdot g_0(x). \tag{4.21}$$

Now let us write

$$g_1(x) = P_3^{-1}\cdot g_0(x). \tag{4.22}$$

From (4.21) it follows that the function $g_1(x)$ satisfies the equation

$$g_1(x\cdot y) = F_1(x)\cdot g_1(y)+g_1(x), \tag{4.23}$$

where $F_1(x)$ is equal to $F_0(x)$ defined by (1.4) with $a = a_2$ and $\varphi \equiv \varphi_3 \equiv 1$, i.e., this case is reduced to (3.9).

Using (3.9) and (4.22) we get

$$g(x) = C\cdot g_0(x) = C\cdot P_3\cdot g_1(x) = C^*\cdot g_1(x) \tag{4.24}$$

$$= C^*\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\omega a_2^2(\Delta)\\ 2\omega a_2(\Delta)\\ \ln|\Phi_3(\Delta)|\end{bmatrix},$$

$C\in \mathrm{GL}(3,R)$ and $C\cdot P_3 = C^*\in \mathrm{GL}(3,R)$.

Now let a_1 and a_2 be linearly dependent and let $a_1 \not\equiv 0$. Thus, there exist numbers η_1, η_2 such that

$$\eta_1^2+\eta_2^2>0 \quad \text{and} \quad \eta_1 a_1+\eta_2 a_2 \equiv 0.$$

It follows from $a_1\not\equiv 0$ that $\eta_2\neq 0$ and consequently we obtain $a_2\equiv \tau a_1$, where $\tau = -\eta_1/\eta_2$, and so

$$F_0(x) = \begin{bmatrix}1 & 0 & a_1\\ 0 & 1 & \tau a_1\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ \tau & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & a_1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -\tau & 1 & 0\end{bmatrix}.$$

Putting

$$U = \begin{bmatrix}1 & 0 & 0\\ \tau & 0 & 1\\ 0 & 1 & 0\end{bmatrix},\quad U^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -\tau & 1 & 0\end{bmatrix},\quad F_2(x) = \begin{bmatrix}1 & a_1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

we get

$$F_0(x) = U\cdot F_2(x)\cdot U^{-1} \quad\text{or}\quad F_2(x) = U^{-1}\cdot F_0(x)\cdot U$$

and equation (2.14) has the form

$$g_0(x\cdot y) = \big(U\cdot F_2(x)\cdot U^{-1}\big)\cdot g_0(y)+g_0(x).$$

Consequently,

$$U^{-1}\cdot g_0(x\cdot y) = F_2(x)\cdot\big(U^{-1}\cdot g_0(y)\big)+U^{-1}\cdot g_0(x). \tag{4.25}$$

Now let us define

$$(4.26) \qquad g_2(x) = U^{-1}\cdot g_0(x).$$

Using (4.25) we see that the function $g_2(x)$ satisfies the equation

$$(4.27) \qquad g_2(x\cdot y) = F_2(x)\cdot g_2(y)+g_2(x),$$

where $F_2(x)$ is equal to $F_0(x)$ defined by (1.4), with $a = a_1$ and $\varphi \equiv \varphi_3 \equiv 1$, i.e., the case considered is reduced to (3.9). Using (3.9) and (4.26), we get

$$(4.28) \qquad g(x) = C\cdot g_0(x) = (C\cdot U)\cdot g_2(x) = C^{**}\cdot g_2(x)$$

$$= C^{**}\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\omega a_1^2(\Delta)\\ 2\omega a_1(\Delta)\\ \ln|\Phi_3(\Delta)|\end{bmatrix},$$

where $C\in \mathrm{GL}(3,R)$ and $C\cdot U = C^{**}\in \mathrm{GL}(3,R)$.

We omit the deduction in the other cases as fully analogous to that just performed.

It remains to consider the case of the function $F(x) = C\cdot F_0(x)\cdot C^{-1}$ with $F_0(x)$ defined by (1.9). This problem will be considered in subsequent sections.

5. Determination of the function $g(x)$ in the case where $F(x) = C\cdot F_0(x)\cdot C^{-1}$ with $F_0(x)$ defined by (1.9). First of all we consider the case when the function $F_0(x)$ takes the form (1.9) with the additional restriction that the multiplicative function is not trivial:

$$(5.1) \qquad \xi\varphi(\xi)\not\equiv 1 \quad \text{for } \xi>0.$$

By (5.1) there exists a number $\xi_1>0$ such that

$$(5.2) \qquad \xi_1\varphi(\xi_1)\neq 1$$

(of course, $\xi_1\neq 1$).

Denote, for brevity,

$$(5.3) \qquad x_0 = \begin{bmatrix}\sqrt{\xi_1} & 0\\ 0 & \sqrt{\xi_1}\end{bmatrix}.$$

Since

$$(5.4) \qquad \det x_0 = \xi_1 > 0,$$

we have that $x_0\in \mathrm{GL}(2,R)$ and from (1.9), (5.3), (5.2) it follows that the matrix $F_0(x_0)-E$ is non-singular, i.e.,

$$(5.5) \qquad \det[F_0(x_0)-E] = [\xi_1\varphi(\xi_1)-1]^3\neq 0.$$

In fact,

$$F_0(x_0)-E = \varphi(\xi_1)\{\xi_1,\,\xi_1,\,\xi_1\}-E$$
$$= \xi_1\varphi(\xi_1)\,E-E = [\xi_1\varphi(\xi_1)-1]E.$$

Consequently, using Lemma 2.5, we obtain immediately

THEOREM 5.1. *If a pair of functions $F(x)$ and $g(x)$ is a solution of the system of functional equations* (0.1) *and* (0.2), *where* $F_0(x) = C^{-1}\cdot F(x)\cdot C$ *is defined by formula* (1.9) *with the function* $\varphi(\xi) \not\equiv 1/\xi$ *for* $\xi > 0$ (*i.e.,* $\xi\varphi(\xi) \not\equiv 1$ *for* $\xi > 0$), *then*

$$g(x) = [F(x)-E]\cdot q_1, \tag{5.6}$$

where q_1 *is given by* (2.29).

On the other hand, it can easily be verified that any such pair of functions actually satisfies the system of equations (0.1) and (0.2).

In the sequel we shall consider the case where the function $F_0(x)$ has the form (1.9) but condition (5.1) is not fulfilled, i.e., for $\xi > 0$

$$\xi\varphi(\xi) \equiv 1. \tag{5.7}$$

Now we have for the multiplicative function $\xi\varphi(\xi)$ only two possibilities:

$$\xi\varphi(\xi) = \operatorname{sgn}\xi \quad \text{for all } \xi \neq 0 \tag{5.8}$$

or

$$\xi\varphi(\xi) = 1 \quad \text{for every } \xi \neq 0. \tag{5.9}$$

It follows from (5.8) for $\xi \neq 0$

$$\varphi(\xi) = 1/|\xi|. \tag{5.10}$$

From (5.9) we have for $\xi \neq 0$

$$\varphi(\xi) = 1/\xi. \tag{5.11}$$

Using on formulae (5.10) and (5.11), we obtain for $F_0(x)$ defined by (1.9):

$$F_0(x) = \frac{1}{|\Delta|}\begin{bmatrix} x_{11}^2 & 2x_{11}x_{12} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22}+x_{12}x_{21} & x_{12}x_{22} \\ x_{21}^2 & 2x_{21}x_{22} & x_{22}^2 \end{bmatrix} \tag{5.12}$$

and

$$F_0(x) = \frac{1}{\Delta}\begin{bmatrix} x_{11}^2 & 2x_{11}x_{12} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22}+x_{12}x_{21} & x_{12}x_{22} \\ x_{21}^2 & 2x_{21}x_{22} & x_{22}^2 \end{bmatrix}. \tag{5.13}$$

Introducing the function $g_0(x)$ defined by (2.13), i.e.,

$$g_0(x) = C^{-1}\cdot g(x) = [\gamma_i(x)] \quad (i = 1, 2, 3),$$

we obtain a solution of equation (2.14), where $F_0(x)$ denotes the corresponding matrix-function (5.12) or (5.13) (according to the occurring case). All solutions of this equation (2.14) in both cases (i.e., when $F_0(x)$

is defined either by (5.12) or by (5.13) have been given by the author in paper [4] (cf. p. 5, Theorem) and [5] (cf. p. 219–220, Theorem 0.1).

The method used in papers [4] and [5] is analogous. However, the situation in [5] is more interesting and more complicated, and where exist non-measurable solutions whereas all solutions in [4] are measurable.

Applying the results of papers [4] and [5], we know all solutions $g_0(x)$ in both cases; hence we know also $g(x) = C\cdot g_0(x)$. Thus we obtain the two following theorems:

THEOREM 5.1. *If a pair of functions $F(x)$ and $g(x)$ is a solution of the system* (0.1) *and* (0.2), *where $F_0(x) = C^{-1}\cdot F(x)\cdot C$ is defined by formula* (1.9) *with $\varphi(\Delta) = 1/|\Delta|$ (i.e., $F_0(x)$ is defined by formula* (5.12)), *then*

$$g(x) = [F(x)-E]\cdot q, \tag{5.14}$$

where $q = C\cdot q_0$.

Now

$$\begin{aligned} g(x) &= C\cdot g_0(x) = C\cdot[F_0(x)-E]\cdot q_0 \\ &= [C\cdot F_0(x)\cdot C^{-1}\cdot C - C\cdot E\cdot C^{-1}\cdot C]\cdot q_0 = [F(x)-E]\cdot(C\cdot q_0) \\ &= [F(x)-E]\cdot q. \end{aligned}$$

THEOREM 5.2. *If a pair of functions $F(x)$ and $g(x)$ is a solution of the system* (0.1), (0.2), *where $F_0(x) = C^{-1}\cdot F(x)\cdot C$ is defined by formula* (1.9) *with $\varphi(\Delta) = 1/\Delta$ (i.e., $F_0(x)$ is defined by* (5.13)), *then*

$$g(x) = [F(x)-E]\cdot q + g^*(x), \tag{5.15}$$

where

$$g^*(x) = C\cdot g_0^*(x) = C\cdot\frac{1}{\Delta}\begin{bmatrix} \begin{vmatrix} \psi(x_{11}) & \psi(x_{12}) \\ x_{11} & x_{12} \end{vmatrix} \\ \frac{1}{2}\left\{\begin{vmatrix} \psi(x_{11}) & \psi(x_{22}) \\ x_{11} & x_{22} \end{vmatrix} + \begin{vmatrix} \psi(x_{21}) & \psi(x_{12}) \\ x_{21} & x_{12} \end{vmatrix}\right\} \\ \begin{vmatrix} \psi(x_{21}) & \psi(x_{22}) \\ x_{21} & x_{22} \end{vmatrix} \end{bmatrix}. \tag{5.16}$$

In formula (5.16) ψ is a derivation of R.

By a derivation of R we mean any function $\psi\colon R\to R$ satisfying the conditions:

$$\psi(\xi+\eta) = \psi(\xi)+\psi(\eta), \tag{5.17}$$

$$\psi(\xi\eta) = \eta\psi(\xi)+\xi\psi(\eta) \tag{5.18}$$

for all ξ, η in R.

Numerous and interesting properties of derivations are given and proved in papers [5], p. 220, [3] and [15], p. 120.

In particular, if ψ is a derivation, then it is well known that $\psi(0) = \psi(1) = 0$, $\psi(-\xi) = -\psi(\xi)$ and $\psi(\xi^p) = p\xi^{p-1}\psi(\xi)$ for ξ in R and $p = 1, 2, \ldots$ If $\eta \neq 0$, then

$$\psi\left(\frac{1}{\eta}\right) = -\frac{\psi(\eta)}{\eta^2} \quad \text{and} \quad \psi\left(\frac{\xi}{\eta}\right) = \frac{\eta\psi(\xi)-\xi\psi(\eta)}{\eta^2}.$$

Evidently, the function ψ: $R \to \{0\}$ (i.e., the function identically 0) is a derivation of R. This function will be called the *trivial derivation* of R.

Furthermore, every derivation of R vanishes on the algebraic closure $\bar{Q}$ of the field Q of rationals in R and has a dense set of periods. Every derivation of R, according to the terminology introduced by S. Gołąb, is a microperiodical function. Every rational number is a period of an arbitrary derivation of R. It is well known that any measurable derivation of R is trivial (cf. [3]), but there are non-trivial derivations of R (cf. [15], p. 124, Corollaries 1 and 1'). Every non-trivial derivation of R must be a non-measurable function.

Thus we get the following

COROLLARY 5.1. *Any measurable solution of* (2.14), *where* $F_0(x)$ *denotes the matrix-function* (5.13), *is of the form* $g_0(x) = [F_0(x)-E]\cdot q_0$.

In fact, under the assumption that the solution $g_0(x)$ is measurable we infer that ψ is a measurable derivation of R (i.e., ψ is identically 0) and consequently we obtain

$$g_0^*(x) = [0] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The assertion of Corollary 5.1 is thus proved.

In the case considered we have $g^*(x) = C\cdot g_0^*(x) = [0]$ and $g(x) = C\cdot g_0(x) = C\cdot[F_0(x)-E]\cdot q_0 = [F(x)-E]\cdot q$, where $q = C\cdot q_0$.

Remark 5.1. In both cases (those of Theorem 5.1 and 5.2) we can easily verify that each of the pairs of functions $F(x)$ and $g(x)$ actually satisfy the system of (0.1) and (0.2).

Thus, we have considered all the possible cases and so the proof of the following theorem has been completed.

THEOREM 5.3. *All solutions $F(x)$, $g(x)$ of the system of functional equations* (0.1) *and* (0.2) *for $x, y \in \mathrm{GL}(2, R)$ are given by the following formulae:*

1–9 *formulae* (3.1)–(3.9),

10. $F(x) = C\cdot F_0(x)\cdot C^{-1}$, *where $F_0(x)$ is an arbitrary function of* (1.2)–(1.8) *with additional restrictions: $\varphi(\xi)\not\equiv 1$, $\varphi_i(\xi)\not\equiv 1$ $(i\not\equiv 1,2,3)$ for $\xi\neq 0$,*

$g(x) = [F(x)-E]\cdot q$;

11. $F(x) = C\cdot F_0(x)\cdot C^{-1}$, *where $F_0(x)$ defined by formula* (1.9) *with additional restriction $\xi\varphi(\xi)\not\equiv 1$ (i.e., $\varphi(\xi)\not\equiv 1/\xi$ for $\xi\neq 0$),*

$g(x) = [F(x)-E]\cdot q$;

12. $$F(x) = C\cdot\begin{bmatrix}1 & 0 & a_1(\Delta)\\ 0 & 1 & a_2(\Delta)\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1},$$ *where a_1, a_2 are arbitrary linearly independent solutions of* (1.12),

$$g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|\\ \ln|\Phi_2(\Delta)|\\ 0\end{bmatrix},$$ *where Φ_1, Φ_2 are non-trivial multiplicative functions;*

13. $$F(x) = C\cdot\begin{bmatrix}1 & a_1(\Delta) & a_2(\Delta)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1},$$ *where a_1, a_2 as above,*

$$g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\frac{1}{2}\varepsilon_1 a_1^2(\Delta)+\frac{1}{2}\bar\varepsilon_2 a_2^2(\Delta)+\bar\varepsilon_1 a_1(\Delta)a_2(\Delta)\\ \varepsilon_1 a_1(\Delta)+\bar\varepsilon_1 a_2(\Delta)\\ \bar\varepsilon_1 a_1(\Delta)+\bar\varepsilon_2 a_2(\Delta)\end{bmatrix},$$

where Φ_1 is a non-trivial multiplicative function;

14. $$F(x) = C\cdot\begin{bmatrix}1 & a_1(\Delta) & \frac{1}{2}a_1^2(\Delta)+a_2(\Delta)\\ 0 & 1 & a_1(\Delta)\\ 0 & 0 & 1\end{bmatrix}\cdot C^{-1},$$

$$g(x) = C\cdot\begin{bmatrix}\ln|\Phi_1(\Delta)|+\frac{1}{3}\omega a_1^3(\Delta)+\bar\tau a_1^2(\Delta)+2\omega a_1(\Delta)a_2(\Delta)\\ \omega a_1^2(\Delta)+2\bar\tau a_1(\Delta)+2\omega a_2(\Delta)\\ 2\omega a_1(\Delta)\end{bmatrix};$$

$$15. \qquad F(x) = C\cdot\frac{1}{\Delta}\begin{bmatrix} x_{11}^2 & 2x_{11}x_{12} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22}+x_{12}x_{21} & x_{12}x_{22} \\ x_{21}^2 & 2x_{21}x_{22} & x_{22}^2 \end{bmatrix}\cdot C^{-1},$$

$$g(x) = [F(x)-E]\cdot q + C\cdot g_0^*(x)$$

$$= [F(x)-E]\cdot q + C\cdot\frac{1}{\Delta}\begin{bmatrix} \begin{vmatrix} \psi(x_{11}) & \psi(x_{12}) \\ x_{11} & x_{12} \end{vmatrix} \\ \frac{1}{2}\left\{\begin{vmatrix} \psi(x_{11}) & \psi(x_{22}) \\ x_{11} & x_{22} \end{vmatrix} + \begin{vmatrix} \psi(x_{21}) & \psi(x_{12}) \\ x_{21} & x_{12} \end{vmatrix}\right\} \\ \begin{vmatrix} \psi(x_{21}) & \psi(x_{22}) \\ x_{21} & x_{22} \end{vmatrix} \end{bmatrix},$$

where ψ *is an arbitrary derivation of* R, $q = C\cdot q_0$.

In formulae 1–15, $\lambda, \omega, \varepsilon_1, \bar{\varepsilon}_2, \bar{\tau}, \bar{\varepsilon}_1, \lambda_i$ $(i = 1, 2, 3)$ are arbitrary constants; C is an arbitrary constant matrix $\in \mathrm{GL}(3, R)$; E is the 3×3 unit matrix; Δ (as usual) denotes the determinant of the matrix $x \in \mathrm{GL}(2, R)$; φ, φ_i $(i = 1, 2, 3)$ are an arbitrary multiplicative functions not vanishing identically; a, a_i $(i = 1, 2)$ are arbitrary functions satisfying equation (1.12) and not vanishing identically; a_1 and a_2 are linearly independent in formulae 12, 13. (In formula 14, $a_1 \not\equiv 0$.) $q = C\cdot q_0$, where q_0 is an arbitrary constant vector.

On the other hand, it can be easily verified that each of the pairs of functions 1–15 actually satisfy the system (0.1), (0.2).

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