ANNALES
POLONICI MATHEMATICI
XXII (1970)

A generalization of the Haruki functional equation*

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The Haruki functional equation

$$(1)\qquad f(x+t,y+t)+f(x+t,y-t)+f(x-t,y+t)+f(x-t,y-t) = 4f(x,y)$$

was considered by many authors under different assumptions (cf. lectures by Aczél, Choquet, Haruki, and McKiernan during the Conference on Functional Equations at Oberwolfach in 1966). Its general solution was found later by McKiernan and by Sakovič (cf. [1]).

The most general continuous solution of equation (1) has the form

$$f(x,y) = axy(x^2-y^2)+bx(3y^2-x^2)+cy(3x^2-y^2)+ d(x^2-y^2)+exy+fx+gy+h\,,$$

where a, b, c, d, e, f, g, h are arbitrary constants.

A natural generalization of equation (1) is the equation

$$(2)\qquad f\big(x+\varphi(t),y+\psi(t)\big)+f\big(x+\varphi(t),y-\psi(t)\big)+f\big(x-\varphi(t),y+\psi(t)\big)+ f\big(x-\varphi(t),y-\psi(t)\big) = 4f(x,y)$$

which has a simple geometrical interpretation (similar to that for the Haruki equation).

It can be easily shown that in the case $\varphi(t) = at$, $\psi(t) = \beta t$ ($a \neq 0$, $\beta \neq 0$) all the solutions of equation (2) can be obtained by the linear mapping $x' = \beta x$, $y' = ay$ from those of the Haruki equation. But it is not always so. Usually the most general continuous solution of equation (2) has the form

$$(3)\qquad f(x,y) = Axy+Bx+Cy+D\,,$$

* Results of this paper were presented without proofs during the 1-st International Conference on Functional Equations at Zakopane, October 1967.

where A, B, C, D are arbitrary constants. This fact is very interesting since functions (3) are also the most general continuous solutions of the equation

$$f(x+t, y+\tau)+f(x+t, y-\tau)+f(x-t, y+\tau)+f(x-t, y-\tau) = 4f(x, y)$$

with 4 variables which was considered by Aczél.

Notation:

$$\varphi_a = \varphi(a), \quad \varphi_t' = \varphi'(t), \quad \varphi_t'' = \varphi''(t), \quad \varphi_t''' = \varphi'''(t), \quad \varphi_t^{(4)} = \varphi^{(4)}(t),$$
$$\psi_a = \psi(a), \quad \psi_t' = \psi'(t), \quad \psi_t'' = \psi''(t), \quad \psi_t''' = \psi'''(t), \quad \psi^{(4)} = \psi^{(4)}(t),$$
$$(\ ) = \big(x+\varphi(t), y+\psi(t)\big),$$
$$[\ ] = \big(x+\varphi(t), y-\psi(t)\big),$$
$$\{\ \} = \big(x-\varphi(t), y+\psi(t)\big),$$
$$\langle\ \rangle = \big(x-\varphi(t), y-\psi(t)\big).$$

We shall make use of the following theorems:

I (cf. H. Świątak [3]). *Suppose that* $n > 1$, *and*

1° $a_i(x, t) \in C^\infty$ *in* R^n *for every fixed* t *from an open interval* $\Delta \subset R$, $i = 1, \ldots, k$,

2° $a_i(x, t) \in C^m$ *in* $R^n \times \Delta$ $(i = 1, \ldots, k)$,

3° $\varphi_i(t) = \big(\varphi_{i1}(t), \ldots, \varphi_{in}(t)\big) \in C^m$ *in* Δ $(i = 1, \ldots, k)$,

4° *there exists an* $a \in \Delta$ *such that* $\varphi_i(a) = 0$ *for* $i = 1, \ldots, k$,

5° *the equation*

$$\frac{\partial^m}{\partial t^m}\Big(\sum_{i=1}^{k} a_i(x, t) f\big(x+\varphi_i(t)\big)\Big) = 0$$

(*where the unknown function* $f \in C^m$ *in* R^n) *is elliptic for* $t = a$.

Then every locally integrable solution f *of the equation*

$$\sum_{i=1}^{k} a_i(x, t) f\big(x+\varphi_i(t)\big) = 0$$

is equal almost everywhere to a function of class C^∞ *and every continuous solution* f *is a function of class* C^∞.

II (cf. H. Świątak [4]). *Suppose that* $n > 1$, *and*

1° $b(x, t) \in C^\infty$ *in* R^n *for every fixed* t *from an open set* $\Delta \subset R$,

2° $b(x, t) \in C^m$ *in* $R^n \times \Delta$ *and* $a_i(t) \in C^m$ *in* Δ $(i = 1, \ldots, k)$,

3° $\varphi_i(t) = \big(\varphi_{i1}(t), \ldots, \varphi_{in}(t)\big) \in C^m$ *in* Δ $(i = 1, \ldots, k)$, 4° *there exists an* $a \in \Delta$ *such that* $\varphi_i(a) = \theta$ *for* $i = 1, \ldots, k$,

5° *the equation*

$$\frac{\partial^m}{\partial t^m}\Big(\sum_{i=1}^{k} a_i(t) f\big(x+\varphi_i(t)\big)\Big) = 0$$

(*where the unknown function* $f \in C^m$ *in* R^n) *is hypoelliptic for* $t = a$.

Then every locally integrable solution f *of the equation*

$$\sum_{i=1}^{k} a_i(t) f\big(x+\varphi_i(t)\big) = b(x, t)$$

is equal almost everywhere to a function of class C^∞ *and every continuous solution* f *is a function of class* C^∞.

THEOREM I. *If there exists an* a *such that* $\varphi_a = \psi_a = 0$, $\varphi'_a \psi'_a \neq 0$, *and if* $\varphi(t)$, $\psi(t) \in C^2$ *in an interval* Δ *such that* $a \in \operatorname{int} \Delta$, *then every locally integrable solution* f *of equation* (2) *is equal almost everywhere to a function of class* C^∞ *and every continuous solution* f *is a function of class* C^∞.

Proof. To prove this theorem it is enough to show that equation (2) satisfies all the assumptions of I.

In our case $n = 2$, $m = 2$, $k = 5$, $a_i(x, t) \equiv 1$ for $i = 1, 2, 3, 4$, $a_5(x, t) \equiv -4$, $\varphi_1(t) = \big(\varphi(t), \psi(t)\big)$, $\varphi_2(t) = \big(\varphi(t), -\psi(t)\big)$, $\varphi_3(t) = \big(-\varphi(t), \psi(t)\big)$, $\varphi_4(t) = \big(-\varphi(t), -\psi(t)\big)$, $\varphi_5(t) = (0, 0)$, $\varphi_i(a) = (0, 0)$ for $i = 1, 2, 3, 4, 5$ and satisfying of 1°, 2°, 3°, and 4° is obvious.

Now, let us assume for a moment that $f \in C^2$. Differentiating equation (2) twice by t we obtain

$$\begin{aligned}(4)\quad & \varphi''_t f'_x(\ ) + \varphi'^2_t f''_{xx}(\ ) + 2\varphi'_t\psi'_t f''_{xy}(\ ) + \psi''_t f'_y(\ ) + \psi'^2_t f''_{yy}(\ ) + \\ & + \varphi''_t f'_x[\ ] + \varphi'^2_t f''_{xx}[\ ] - 2\varphi'_t\psi'_t f''_{xy}[\ ] - \psi''_t f'_y[\ ] + \psi'^2_t f''_{yy}[\ ] - \\ & - \varphi''_t f'_x\{\ \} + \varphi'^2_t f''_{xx}\{\ \} - 2\varphi'_t\psi'_t f''_{xy}\{\ \} + \psi''_t f'_y\{\ \} + \psi'^2_t f''_{yy}\{\ \} - \\ & - \varphi''_t f'_x\langle\ \rangle + \varphi'^2_t f''_{xx}\langle\ \rangle + 2\varphi'_t\psi'_t f''_{xy}\langle\ \rangle - \psi''_t f'_y\langle\ \rangle + \psi'^2_t f''_{yy}\langle\ \rangle = 0\,.\end{aligned}$$

Hence, in view of the assumption $\varphi_a = \psi_a = 0$, we obtain for $t = a$ the equation

$$\varphi'^2_a f''_{xx}(x, y) + \psi'^2_a f''_{yy}(x, y) = 0\,. \tag{5}$$

Equation (5) is elliptic since $\varphi'_a \psi'_a \neq 0$. Thus also assumption 5° of I is satisfied.

This finishes the proof.

THEOREM II. *If there exists an* a *such that* $\varphi_a = \psi_a = 0$, $\varphi'_a \psi''_a \neq 0$, $\varphi'_a \psi''_a \neq \varphi''_a \psi'_a$, *and if* $\varphi(t), \psi(t) \in C^3$ *in an interval* Δ *such that* $a \in \operatorname{int} \Delta$, *then the most general continuous solution of equation* (2) *has form* (3).

Proof. Notice that all the assumptions of Theorem I are satisfied and therefore every continuous solution f of equation (2) is a function of class C^∞.

Differentiating equation (2) three times by t (we make use of (4)), we get

$$
\begin{aligned}
(6)\qquad & \varphi_t'''f_x'(\;)+3\varphi_t'\varphi_t''f_{xx}''(\;)+3\varphi_t''\psi_t'f_{xy}''(\;)+\varphi_t'^3f_{xxx}'''(\;)+\\
& +3\varphi_t'^2\psi_t'f_{xxy}'''(\;)+3\varphi_t'\psi_t'^2f_{xyy}'''(\;)+3\psi_t'\psi_t''f_{yy}''(\;)+\psi_t'''f_y'(\;)+\\
& +3\varphi_t'\psi_t''f_{xy}''(\;)+\psi_t'^3f_{yyy}'''(\;)+\varphi_t'''f_x'[\;]+3\varphi_t'\varphi_t''f_{xx}''[\;]-\\
& -3\varphi_t''\psi_t'f_{xy}''[\;]+\varphi_t'^3f_{xxx}'''[\;]-3\varphi_t'^2\psi_t'f_{xxy}'''[\;]+3\varphi_t'\psi_t'^2f_{xyy}'''[\;]+\\
& +3\psi_t'\psi_t''f_{yy}''[\;]-\psi_t'''f_y'[\;]-3\varphi_t'\psi_t''f_{xy}''[\;]-\psi_t'^3f_{yyy}'''[\;]-\\
& -\varphi_t'''f_x'\{\;\}+3\varphi_t'\varphi_t''f_{xx}''\{\;\}-3\varphi_t''\psi_t'f_{xy}''\{\;\}-\varphi_t'^3f_{xxx}'''\{\;\}+\\
& +3\varphi_t'^2\psi_t'f_{xxy}'''\{\;\}-3\varphi_t'\psi_t'^2f_{xyy}'''\{\;\}+3\psi_t'\psi_t''f_{yy}''\{\;\}+\psi_t'''f_y'\{\;\}-\\
& -3\varphi_t'\psi_t''f_{xy}''\{\;\}+\psi_t'^3f_{yyy}'''\{\;\}-\varphi_t'''f_x'\langle\;\rangle+3\varphi_t'\varphi_t''f_{xx}''\langle\;\rangle+\\
& +3\varphi_t''\psi_t'f_{xy}''\langle\;\rangle-\varphi_t'^3f_{xxx}'''\langle\;\rangle-3\varphi_t'^2\psi_t'f_{xxy}'''\langle\;\rangle-3\varphi_t'\psi_t'^2f_{xyy}'''\langle\;\rangle+\\
& \qquad +3\psi_t'\psi_t''f_{yy}''\langle\;\rangle-\psi_t'''f_y'\langle\;\rangle+3\varphi_t'\psi_t''f_{xy}''\langle\;\rangle-\psi_t'^3f_{yyy}'''\langle\;\rangle=0\,.
\end{aligned}
$$

Hence, in view of the assumption $\varphi_a=\psi_a=0$, we get the equation

$$(7)\qquad \varphi_a'\varphi_a''f_{xx}''(x,y)+\psi_a'\psi_a''f_{yy}''(x,y)=0\,.$$

We had shown in the proof of Theorem I that the function f satisfies also equation (5). Multiplying equation (7) by φ_a' and taking into account (5), we obtain

$$\psi_a'(\varphi_a'\psi_a''-\varphi_a''\psi_a')f_{yy}''(x,y)=0\,.$$

In view of the assumptions $\psi_a'\neq 0$ and $\varphi_a'\psi_a''\neq\varphi_a''\psi_a'$,

$$(8)\qquad f_{yy}''(x,y)\equiv 0\,.$$

In view of $\varphi_a'\neq 0$, it follows from (5) that also

$$(9)\qquad f_{xx}''(x,y)\equiv 0\,.$$

By (8) and (9)

$$f(x,y)=Axy+Bx+Cy+D\,,$$

where A, B, C, D are arbitrary constants. Q.E.D.

THEOREM III. *If there exists an a such that $\varphi_a=\psi_a=0$, $\psi_a'=0$ (or $\varphi_a'=0$), $\varphi_a'\psi_a''\neq 0$ (or $\varphi_a''\psi_a'\neq 0$), $4\varphi_a'\varphi_a'''+3\varphi_a''^2\geqslant 0$ (or $4\psi_a'\psi_a'''+3\psi_a''^2\geqslant 0$), and if $\varphi(t),\psi(t)\in C^4$ in an interval Δ such that $a\in\operatorname{int}\Delta$, then the most general continuous solution f of equation* (2) *has form* (3).

Proof. To prove this theorem we shall make use of II.

It is easy to verify that assumptions 1°, 2°, 3° and 4° are satisfied. Now, suppose for a moment that $f \in C^4$. Differentiating equation (2) 4 times by t (we make use of (6)) and putting $t = a$ yields

$$\varphi_a'^4 f^{(4)}_{xxxx}(x, y) + (4\varphi_a'\varphi_a''' + 3\varphi_a''^2) f''_{xx}(x, y) + 6\varphi_a'^2\psi_a'^2 f^{(4)}_{xxyy}(x, y) + \\ + (4\psi_a'\psi_a''' + 3\psi_a''^2) f''_{yy}(x, y) + \psi_a'^4 f^{(4)}_{yyyy}(x, y) = 0 .$$

Since $\psi_a' = 0$, we obtain hence

$$\varphi_a'^4 f^{(4)}_{xxxx}(x, y) + (4\varphi_a'\varphi_a''' + 3\varphi_a''^2) f''_{xx}(x, y) + 3\psi_a''^2 f''_{yy}(x, y) = 0 . \tag{10}$$

In view of our assumptions

$$P(\xi) = \varphi_a'^4 \xi_1^4 + (4\varphi_a'\varphi_a''' + 3\varphi_a''^2)\xi_1^2 + 3\psi_a''^2 \xi_2^2 \neq 0$$

for every $\xi = (\xi_1, \xi_2) \neq (0, 0)$.

It is easy to verify that

$$\frac{D_\xi^p P(\xi)}{P(\xi)} \to 0 \quad \text{when } |\xi| = \sqrt{\xi_1^2 + \xi_2^2} \to \infty$$

for each couple of integers p_1, p_2 such that $p_1 \geqslant 0$, $p_2 \geqslant 0$, and $p_1 + p_2 > 0$. (Here $p = (p_1, p_2)$ and $D^p = \dfrac{\partial^{p_1+p_2}}{\partial\xi_1^{p_1}\partial\xi_2^{p_2}}$.) It means that equation (10) is hypoelliptic (cf. Hörmander [2]). Thus assumption 5° of II is satisfied as well.

Now, it follows from II that every continuous solution f of equation (2) is a function of class C^∞ and all the continuous solutions of this equation can be obtained by solving the resulting differential equations.

In view of $\varphi_a' \neq 0$ and $\psi_a' = 0$, it follows from (5) that $f''_{xx}(x, y) \equiv 0$. In view of $\psi_a'' \neq 0$, we conclude from (10) that $f''_{yy}(x, y) \equiv 0$. Therefore

$$f(x, y) = Axy + Bx + Cy + D ,$$

where A, B, C, D are arbitrary constants. Q.E.D.

LEMMA. *If a continuous function $\tilde{f}$ is equal almost everywhere to a solution f of equation (2) it is a solution of this equation, too.*

Proof. Let us fix a point (x^*, y^*) and a value t and define

$$K_\nu = \{(x, y)\colon \sqrt{(x - x^*)^2 + (y - y^*)^2} < 1/\nu\} ,$$

$$A = \{(x, y)\colon \tilde{f}(x, y) = f(x, y)\} ,$$

$$A_1 = \{(x, y)\colon \tilde{f}(x + \varphi(t), y + \psi(t)) = f(x + \varphi(t), y + \psi(t))\} ,$$

$$A_2 = \{(x, y)\colon \tilde{f}(x + \varphi(t), y - \psi(t)) = f(x + \varphi(t), y - \psi(t))\} ,$$

$$A_3 = \{(x, y)\colon \tilde{f}(x - \varphi(t), y + \psi(t)) = f(x - \varphi(t), y + \psi(t))\} ,$$

$$A_4 = \{(x, y)\colon \tilde{f}(x - \varphi(t), y - \psi(t)) = f(x - \varphi(t), y - \psi(t))\} .$$

It is

$$\mu(A \cap K_\nu) = \mu(A_i \cap K_\nu) = \pi/\nu^2 \quad \text{for } i = 1, 2, 3, 4 .$$

Therefore

$$B_\nu = K_\nu \cap A \cap A_1 \cap A_2 \cap A_3 \cap A_4 \neq \emptyset \quad \text{and} \quad \mu(B_\nu) = \pi/\nu^2 .$$

Let us consider a sequence (x_ν, y_ν) such that $(x_\nu, y_\nu) \in B_\nu$. At the points (x_ν, y_ν) equation (2) can be written as

$$\tilde{f}\big(x_\nu + \varphi(t), y_\nu + \psi(t)\big) + \tilde{f}\big(x_\nu + \varphi(t), y_\nu - \psi(t)\big) + \tilde{f}\big(x_\nu - \varphi(t), y_\nu + \psi(t)\big) + \\ + \tilde{f}\big(x_\nu - \varphi(t), y_\nu - \psi(t)\big) = 4\tilde{f}(x_\nu, y_\nu) .$$

Since $(x_\nu, y_\nu) \to (x^*, y^*)$ when $\nu \to \infty$ and since the function $\tilde{f}$ is continuous,

$$\tilde{f}\big(x^* + \varphi(t), y^* + \varphi(t)\big) + \tilde{f}\big(x^* + \varphi(t), y^* - \psi(t)\big) + \tilde{f}\big(x^* - \varphi(t), y^* + \psi(t)\big) + \\ + \tilde{f}\big(x^* - \varphi(t), y^* - \psi(t)\big) = 4\tilde{f}(x^*, y^*)$$

that is the function $\tilde{f}$ satisfies equation (2) at an arbitrary point (x^*, y^*) and for an arbitrary t.

This finishes the proof.

THEOREM IV. *Every locally integrable solution f of equation* (2) *satisfying either the assumptions of Theorem* II *or these of Theorem* III *is equal almost everywhere to a function* (3).

Proof follows immediately from the fact that every locally integrable solution f of equation (2) is (under our assumptions) equal almost everywhere to a function of class C^∞ which, in view of Lemma, is a solution of this equation.

References

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Reçu par la Rédaction le 4. 11. 1968