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On uniqueness and successive approximations in the Darboux problem for the equation

$$u_{xy} = f\Big(x, y, u, u_x, u_y, \int_0^x\int_0^y g\big(x, y, s, t, u(s,t), u_s(s,t), u_t(s,t)\big)\,ds\,dt\Big)$$

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This paper contains certain generalizations of the results presented in [5], namely, those concerning the uniqueness and the convergence of successive approximations in the Darboux problem. The conditions quoted in [5] and sufficient for the uniqueness and the convergence of successive approximations referred to the conditions of the Krasnosielski-Krein type, transferred to the field of hyperbolic differential equations from the sphere of ordinary differential equations ([4]).

In note [1] F. Brauer has quoted certain more general conditions ensuring the uniqueness and the convergence of successive approximations in the field of ordinary differential equations and including the conditions of Krasnosielski and Krein.

Below, conditions of the Brauer type are used in reference to vector equations of the form

$$(1)\qquad u_{xy} = f\Big(x, y, u, u_x, u_y, \int_0^x\int_0^y g\big(x, y, s, t, u(s,t), u_s(s,t)\; u_t(s,t)\big)\,ds\,dt\Big)$$

with boundary conditions

$$(2)\qquad u(x, 0) = \sigma(x)+\tau(0), \qquad u(0, y) = \tau(y)+\sigma(0).$$

The problem consisting in finding a solution of equation (1) fulfilling conditions (2) will briefly be called *problem* (D).

The functions $f(x, y, u, p, q, r)$, $g(x, y, s, t, u, p, q)$, $\sigma(x)$, $\tau(y)$, appearing in it, are suitably defined on $E_1 = R\times\mathfrak{X}^4$, $E_2 = R^2\times\mathfrak{X}^3$, $\langle 0, a\rangle$, $\langle 0, b\rangle$, $a, b > 0$, where $R = \langle 0, a\rangle\times\langle 0, b\rangle$, with values in $\mathfrak{X}$. $\mathfrak{X}$ denotes a Banach space.

We say that the vector function $u(x, y)$ with values in $\mathfrak{X}$ *belongs to the class* $C^*(R)$, if it is continuous on R together with its partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial x\partial y}$.

By a *solution of problem* (D) we mean the vector function $u(x, y) \in C^*(R)$ fulfilling equation (1) on R and conditions (2), where σ and τ are of class C^1 on $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant b$, respectively.

1. The formulation of the theorems on uniqueness and successive approximations will be preceded by some auxiliary data.

Brauer's conditions ([1].)

A pair of real functions $\{\psi_1, \psi_2\}$ is, according to the definition, *included in class* (B_a), $a > 0$, if the following conditions are simultaneously fulfilled:

(3^1) the functions $\psi_i(t, r)$ with $i = 1, 2$ are continuous and non-negative for $t \in (0, a\rangle$ and $r \geqslant 0$,

(3^2) there exist functions $A_i(t)$, $i = 1, 2$, defined for $t \in \langle 0, a\rangle$ and such that $A_i(0) = 0$, $A_i(t) > 0$ for $t \in (0, a\rangle$ and $\lim\limits_{t \to 0+} \dfrac{A_1(t)}{A_2(t)} = 0$,

(3^3) if a function $u(t)$, continuous and non-negative on $\langle 0, a\rangle$ and continuously differentiable on $(0, a\rangle$, is such that $u'(t) = \psi_1(t, u(t))$ for $t \in (0, a)$ and $u(0) = 0$, then $u(t) \leqslant A_1(t)$ for $t \in \langle 0, a\rangle$,

(3^4) if a function $v(t)$, continuous and non-negative on $\langle 0, a\rangle$ and continuously differentiable on $(0, a\rangle$, is such that $v'(t) = \psi_2(t, v(t))$ for $t \in (0, a\rangle$ and $\lim\limits_{t \to 0+} \dfrac{v(t)}{A_2(t)} = 0$, then $v(t) \equiv 0$ on $\langle 0, a\rangle$.

In order to apply the pair $\{\psi_1, \psi_2\} \in (B_a)$ to the investigation of problem (D) we discover some further properties of functions ψ_i:

(3^5) functions $r \to \psi_i(t, r)$, $i = 1, 2$, are non-decreasing on $\langle 0, +\infty)$ for each fixed $t \in (0, a\rangle$,

(3^6) functions $t \to \psi_i(t, rt)$, $i = 1, 2$, are non-decreasing on $(0, a\rangle$ for each fixed $r \geqslant 0$ (cf. [7], Satz 4, condition ϑ).

By (B_a^*) we understand the class of all pairs $\{\psi_1, \psi_2\}$ of functions fulfilling simultaneously the conditions (3^1)-(3^6). The following facts easily may be checked.

LEMMA 1. *If* $\{\psi_1, \psi_2\} \in (B_a^*)$, *then the functions* $t \to \psi_i\left(t, \int_0^t w(s)\,ds\right)$ *for* $i = 1, 2$ *are non-decreasing with any function* $w(s) \geqslant 0$ *non-decreasing on* $\langle 0, a\rangle$.

Indeed, from the fact that the function $z(t) = t^{-1} \int_0^t w(s)\,ds$ is non-decreasing it follows that the condition $\{\psi_1, \psi_2\} \in (B_a^*)$ implies the function $\psi_i(t, tz(t))$ for $i = 1, 2$ to be monotone.

LEMMA 2. *If the function $v(t)$ is non-negative, absolutely continuous on $\langle 0, a\rangle$ and such that $v(0) = 0$ and almost everywhere on $(0, a\rangle$ we have the inequality*

$$v'(t) \leqslant \psi_i(t, v(t)) \quad \textit{for} \quad i = 1, 2, \tag{4}$$

where $\{\psi_1, \psi_2\} \in (B_a)$ and the condition (3^5) *is fulfilled, then $v(t) \equiv 0$ on the interval $\langle 0, a\rangle$.*

The proof is analogous to the proof of Theorem 1 of [1].

No assumptions concerning the right side of equation (1) have been considered as yet. We shall now discuss the case where $g(x, y, s, t, u, p, q)$ fulfils the Carathéodory conditions.

LEMMA 3. *Let the vector function $g(x, y, s, t, z_1, \ldots, z_m)$ defined for (x, y), $(s, t) \in R$, $z_i \in \mathfrak{X}$, $i = 1, \ldots, m$, with values from $\mathfrak{X}$ fulfil the following conditions:*

1° *the function $(s, t) \to g(x, y, s, t, z_1, \ldots, z_m)$ is measurable in Bochner's sense for each fixed system $(x, y, z_1, \ldots, z_m)$,*

2° *with $l = 1, \ldots, m$ the function $(x, y, z_l) \to g(x, y, s, t, z_1, \ldots, z_m)$ is continuous for each fixed system $(s, t, z_1, \ldots, z_{l-1}, z_{l+1}, \ldots, z_m)$,*

3° *there exists a function φ, non-negative and summable on R, and the inequality*

$$\|g(x, y, s, t, z_1, \ldots, z_m)\| \leqslant \varphi(s, t)$$

holds for (x, y), $(s, t) \in R$, $z_1, \ldots, z_m \in \mathfrak{X}$.

Then for each $(x, y) \in R$

$$(s, t) \to g(x, y, s, t, z_1(s, t), \ldots, z_m(s, t))$$

is on R a measurable and integrable function in Bochner's sense (see, for instance, [2], Chap. III) *and*

$$(x, y) \to \int_0^x \int_0^y g(x, y, s, t, z_1(s, t), \ldots, z_m(s, t))\, ds\, dt$$

constitutes a continuous function on R, while $\{z_1(s, t), \ldots, z_m(s, t)\}$ is an arbitrary system of functions continuous on R.

We may easily leave out the simple proof of this lemma.

Assuming at present that the function $f(x, y, u, p, q, r)$ is continuous on E_1 while $g(x, y, s, t, u, p, q)$ fulfils the assumptions of Lemma 3, we may say that problem (D) for equation (1) is equivalent to solving the integral equation

$$(5)\quad s(x,y)=f\Big(x,y,\sigma(x)+\tau(y)+\int_0^y\int_0^x s(u,v)\,du\,dv,$$
$$\sigma'(x)+\int_0^y s(x,v)\,dv,\ \tau'(y)+\int_0^x s(u,y)\,du,\ \int_0^x\int_0^y g\Big(x,y,u,v,\sigma(u)+\tau(v)+$$
$$+\int_0^v\int_0^u s(r,t)\,dr\,dt,\ \sigma'(u)+\int_0^v s(u,t)\,dt,\ \tau'(v)+\int_0^u s(r,v)\,dr\Big)du\,dv\Big)$$

in the range of the functions $s\in C(R)$.

Among the assumptions that we make for the functions f and g the following conditions play the most important role (cf. [8], Beispiel 4, p. 203):

$$(6_f)\quad \|f(x,y,u,p,q,r)-f(x,y,\bar u,\bar p,\bar q,\bar r)\|$$
$$\leqslant \lambda_i^{(1)}(x,y)\psi_i(xy,\|u-\bar u\|)+\mu_i^{(1)}(x,y)\psi_i(xy,x\|p-\bar p\|)+$$
$$+\nu_i^{(1)}(x,y)\psi_i(xy,y\|q-\bar q\|)+\frac{\eta_i(x,y)}{xy}\|r-\bar r\|,\qquad i=1,2,$$

$$(6_g)\quad \|g(x,y,s,t,u,p,q)-g(x,y,s,t,\bar u,\bar p,\bar q)\|$$
$$\leqslant \lambda_i^{(2)}(x,y)\psi_i(xy,\|u-\bar u\|)+\mu_i^{(2)}(x,y)\psi_i(xy,x\|p-\bar p\|)+$$
$$+\nu_i^{(2)}(x,y)\psi_i(xy,y\|q-\bar q\|),\qquad i=1,2,$$

where $(x,y),(s,t)\in R$, $xy>0$ and for $i,j=1,2$, $\lambda_i^{(j)}$, $\mu_i^{(j)}$, $\nu_i^{(j)}$, η_i are arbitrary non-negative functions on R for which

$$\lambda_i^{(1)}(x,y)+\mu_i^{(1)}(x,y)+\nu_i^{(1)}(x,y)+\eta_i(x,y)\equiv 1\quad\text{on}\quad R$$

and

$$\lambda_i^{(2)}(x,y)+\mu_i^{(2)}(x,y)+\nu_i^{(2)}(x,y)\equiv 1\quad\text{on}\quad R\quad\text{for}\quad i=1,2.$$

2. We now proceed to presenting the theorem on uniqueness.

THEOREM 1. *If the vector function $f(x,y,u,p,q,r)$, defined and continuous on E with values from the space $\mathfrak{X}$, fulfils inequality* (6_f) *while the function g fulfils the assumptions of Lemma* 3 *and inequality* (6_g) *and $\{\psi_1,\psi_2\}\in(B^*_{ab})$, then problem* (D) *has at most one solution in the class $C^*(R)$.*

Proof. If $s_1(x, y)$ and $s_2(x, y)$ were two solutions of equation (5), then putting $z(x, y) = \|s_2(x, y) - s_1(x, y)\|$ for $(x, y) \in R$ we should obtain on the basis of (6_f) and (6_g) the inequality

$$(7)\qquad z(x, y) \leqslant \lambda_i^{(1)}(x, y)\psi_i\Big(xy, \int_0^y\int_0^x z(u, v)\,du\,dv\Big) + \mu_i^{(1)}(x, y)\psi_i\Big(xy, x\int_0^y z(x, v)\,dv\Big) + \nu_i^{(1)}(x, y)\psi_i\Big(xy, y\int_0^x z(u, y)\,du\Big) + \frac{\eta_i(x, y)}{xy}\int_0^x\int_0^y \Big[\lambda_i^{(2)}(x, y)\psi_i\Big(xy, \int_0^v\int_0^u z(t, r)\,dt\,dr\Big) + \ldots + \nu_i^{(2)}(x, y)\psi_i\Big(xy, y\int_0^u z(t, v)\,dt\Big)\Big]du\,dv, \qquad i = 1, 2.$$

Now putting $A_\sigma = \{(x, y)\colon 0 \leqslant xy \leqslant \sigma\} \cap R$ for $\sigma \geqslant 0$ we define on $\langle 0, ab\rangle$ a continuous and non-decreasing function $\omega(\sigma) \geqslant 0$, $\omega(0) = 0$, in the following manner:

$$(8)\qquad \omega(\sigma) = \sup_{(x,y)\in A_\sigma} z(x, y)$$

for which the inequality $z(x, y) \leqslant \omega(xy)$ occurs on R and this gives the following estimation on the basis of (7):

$$(9)\qquad z(x, y) \leqslant \lambda_i^{(1)}(x, y)\psi_i\Big(xy, \int_0^y\int_0^x \omega(uv)\,du\,dv\Big) + \ldots + \frac{\eta_i(x, y)}{xy}\int_0^x\int_0^y \Big[\lambda_i^{(2)}(x, y)\psi_i\Big(xy, \int_0^v\int_0^u \omega(tr)\,dt\,dr\Big) + \ldots + \nu_i^{(2)}(x, y)\psi_i\Big(xy, y\int_0^u \omega(tv)\,dt\Big)\Big]du\,dv \leqslant \psi_i\Big(xy, \int_0^{xy} \omega(\sigma)\,d\sigma\Big), \qquad i = 1, 2.$$

On the basis of Lemma 1 and (8) we obtain the following inequality for any point $(x, y) \in A_t$:

$$z(x, y) \leqslant \psi_i\Big(t, \int_0^t \omega(\sigma)\,d\sigma\Big), \qquad i = 1, 2;$$

hence it follows that

$$(10) \qquad \omega(t) \leqslant \psi_i\Big(t, \int_0^t \omega(\sigma)d\sigma\Big), \qquad i = 1, 2,$$

for $t \in (0, ab\rangle$.

Putting $v(t) = \int_0^t \omega(\sigma)d\sigma$ in (10) we see that the function $v(t)$ fulfils the assumptions of Lemma 2 on $\langle 0, ab\rangle$, hence it follows that $v(t) \equiv 0$ on the interval $\langle 0, ab\rangle$, which on the basis of continuity $\omega(\sigma)$ gives $\omega(\sigma) = 0$ for $\sigma \in \langle 0, ab\rangle$ and because of (8) also $z(x, y) = 0$ for $(x, y) \in R$. This proves the "uniqueness".

Note 1. Assuming $\bar{\psi}_1(t, r) = Cr^a$, $\bar{\psi}_2(t, r) = kt/r$, where C, k and a are such positive constants that $0 < a < 1$ and $k(1-a) < 1$, we can easily state that $\{\bar{\psi}_1, \bar{\psi}_2\} \in (B_a^*)$. For this purpose it is sufficient to assume that

$$A_1(t) = [C(1-a)t]^{1/(1-a)} \quad \text{and} \quad A_2(t) = t^k.$$

From Theorem 1 it follows that the conditions

$$\|f(x, y, \bar{u}, \bar{w}) - f(x, y, u, w)\| \leqslant \lambda_i(x, y)\bar{\psi}_i(xy, \|u-\bar{u}\|) + \frac{\mu_i(x, y)}{xy}\|w - \bar{w}\|$$

and

$$\|g(x, y, s, t, \bar{u}) - g(x, y, s, t, u)\| \leqslant \bar{\psi}_i(xy, \|u-\bar{u}\|), \qquad i = 1, 2,$$

where $(x, y), (s, t) \in R$, $xy > 0$, and the functions λ_i, μ_i are non-negative and integrable on R, while

$$\lambda_i(x, y) + \mu_i(x, y) \equiv 1 \quad \text{on} \quad R \quad \text{for} \quad i = 1, 2,$$

secure the uniqueness of problem (D) for the equation

$$(1') \qquad \frac{\partial^2 u}{\partial x \partial y} = f\Big(x, y, u, \int_0^x \int_0^y g\big(x, y, s, t, u(s, t)\big)\, ds\, dt\Big),$$

if the inequalities $0 < a < 1$ and $k(1-a) < 1$ are fulfilled.

On the other hand, we shall show that it is sufficient to assume $0 < a < 1$ and $k(1-a)^2 < 1$. Indeed, if $z(x, y) = \|u_2(x, y) - u_1(x, y)\|$, where u_1 and u_2 constitute the solutions of the equation

$$(1'') \qquad u(x, y) = \sigma(x) + \tau(y) + \int_0^y \int_0^x f\Big[s, t, u(s, t), \int_0^s \int_0^t g\big(s, t, v, w, u(v, w)\big)\, dv\, dw\Big]\, ds\, dt,$$

we have with $z^*(x, y) = \max\limits_{\substack{0 \leqslant v \leqslant x \\ 0 \leqslant w \leqslant y}} z(v, w)$ the estimation

$$z^*(x, y) \leqslant \int\limits_0^x \int\limits_0^y \overline{\varphi}_i[st, z^*(s, t)]\,ds\,dt, \qquad i = 1, 2,$$

where on the basis of Lemma 1 of [5] we obtain $z^*(x, y) \equiv 0$ on R, and thus $u_1(x, y) \equiv u_2(x, y)$ on R. Theorem 1 does not thus include the optimal result. The above situation shows that class (B_a^*) is too narrow if we want to apply it to equation (1) and this suggests that it would require a consideration of comparative functions of five variables of the type that has been dealt with by Shanahan ([6]) and Walter ([8], Satz 7, p. 201).

3. We shall proceed now to the presentation of the theorem on successive approximations including the argumentation concerning their convergence, following the general conception of T. Ważewski ([9]). The successive approximations can be defined recurrently by

$$(11) \qquad s_0(x, y) = \varphi(x, y), \qquad s_{n+1}(x, y) = Fs_n(x, y), \qquad n = 0, 1, 2, \ldots,$$

where the operation F is defined for any continuous function $s(x, y)$ by the right side of equation (5) while $\varphi \in C(R)$.

THEOREM 2. *If the vector functions $f(x, y, u, p, q, r)$ and $g(x, y, s, t, u, p, q)$ fulfil the assumptions of Theorem* 1 *and, moreover,* $M = \sup\limits_{E_1}\|f(x, y, u, p, q, r)\| < \infty$, *then the successive approximations* (11) *are uniformly convergent on R to the unique solution of problem* (D) *for equation* (1).

Proof (cf. [3]). Let us first emphasize that to prove the correctness of the statement it is sufficient to show, the space $\mathfrak{X}$ being complete, that the sequence (11)—$\{s_n(x, y)\}$—fulfils Cauchy's condition of uniform convergence on R. Let $\overline{M} = \max\limits_R\|\varphi(x, y)\|$ and $N = \max[M, \overline{M}]$. Then

$$\max_R\|s_n(x, y)\| \leqslant N \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Further we assume

$$(12) \qquad \delta_n(x, y) = \sup_{1 \leqslant m < \infty} \|s_{n+m}(x, y) - s_n(x, y)\|, \qquad n = 0, 1, 2, \ldots,$$

$$(13) \qquad \omega_n(\sigma) = \sup_{(x,y) \in A_\sigma} \delta_n(x, y) \quad \text{for} \quad 0 \leqslant \sigma \leqslant ab$$

and

$$(14) \qquad \omega(\sigma) = \limsup_{n \to \infty} \omega_n(\sigma) \quad \text{for} \quad \sigma \in \langle 0, ab\rangle.$$

It follows from the above that $\{\delta_n(x, y)\}$ is a bounded sequence of measurable functions, $\{\omega_n(\sigma)\}$ is also a bounded sequence of measurable functions and $\omega(\sigma)$ is a non-decreasing and bounded function, thus being integrable.

On the basis of (6_f) and (6_g), (11) and (12)-(14) we obtain

$$\|s_{n+m+1}(x,y)-s_{n+1}(x,y)\| = \|Fs_{n+m}(x,y)-Fs_n(x,y)\|$$

$$\leqslant \lambda_i^{(1)}(x,y)\psi_i\Big(xy, \int_0^y\int_0^x \|s_{n+m}(u,v)-s_n(u,v)\|du\,dv\Big)+$$

$$+\mu_i^{(1)}(x,y)\psi_i\Big(xy, x\int_0^y \|s_{n+m}(x,v)-s_n(x,v)\|dv\Big)+$$

$$+\nu_i^{(1)}(x,y)\psi_i\Big(xy, y\int_0^x \|s_{n+m}(u,y)-s_n(u,y)\|du\Big)+$$

$$+\frac{\eta_i(x,y)}{xy}\int_0^x\int_0^y \Big[\lambda_i^{(2)}(x,y)\psi_i\Big(xy, \int_0^v\int_0^u \|s_{n+m}(t,r)-s_n(t,r)\|dt\,dr\Big)+\ldots+$$

$$+\nu_i^{(2)}(x,y)\psi_i\Big(xy, y\int_0^u \|s_{n+m}(t,v)-s_n(t,v)\|dt\Big)\Big]du\,dv, \qquad i=1,2,$$

whence in accordance with (12) and the inequality $\delta_n(x,y)\leqslant\omega_n(xy)$, we further obtain

$$\delta_{n+1}(x,y)\leqslant \lambda_i^{(1)}(x,y)\psi_i\Big(xy, \int_0^y\int_0^x \delta_n(u,v)\,du\,dv\Big)+\ldots+ \tag{15}$$

$$+\frac{\eta_i(x,y)}{xy}\int_0^x\int_0^y \Big[\lambda_i^{(2)}(x,y)\psi_i\Big(xy, \int_0^v\int_0^u \delta_n(t,r)\,dt\,dr\Big)+\ldots+$$

$$+\nu_i^{(2)}(x,y)\psi_i\Big(xy, y\int_0^u \delta_n(t,v)\,dt\Big)\Big]du\,dv$$

$$\leqslant \lambda_i^{(1)}(x,y)\psi_i\Big(xy, \int_0^y\int_0^x \omega_n(uv)\,du\,dv\Big)+\ldots+$$

$$+\frac{\eta_i(x,y)}{xy}\int_0^x\int_0^y \Big[\lambda_i^{(2)}(x,y)\psi_i\Big(xy, \int_0^v\int_0^u \omega_n(tr)\,dt\,dr\Big)+\ldots+$$

$$+\nu_i^{(2)}(x,y)\psi_i\Big(xy, y\int_0^u \omega_n(tv)\,dt\Big)\Big]du\,dv$$

$$\leqslant \psi_i\Big(xy, \int_0^{xy}\omega_n(\sigma)\,d\sigma\Big), \qquad i=1,2.$$

From (15), just as in the proof of Theorem 1, it follows that

$$\omega_{n+1}(t) \leqslant \psi_i\Big(t, \int\limits_0^t \omega_n(\sigma)\,d\sigma\Big) \tag{16}$$

with $n = 0, 1, 2, \ldots,$ $i = 1, 2,$ and $t \in (0, ab\rangle$.

Relation (16) in accordance with (14) and the continuity and monotonicity of the function $\psi_i(t, r)$ with respect to r, $i = 1, 2$, and Fatou's lemma results in

$$\omega(t) \leqslant \psi_i\Big(t, \int\limits_0^t \omega(\sigma)\,d\sigma\Big), \qquad i = 1, 2, \tag{17}$$

for $t \in (0, ab\rangle$.

Assuming now $v(t) = \int\limits_0^t \omega(\sigma)\,d\sigma$ in (17) we obtain inequality (4) almost everywhere on $\langle 0, ab\rangle$, which in accordance with Lemma 2 and the monotonicity of the function $\omega(\sigma)$ results in $\omega(\sigma) \equiv 0$ on $\langle 0, ab\rangle$.

It thus follows from (13) and (14) that the sequence $\{\delta_n(x, y)\}$ is uniformly convergent to zero on R, which in relation to (12) proves that the sequence $\{s_n(x, y)\}$ is uniformly convergent to a vector function $\tilde{s}(x, y)$ which is continuous on R. In this situation it is sufficient to pass from n in (11) to infinity, to see that $\tilde{s}$ is the only fixed point of the operation F and thus the solution of problem (D) for equation (1).

Similarly, we may prove the uniqueness and the convergence of successive approximations in problem (D) for equation (1) if its right side fulfils conditions:

$$\begin{aligned} (18_f) \qquad & \|f(x, y, u, p, q, r) - f(x, y, \bar{u}, \bar{p}, \bar{q}, \bar{r})\| \\ & \leqslant \lambda_i(x, y)\psi_i(x+y, \|u-\bar{u}\| + \|p-\bar{p}\| + \|q-\bar{q}\|) + \frac{\mu_i(x, y)}{xy}\|r-\bar{r}\|, \qquad i = 1, 2, \end{aligned}$$

and also

$$\begin{aligned} (18_g) \qquad & \|g(x, y, s, t, u, p, q) - g(x, y, s, t, \bar{u}, \bar{p}, \bar{q})\| \\ & \leqslant \psi_i(x+y, \|u-\bar{u}\| + \|p-\bar{p}\| + \|q-\bar{q}\|), \qquad i = 1, 2, \end{aligned}$$

with $(x, y), (s, t) \in R$, $x + y > 0$, $\{(a+2)\psi_1, (a+2)\psi_2\} \in (B^*_{a+b})$, and

$$\lambda_i(x, y),\ \mu_i(x, y) \geqslant 0 \quad \text{on} \quad R,$$

$$\lambda_i(x, y) + \mu_i(x, y) \equiv 1 \quad \text{on} \quad R \quad \text{for} \quad i = 1, 2.$$

Note 2. Obviously the theorems obtained above include a series of particular cases which may be obtained by a proper modelling of the space $\mathfrak{X}$ as well as a few comparative functions $\{\psi_1, \psi_2\}$. If for instance $\mathfrak{X} = l^p_\nu$, $p \geqslant 1$, constitutes a space of ν-vectors $u = \{u_1, \ldots, u_\nu\}$ of scalars $u_1, \ldots, u_\nu$ and has the norm $\|u\| = \big(\sum\limits_{k=1}^{\nu} |u_k|^p\big)^{1/p}$, then we can easily formulate analogies of Theorems 1 and 2 for the finite systems of equations of the form

$$\frac{\partial^2 u_k}{\partial x \partial y} = f_k\Big(x, y, u_1, \ldots, u_\nu, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_\nu}{\partial x}, \frac{\partial u_1}{\partial y}, \ldots$$

$$\ldots, \frac{\partial u_\nu}{\partial y}, \int_0^x \int_0^y g_1\Big(x, y, s, t, u_1(s,t), \ldots, u_\nu(s,t),$$

$$\frac{\partial u_1(s,t)}{\partial s}, \ldots, \frac{\partial u_\nu(s,t)}{\partial s}, \frac{\partial u_1(s,t)}{\partial t}, \ldots, \frac{\partial u_\nu(s,t)}{\partial t}\Big) ds\, dt, \ldots$$

$$\ldots, \int_0^x \int_0^y g_\nu\Big(x, y, s, t, u_1(s,t), \ldots, u_\nu(s,t), \frac{\partial u_1(s,t)}{\partial s}, \ldots$$

$$\ldots, \frac{\partial u_\nu(s,t)}{\partial t}, \frac{\partial u_1(s,t)}{dt}, \ldots, \frac{\partial u_\nu(s,t)}{\partial t}\Big) ds\, dt\Big), \qquad k = 1, 2, \ldots, \nu,$$

or if $\mathfrak{X} = C_\nu(A)$ is a space of systems $\{u_1(\sigma), \ldots, u_\nu(\sigma)\} = u(\sigma)$ of functions continuous on a self-compact subset A of an n-dimensional Euclidean space $\mathcal{E}^n$, with the norm $\|u(\cdot)\| = \max\limits_{\sigma \in A} \max\limits_{1 \leqslant k \leqslant \nu} |u_k(\sigma)|$, then we can similarly obtain theorems on uniqueness and successive approximations for the system of integro-differential equations of the form

$$\frac{\partial^2 u_s(x, y, \sigma)}{\partial x \partial y} = \int_A K_s\Big(x, y, \sigma, \xi, u_1(x, y, \xi), \ldots, u_\nu(x, y, \xi), \frac{\partial u_1(x, y, \xi)}{\partial x}, \ldots$$

$$\ldots, \frac{\partial u_\nu(x, y, \xi)}{\partial x}, \frac{\partial u_1(x, y, \xi)}{\partial y}, \ldots, \frac{\partial u_\nu(x, y, \xi)}{\partial y}\Big) d\xi, \qquad s = 1, 2, \ldots, \nu.$$

Finally, applying here the method contained in paper [3] we can, on the basis of Theorems 1 and 2, formulate a statement on the continuous dependence of the solutions of equation (1) on the functions f, g, σ and τ.

References

[1] F. Brauer, *Some results on uniqueness and successive approximations*, Canadian Journ. Math. 11 (1959), pp. 527-533.

[2] E. Hill, *Functional Analysis and Semigroups*, New York 1949.

[3] J. Kisyński, *Application de la méthode des approximations successives dans la théorie de l'équation* $\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$, Ann. Univ. M. Curie-Skłodowska Sectio A, 14 (1960), pp. 65-86.

[4] М. А. Красносельский и С. Г. Крейн, *Об одном классе теорем единственности, для уравнения* $y' = f(x, y)$, Усп. Мат. Наук, т. XI, вып. 1 (67), (1956), pp. 209-213.

[5] B. Palczewski, *On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem under the conditions of the Krasnosielski and Krein type*, Ann. Polon. Math. 14 (1964), pp. 183-190.

[6] J. P. Shanahan, *On uniqueness questions for hyperbolic differential equations*, Pacific Math. Journ. 10 (1960), pp. 677-688.

[7] W. Walter, *Über die Differentialgleichung* $u_{xy} = f(x, y, u, u_x, u_y)$, *I*, Math. Zeitschr. 71 (3) (1959), pp. 308-324.

[8] — *Eindeutigkeitssätze für gewöhnliche, parabolische und hyperbolische Differentialgleichungen*, Math. Zeitschr. 74 (3) (1960), pp. 191-203.

[9] T. Ważewski, *Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison*, Bull. Acad. Polon. Sci. des sci. math., astr. et phys., 8 (1) (1960), pp. 47-52.

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