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The Orlicz type theorem for differential-integral equations with a lagging argument

by MICHAŁ KISIELEWICZ (Zielona Góra)

Abstract. The purpose of this paper is to give the proof of some category theorem concerning the differential-integral equations of the form

$$(1)\qquad \begin{aligned} y(t) &= \varphi(t) \quad \text{for } t < t_0, \\ y'(t) &= \int_0^\infty f(t, y(t-s))\, d_s r(t, s) + g(t) \\ &\qquad \text{for almost every } t \in \langle t_0, T\rangle. \end{aligned}$$

It is shown that non-uniqueness of solutions of (1) is in some sense a rare case.

The aim of this paper is to give a proof of some category-theorem concerning differential-integral equations of the form

$$(1)\qquad \begin{aligned} y(t) &= \varphi(t) \quad \text{for } t \leqslant t_0, \\ y'(t) &= \int_0^\infty f\big(t, y(t-s)\big)\, d_s r(t, s) + g(t) \\ &\qquad \text{for almost every } t \in [t_0, T], \end{aligned}$$

where the integration is of a Riemann–Stieltjes type with respect to $s \geqslant 0$ and φ, r, g, f are given functions. This theorem for ordinary equations has been proved by W. Orlicz in [5] and for partial equations of the hyperbolic type by A. Alexiewicz and W. Orlicz in [2].

By a solution of (1) we mean a function y which is continuous for $t \leqslant t_0$, absolutely continuous for $t_0 \leqslant t \leqslant T$ and satisfies conditions (1).

Let R denote the real line, and let R^n be an n-dimensional linear vector space with the norm $\|x\| = \max(|x_1|, |x_2|, \ldots, |x_n|)$ for $x = (x_1, x_2, \ldots, x_n)$. Let P denote the set in R^{n+1} defined by $P = \{(t, y)\colon t_0 \leqslant t \leqslant T;\ y \in R^n\}$ and let $Q = \{(t, y) \in P;\ \|y - \eta\| \leqslant a\}$, where $\eta \in R^n$ and $a > 0$. Let us denote by G the Banach space of all Lebesgue-integrable functions $g\colon [t_0, T] \to R^n$ with the norm $\|g\|_G = \int_{t_0}^{T} \|g(t)\|\, dt$ and let $\bigvee_{s=0}^{\infty} r(t, s)$ denote the variation of $r(s, t)$ with respect to $s \geqslant 0$.

DEFINITION 1. A function $f\colon P\to R^n$ is said to *satisfy the Carathéodory condition on P* if $f(t, y)$ is measurable in t for each fixed y and continuous in y for each fixed t, and if there is a Lebesgue-integrable function $m(t)$ such that $\|f(t, y)\|\leqslant m(t)$ for $(t, y)\in P$.

DEFINITION 2. A function $f(t, y)$ defined on the set P is said to be *uniformly Lipschitz continuous on P* with respect to y if there exists a constant L satisfying

$$\|f(t, y_2)-f(t, y_1)\|\leqslant L\|y_2-y_1\|$$

for all $(t, y_i)\in P$; $i=1, 2$.

This is proved in [1].

THEOREM 1. *If the function $f\colon Q\to R^n$ satisfies the Carathéodory condition on Q, then there exist continuous functions f_n such that*

(i) $\|f_n(t, y)\|\leqslant m(t)$,

(ii) $\lim\limits_{n\to\infty}\max\limits_{y}\{\|f_n(t, y)-f(t, y)\|;\ (t, y)\in Q\}=0$

for almost every $t\in[t_0, T]$.

As an immediate corollary of Theorem 1 we get the next theorem (see [4]):

THEOREM 2. *Let the function $f\colon Q\to R^n$ satisfy the Carathéodory condition on P. For every $\varepsilon>0$ and for almost every $t\in[t_0, T]$ there exists a function $f^\varepsilon\colon Q\to R^n$ such that*

(a) $\max\limits_{y}\{\|f^\varepsilon(t, y)-f(t, y)\|\colon (t, y)\in Q\}\to 0$ *as* $\varepsilon\to 0$,

(b) $\|f^\varepsilon(t, y)\|\leqslant m(t)$ *for* $(t, y)\in Q$,

(c) $f^\varepsilon(t, y)$ *has continuous partial derivatives of all orders with respect to* $y_1, y_2, \ldots, y_n$.

1. Let us denote by $F(P)$ the set of all functions $f\colon P\to R^n$ satisfying the Carathéodory condition on P and let $\tilde{f}$ be the class of all the functions of $F(P)$ which are different only on a set of measure zero for fixed y. Let us denote by $\mathscr{F}(P)$ the set of all classes $\tilde{f}$. If we define the distance of two elements $\tilde{f}_1, \tilde{f}_2$ of $\mathscr{F}(P)$ as $\varrho_{\mathscr{F}}(\tilde{f}_1, \tilde{f}_2)=\|\tilde{f}_1-\tilde{f}_2\|_{\mathscr{F}}$, where $\|\tilde{f}\|_{\mathscr{F}}=\int_{t_0}^{T}\sup\limits_{y}\{\|f(t, y)\|\colon (t, y)\in P\}\,dt$, then $(\mathscr{F}(P), \varrho_{\mathscr{F}})$ is a metric space. The metric of $(\mathscr{F}(Q), \varrho_{\mathscr{F}})$ has the form $\varrho_{\mathscr{F}}(\tilde{f}_1, \tilde{f}_2)=\int_{t_0}^{T}\sup\limits_{y}\{\|f_1(t, y)-f_2(t, y)\|\colon (t, y)\in Q\}\,dt$.

Let $r\colon [t_0, T]\times[0, \infty)\to R$ be a given function such that the following assumptions are fulfilled:

(I) $r(t, 0)=0$ for $t\in[t_0, T]$,

(II) there exists a number $\vartheta\in(0, \infty)$ such that

$$\bigvee_{s=0}^{\infty} r(t, s)\leqslant\vartheta\qquad\text{for } t\in[t_0, T],$$

(III) for every $\varepsilon > 0$ there exists a number $K > 0$ such that

$$\sum_{s=K}^{\infty} r(t, s) < \varepsilon \quad \text{for } t \in [t_0, T],$$

(IV) for every $a > 0$ and $u \in [t_0, T]$

$$\lim_{t\to u} \int_0^a |r(t, s) - r(u, s)|\, ds = 0.$$

Denote by Φ the space of continuous and bounded functions defined on $(-\infty, 0]$ with the values in R^n. With the metric $\varrho_\Phi(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_\Phi$, where $\|\varphi\|_\Phi = \sup_{t\leqslant t_0} \|\varphi(t)\|$, (Φ, ϱ_Φ) becomes a complete metric space. Finally $\mathscr{H}_P$ will denote the metric space $\Phi \times \mathscr{F}(P) \times G$ with the metric $\varrho = \max(\varrho_\Phi, \varrho_{\mathscr{F}}, \varrho_G)$.

We shall further need the following lemmas:

Lemma 1. *If $(\varphi, f, g) \in \mathscr{H}_P$ and r satisfies assumptions* (I)–(IV), *then equation* (1) *has at least one solution defined on* $(-\infty, T]$.

Lemma 2. *Suppose $(\varphi, f, g) \in \mathscr{H}_P$, r satisfies assumptions* (I)–(IV) *and f is uniformly Lipschitz continuous on P with respect to y. Then there is a unique solution of* (1) *defined on* $(-\infty, T]$.

The proofs of these lemmas are given in [3] and [4], respectively.

Lemma 3. *$(\mathscr{F}(P), \varrho_{\mathscr{F}})$ is a complete metric space.*

Proof. Let $\{\tilde{f}_n\}$ be a sequence of $\mathscr{F}(P)$ such that $\|\tilde{f}_n - \tilde{f}_m\|_{\mathscr{F}} \to 0$ as $n, m \to \infty$ and let $f_n \in \tilde{f}_n, f_m \in \tilde{f}_m$. For every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that

$$\int_{t_0}^{T} \sup_y \{\|f_n(t, y) - f_m(t, y)\|:\ (t, y) \in P\}\, dt \leqslant \varepsilon$$

for $n, m \geqslant N(\varepsilon)$. Suppose $\{n_k\}$ is such that $n_1 < n_2 < \dots$ and $n_k \geqslant N(1/2^{2k})$. Then

$$\int_{t_0}^{T} \sup_y \{\|f_{n_k}(t, y) - f_{n_{k-1}}(t, y)\|:\ (t, y) \in P\}\, dt \leqslant 1/2^{2k} \quad \text{for } k = 1, 2, \dots$$

Taking $A_k = \{t:\ \sup_y [\|f_{n_k}(t, y) - f_{n_{k-1}}(t, y)\|:\ (t, y) \in P] > 1/2^k\}$, we have

$$1/2^{2k} \geqslant \int_{A_k} \sup_y \{\|f_{n_k}(t, y) - f_{n_{k-1}}(t, y)\|:\ (t, y) \in P\}\, dt \geqslant 1/2^k \mu(A_k).$$

Then $\mu(A_k) \leqslant 1/2^k$. Let $A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k$. Since

$$\mu(A) \leqslant \mu\Big(\bigcup_{k=i}^{\infty} A_k\Big) \leqslant \sum_{k=i}^{\infty} \mu(A_k) < \sum_{k=i}^{\infty} 1/2^k = 1/2^{i-1} \quad \text{for } i = 1, 2, \dots,$$

we have $\mu(A) = 0$. Let $A^{\sim} = [t_0, T]\setminus A$ and $A_k^{\sim} = [t_0, T]\setminus A_k$. We have $A^{\sim} = \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} A_k^{\sim}$. Then $t \in A^{\sim}$ implies the existence of a number i such that, for every $k \geqslant i$, $\sup\{\|f_{n_k}(t, y) - f_{n_{k-1}}(t, y)\|: (t, y) \in P\} \leqslant 1/2^k$.
Therefore

$$\sum_{k=i}^{\infty} \sup_y \{\|f_{n_k}(t, y) - f_{n_{k-1}}(t, y)\|: (t, y) \in P\} < \infty \quad \text{for } t \in A^{\sim}.$$

Then the series $f_{n_0}(t, y) + \sum_{k=1}^{\infty} [f_{n_k}(t, y) - f_{n_{k-1}}(t, y)]$ is absolutely and uniformly convergent on $A^{\sim}$ independent of y. Let $f: P \to R^n$ be defined by

$$f(t, y) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(t, y) & \text{for } t \in A^{\sim},\ y \in R^n, \\ 0 & \text{for } t \in A,\ y \in R^n. \end{cases}$$

The function f satisfies the Carathéodory condition on P. We shall show that $\|\tilde{f}_n - \tilde{f}\|_{\mathscr{F}} \to 0$ as $n \to \infty$. For $n, k \geqslant N(\varepsilon)$ we have

$$\int_{t_0}^{T} \sup_y \{\|f_n(t, y) - f_{n_k}(t, y)\|: (t, y) \in P\}\, dt \leqslant \varepsilon.$$

Taking for fixed n

$$\Psi_k(t) = \sup_y \{\|f_n(t, y) - f_{n_k}(t, y)\|: (t, y) \in P\},$$

we have $\|\tilde{f}_n - \tilde{f}\|_{\mathscr{F}} = \int_{t_0}^{T} \lim\limits_{k\to\infty} \Psi_k(t)\, dt$. In virtue of Fatou's Lemma we obtain

$$\int_{t_0}^{T} \underline{\lim}_{k\to\infty} \Psi_k(t)\, dt \leqslant \underline{\lim}_{k\to\infty} \int_{t_0}^{T} \Psi_k(t)\, dt = \underline{\lim}_{k\to\infty} \|\tilde{f}_n - f_{n_k}\|_{\mathscr{F}} \leqslant \varepsilon \quad \text{for } n \geqslant N(\varepsilon).$$

Hence $\|\tilde{f}_n - \tilde{f}\|_F \leqslant \varepsilon$ for $n \geqslant N(\varepsilon)$. This completes the proof.

2. Now we shall prove that non-uniqueness of solutions of (1) is in some sense a rare case. Namely we shall prove the following theorem

THEOREM 3. *Suppose r satisfies assumptions* (I)–(IV). *The set $\mathscr{A}$ of those $(\varphi, f, g) \in \mathscr{H}_P$ for which equations* (1) *has at least two deifferent solutions is of Baire's first category in the space $(\mathscr{H}_P, \varrho)$, where $\varrho = \max(\varrho_\Phi, \varrho_{\mathscr{F}}, \varrho_G)$.*

Proof. Let us denote by $\Delta(t, \varphi, f, g)$ the supremum of the numbers $y_1(t) - y(t)$, where y_1 and y_2 are solutions of (1) corresponding to (φ, f, g). Let $\{t_\tau\}$ denote the sequence of points of $[t_0, T]$ dense in $[t_0, T]$. Then let $\Omega_{MNpq\tau}$ denote the set of those $(\varphi, f, g) \in \mathscr{H}_p$ for which 1° $\|\varphi\|_\Phi \leqslant N$, 2° $\|f\|_{\mathscr{F}} \leqslant M$, 3° $\|g\|_G \leqslant q$, 4° $\Delta(t_\tau, \varphi, f, g) \geqslant 1/p$.

We shall show that $\Omega_{MNpq\tau}$ are closed. Suppose $(\varphi_n, f_n, g_n) \in \Omega_{MNpq\tau}$ to be such that $\varrho[(\varphi_n, f_n, g_n), (\bar{\varphi}, \bar{f}, \bar{g})] \to 0$ as $n \to \infty$. It is easy to see that the functions $\bar{\varphi}, \bar{f}$ and $\bar{g}$ satisfy conditions 1°–3°. Furthermore, there exists a subsequence $\{(\varphi_{n_k}, f_{n_k}, g_{n_k})\}$ of $\{(\varphi_n, f_n, g_n)\}$ such that $\varphi_{n_k} \rightrightarrows \bar{\varphi}$, for $t \leqslant t_0$, $\sup_y \{\|f_{n_k}(t, y) - \bar{f}(t, y)\| \colon (t, y) \in P\} \to 0$ and $\|g_{n_k}(t) - \bar{g}(t)\| \to 0$ as $k \to \infty$ for almost every $t \in [t_0, T]$. By 4° there exist functions $y_{n_k}^{(1)}, y_{n_k}^{(2)}$ satisfying the equations

$$y_{n_k}^{(i)}(t) = \begin{cases} \varphi_{n_k}(t) & \text{for } t \leqslant t_0, \\ \varphi_{n_k}(t_0) + \int\limits_{t_0}^{t} \Big\{ \int\limits_{0}^{\infty} f_{n_k}\big(u, y_{n_k}^{(i)}(u-s)\big)\, ds\, r(u, s) + g_{n_k}(u) \Big\}\, du & \\ \qquad \text{for } t_0 \leqslant t \leqslant T \ (i = 1, 2) & \end{cases}$$

and such that

$$y_{n_k}^{(1)}(t_\tau) - y_{n_k}^{(2)}(t_\tau) \leqslant 1/p - 1/n. \tag{2}$$

From $\sup_y \{\|f_{n_k}(t, y) - \bar{f}(t, y)\|;\ (t, y) \in P\} \to 0$ and $\|g_{n_k}(t) - \bar{g}(t)\| \to 0$; $k \to \infty$ follows the existence of $N(1)$ such that $\sup_y \{\|f_{n_k}(t, y) - \bar{f}(t, y)\| \colon (t, y) \in P\} < 1$ and $\|g_{n_k}(t) - \bar{g}(t)\| < 1$ for almost every $t \in [t_0, T]$ and $k \geqslant N(1)$. Then for almost every $t \in [t_0, T]$ and $k \geqslant N(1)$ we have

$$\|f_{n_k}(t, y)\| \leqslant \sup_y \{\|f_{n_k}(t, y) - \bar{f}(t, y)\| \colon (t, y) \in P\} + \|\bar{f}(t, y)\| < 1 + m(t)$$

and

$$\|g_{n_k}(t)\| \leqslant \|g_{n_k}(t) - \bar{g}(t)\| + \|\bar{g}(t)\| < 1 + \|\bar{g}(t)\|,$$

where $m(t)$ is a Lebesgue-integrable function such that $\|\bar{f}(t, y)\| \leqslant m(t)$ for $(t, y) \in P$. Taking

$$\Gamma(t) = \max\big(1 + m(t), m_{n_1}(t), \ldots, m_{N(1)}(t)\big),$$

$$H(t) = \max\big(1 + g(t), g_{n_1}(t), \ldots, g_{N(1)}(t)\big)$$

we have

$$\|f_{n_k}(t)\| \leqslant \Gamma(t)$$

and

$$\|g_{n_k}(t)\| \leqslant H(t)$$

for every $k = 1, 2, \ldots$ and almost every $t \in [t_0, T]$. Since

$$\|y_{n_k}^{(i)}(t)\| \leqslant \|\varphi_{n_k}(t_0)\| + \vartheta \int\limits_{t_0}^{t} \sup_{s \geqslant 0} \big\| f_{n_k}\big(u, y_{n_k}^{(i)}(u-s)\big) \big\|\, du + $$

$$+ \int\limits_{t_0}^{t} \|g_{n_k}(u)\|\, du \leqslant N + q + \vartheta M \qquad (i = 1, 2)$$

and

$$\|y_{n_k}^{(i)}(t_1)-y_{n_k}^{(i)}(t_2)\| \leqslant \vartheta\Big|\int_{t_1}^{t_2} \Gamma(u)\,du\Big|+\Big|\int_{t_1}^{t_2} H(u)\,du\Big| \quad (i=1,2)$$

for $t, t_1, t_2 \in [t_0, T]$, $i=1,2$ and every $k=1,2,\ldots$, then by Arzela's theorem there are subsequences, say $\{y_k^{(i)}\}$ of $\{y_{n_k}^{(i)}\}$ $(i=1,2)$, such that $y_k^{(i)}(t) \rightrightarrows y^{(i)}(t)$ on $[t_0, T]$. For $t \in [t_0, T]$ and $i=1,2$ we have

$$(3) \quad y^{(i)}(t)-\bar{\varphi}(t_0)-\int_{t_0}^{t}\Big\{\int_0^\infty \bar{f}\big(u, y^{(i)}(u-s)\big)\,d_s r(u,s)+\bar{g}(u)\Big\}du = \sum_{m=1}^{5} \Lambda_m^{(i)}(t),$$

where

$$\Lambda_1^{(i)}(t) = y^{(i)}(t)-y_k^{(i)}(t),$$

$$\Lambda_2^{(i)}(t) = \varphi_k(t_0)-\bar{\varphi}(t_0),$$

$$\Lambda_3^{(i)}(t) = \int_{t_0}^{t}\Big\{\int_0^\infty \big[f_k\big(u, y_k^{(i)}(u-s)\big)-\bar{f}\big(u, y_k^{(i)}(u-s)\big)\big]\,d_s r(u,s)\Big\}du,$$

$$\Lambda_4^{(i)}(t) = \int_{t_0}^{t}\Big\{\int_0^\infty \big[\bar{f}\big(u, y_k^{(i)}(u-s)\big)-\bar{f}\big(u, y^{(i)}(u-s)\big)\big]\,d_s r(u,s)\Big\}du,$$

$$\Lambda_5^{(i)}(t) = \int_{t_0}^{t}[g_k(u)-\bar{g}(u)]\,dt.$$

It is easy to see that for $t \in [t_0, T]$ we have $\|\Lambda_3^{(i)}(t)\| \leqslant \vartheta\|\tilde{f}_k-\tilde{f}\|_{\mathscr{F}}$. Let us write $W_k^{(i)}(t) = \big\|\bar{f}\big(t, y_k^{(i)}(t-s)\big)-\bar{f}\big(t, y^{(i)}(t-s)\big)\big\|$ for fixed $i=1,2$ and $s \geqslant 0$. The functions $W_k^{(i)}$ are measurable on $[t_0, T]$ and such that $|W_k^{(i)}(t)| < 2m(t)$, where $m \in \mathscr{L}(t_0, T)$. Since $\lim\limits_{k\to\infty} W_k^{(i)}(t) = 0$ uniformly with respect to $s \geqslant 0$, then by the Lebesgue theorem we have $\Lambda_4^{(i)}(t) \to 0$ as $k\to\infty$. It is obvious that $\Lambda_1^{(i)}(t)\to 0$, $\Lambda_2^{(i)}(t)\to 0$ and $\Lambda_5^{(i)}(t)\to 0$ as $k\to\infty$. Passing to the limit in (3) we see that $y^{(1)}, y^{(2)}$ satisfy equation (1) and by (2) $y^{(1)}(t_\tau)-y^{(2)}(t_\tau) \geqslant 1/p$, whence $(\bar{\varphi}, \bar{f}, \bar{g}) \in \Omega_{MNpq\tau}$. The sets $\Omega_{MNpq\tau}$ are non-dense. For, suppose that $\Omega_{MNpq\tau}$ is dense in the sphere S_h with centre (φ_0, f_0, g_0) and radius h. Then $S_h \subset \bar{\Omega}_{MNpq\tau} = \Omega_{MNpq\tau}$. Note that for every $(\varphi, f, g) \in \Omega_{MNpq\tau}$ and $\eta \in \boldsymbol{R}^n$ there exists a number $a > 0$ such that equation (1) corresponding to (φ, f, g) is equivalent to (1) with $(\varphi, f|Q, g)$, where $Q = \{(t, y) \in P\colon \|y-\eta\| \leqslant a\}$. In virtue of Theorem 2 for f_0 and every $\delta > 0$ there exists a function $f^\delta\colon Q \to \boldsymbol{R}^n$ such that conditions (a), (b) of this theorem are fulfilled. Then $\max\limits_{y}\{\|f^\delta(t, y)-f_0(t, y)\|;\ (t, y) \in Q\} < \delta$ for almost every $t \in [t_0, T]$. Taking $\delta < h/(T-t_0)$ we have $\|\tilde{f}^\delta-\tilde{f}_0\|_{\mathscr{F}} < h$. Then $(\varphi_0, f^\delta, g_0) \in S_h \subset \Omega_{MNpq\tau}$. Since f^δ is uniformly Lipschitz-continuous with respect to y, then for $(\varphi_0, f^\delta, g_0)$ equation (1) has a unique solution. Therefore $(\varphi_0, f^\delta, g_0) \notin S_h$. The identity

$$\mathscr{A} = \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcup_{r=1}^{\infty} \Omega_{MNpqr}$$

completes the proof.

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