ANNALES
POLONICI MATHEMATICI
XXVI (1972)

Attractive operators and fixed points

by BEATRIZ MARGOLIS (La Plata, Argentina)

Let (X, d) be a metric space, and T a self-mapping of X. If $x \in X$, let $L(x) = \overline{(T^n x)}$, i.e. the set of limit points of the iterates of x by T. In [2], the authors obtained results concerning $L(x)$ for T continuous, assuming essentially: a) the existence and compactness of $F(T)$ = the set of fixed points of T; b) $d(Tx, F(T)) < d(x, F(T))$ whenever $x \notin F(T)$.

In the present article we want to study $F(T)$ for a given T, without prescribing its existence a priori; we use a property similar to b), but for arbitrary sets. Thus we are able to obtain "bounds" for $F(T)$ and in case $F(T)$ is not empty, we obtain the convergence of suitable sequences of iterates to a fixed point. Several examples will show the necessity of the conditions involved, and will also give an idea of the extent of the results obtained.

The second part of the paper deals with the relationship among the so-called attractive operators and other known classes of operators.

For other results related to a condition of type b), see [3], [5].

1. DEFINITION 1.1. Let (X, d) be a metric space, B a non-empty proper subset of X, and T a self-mapping of X. T is said to be *attractive with respect to B* (in short, *B-attractive*), if B is T-invariant (i.e. $T(B) \subset B$), and $d(Tx, B) < d(x, B)$ whenever x is not in B.

Our aim is to study B-attractive operators; essential for us will be the family of sets with respect to which T is attractive.

Notation. $\mathscr{A} = \{B \subsetneq X / B \neq \emptyset,\ T \text{ is } B\text{-attractive}\}$, $\mathscr{D} = \bigcap_{B \in \mathscr{A}} B$, $F(T) = \{x \in X / Tx = x\}$, $\mathscr{A}' = \{B \in \mathscr{A} / B \text{ is closed}\}$, $\mathscr{D}' = \bigcap_{B \in \mathscr{A}'} B$.

LEMMA 1.1. $B \in \mathscr{A} \Rightarrow F(T) \subset B$.

Proof. We may assume that $F(T) \neq \emptyset$, for otherwise the result is obvious. Let $y \in F(T) \setminus B$. Then: $d(y, B) = d(Ty, B) < d(y, B)$, which is clearly impossible. Hence, $F(T) \subset B$ for all $B \in \mathscr{A}$.

COROLLARY 1.1. $F(T) \subset \mathscr{D}$.

COROLLARY 1.2. *If two sets in $\mathscr{A}$ are disjoint, T does not have any fixed point.*

The converse is not true, as we now show.

EXAMPLE 1.1. Let $X = \{0, 1, 2, 3\}$, and define: $T(0) = 1$, $T(1) = 2$, $T(2) = 1$, $T(3) = 2$.

Clearly, $B = \{1, 2\} \in \mathscr{A}$, for $T(x) \in B$ for each x. Furthermore, $\mathscr{D} = B$; but $F(T) = \emptyset$.

From now on, unless otherwise stated, $\mathscr{A}$ will always mean $\mathscr{A}'$. Same indication for $\mathscr{D}$.

Remark 1.1. T continuous, $B \in \mathscr{A} \Rightarrow \bar{B} \in \mathscr{A}'$.

THEOREM 1.1. *Any two compact elements in $\mathscr{A}$ have a non-empty intersection.*

Proof. Let $B_1, B_2 \in \mathscr{A}$, both compact, and assume $B_1 \cap B_2 = \emptyset$. Then, $d = d(B_1, B_2) > 0$, and we may find $x_1 \in B_1$, $x_2 \in B_2$ such that $d(x_1, x_2) = d$. Since $x_1 \notin B_2$ and $B_2 \in \mathscr{A}$, $d(Tx_1, B_2) < d(x_1, B_2) \leqslant d(x_1, x_2) = d(B_1, B_2)$. But $Tx_1 \in B_1$, so that this contradiction implies $B_1 \cap B_2 \neq \emptyset$.

THEOREM 1.2. *If X is compact and T is continuous, $\mathscr{D} \neq \emptyset$.* (In view of Remark 1.1, since $F(T)$ is closed when T is continuous, we see that working with closed subsets B is by no means a restriction.)

Proof. We preceed by induction on $k(\mathscr{A})$ = cardinality of $\mathscr{A}$.

If $k(\mathscr{A}) = 1$, the result is obvious. If $k(\mathscr{A}) = 2$, this is Theorem 1.1. Assume the result to be true when $k(\mathscr{A}) = n$. Let $k(\mathscr{A}) = n+1$; hence $\mathscr{A} = \{B_1, B_2, \ldots, B_{n+1}\}$, and assume that the Theorem does not hold. Then, $\bigcap_{i=1}^{n+1} B_i = \emptyset$; owing to the inductive hypothesis, for each h, $1 \leqslant h \leqslant n+1$, there exists an $x_h \notin B_h$, $x_h \in C_h = \bigcap_{\substack{i=1 \\ i \neq h}}^{n+1} B_i$. Fix one such h, say h_0, and write for simplicity, $x = x_{h_0}$, $C = C_{h_0}$. Let $y_k = T^k x$. Since elements in $\mathscr{A}$ are T-invariant, we have $y_k \in C$ $(k = 1, 2, \ldots)$. C is compact, hence sequentially compact, and therefore (y_k) has a convergent subsequence (which we call again (y_k)), i.e. $y_k \to y \in C$.

Since T is continuous, $Ty_k \to Ty$. Also, $C \cap B_{h_0} = \emptyset$ implies $y_k \notin B_{h_0}$ $(k = 1, 2, 3, \ldots)$. Let $a_k = d(y_k, B_{h_0})$. Since $y_k \notin B_{h_0}$ and $B_{h_0} \in \mathscr{A}$, (a_k) is a decreasing sequence of positive numbers. Hence, $a_k \to a \geqslant 0$. If $a = 0$, $d(y, B_{h_0}) = 0$, and $y \in B_{h_0}$, which is clearly impossible, for $y \in C$. Hence, $a > 0$. Therefore, $d(Ty, B_{h_0}) < d(y, B_{h_0}) = a$. But we also have: $d(Ty, B_{h_0}) = \lim d(Ty_k, B_{h_0}) = \lim a_{k+1} = a = d(y, B_{h_0})$. This contradiction shows that if $k(\mathscr{A}) = n+1$, then $\mathscr{D} \neq \emptyset$.

Therefore, we have shown that any finite collection of sets in $\mathscr{A}$ has a non-empty intersection. Therefore, since X is compact, all sets in $\mathscr{A}$ have a common element, i.e. $\mathscr{D} \neq \emptyset$.

Remark 1.2. If X were a Banach space, the conclusion of Theorem 1.2 would be valid merely by requiring the existence of a compact convex element in $\mathscr{A}$, for in this case Schauder's Theorem would imply $F(T) \neq \emptyset$, and hence $\mathscr{D} \neq \emptyset$.

We saw in Example 1.1 a case where $\emptyset = F(T) \subsetneqq \mathscr{D}$. We will now give an example where $F(T) \neq \emptyset$, being still a proper subset of $\mathscr{D}$. This example will also show that the connectedness of X plays no significant role in this situation.

EXAMPLE 1.2. Let X be the plane with Euclidean distance, and for $x = re^{it}$, define

$$Tx = \begin{cases} re^{i(t+a)}, & 0 \leqslant r \leqslant 1, \\ e^{i(t+a)}, & r > 1, \end{cases}$$

where $a > 0$ is chosen small enough, and such that $ka \not\equiv 0\ (2\pi)$, $k = 1, 2, ..$ Clearly, T is continuous. Also, $F(T) = \{0\}$. Let B be the closed unit disc All points outside B are mapped into B, so that $B \in \mathscr{A}$, for T restricted to B is just a rotation. Moreover, we cannot have $B_1 \in \mathscr{A}$, $B_1 \subsetneqq B$. For, assuming this to be true, let $x = re^{it} \in B_1$. Then $T^n x = re^{i(t+na)} \in B_1$, $n = 1, 2, \ldots$; i.e., $(T^n x)$ is dense in the boundary of the circle of radius r. Since B_1 is closed, it contains this boundary. Therefore, $d = d(B_1, B) > 0$, and B_1 is precisely the disc of radius r. Hence, if $x \notin B_1$ and $x \in B$, $d(Tx, B_1) = d(x, B_1)$, i.e. $B_1 \notin \mathscr{A}$. Therefore, B is the smaller set in $\mathscr{A}$. Hence, $\mathscr{D} = B \supsetneqq F(T)$. And, moreover, $F(T) = \{0\}$ is in the interior of $\mathscr{D}$.

DEFINITION 1.2. T will be called *attractive* if it is $F(T)$-attractive, i.e. $d(Tx, F(T)) < d(x, F(T))$ if $x \notin F(T)$.

In other words, T is attractive if and only if $F(T) \in \mathscr{A}$ (we remark that for this case $\mathscr{A}$ does not mean $\mathscr{A}'$).

It is clear that T attractive implies $F(T) = \mathscr{D}$. It is natural to ask: does $F(T) = \mathscr{D}$, $F(T) \neq \emptyset$, imply $F(T) \in \mathscr{A}$? In other words, is attractivity characterized by $F(T) = \mathscr{D}$? Unfortunately, we have no satisfactory answer to this question.

The best thing we can say in this case is that a "good" choice of iterates always produces a fixed point, no matter the initial point. (See Lemma 1.3.)

LEMMA 1.2. *Let X be compact, T continuous. Then, for each $x \in X$ and $B \in \mathscr{A}$, $L(x) \cap B \neq \emptyset$.*

In other words, no matter where we start, with a suitable subsequence of iterates, we can always end up in B.

Proof. If $x \in B$ or if $T^h x \in B$ for some h, the result is obvious owing to the T-invariance and compactness of B. Therefore, assume $x \notin B$, $T^h x \notin B$, $h = 1, 2, \ldots$ Let $a_h = d(T^h x, B)$. Since $a_{h+1} = d(T^{h+1} x, B) = d(T(T^h x), B) < d(T^h x, B) = a_h$, (a_h) is a strictly decreasing sequence of positive numbers. Therefore, $a_h \to a \geqslant 0$. The same reasoning as that used in proving Theorem 1.2 shows that $a > 0$ is impossible. Therefore, $a = 0$ and $L(x) \cap B \neq \emptyset$.

LEMMA 1.3. *With the hypothesis of the previous Lemma, $L(x) \cap \mathscr{D} \neq \emptyset$ for each $x \in X$.*

Proof. As before, assume $x \notin \mathscr{D}$, and $T^h x \notin \mathscr{D}$, $h = 1, 2, \ldots$

We proceed by induction on $k(\mathscr{A})$. If $k(\mathscr{A}) = 1$, this is Lemma 1.2. Assume the result to be true for $k(\mathscr{A}) = n$, and let $k(\mathscr{A}) = n+1$, i.e. $\mathscr{A} = \{B_1, B_2, \ldots, B_{n+1}\}$. Since $x \notin \mathscr{D}$, for some $B_i \in \mathscr{A}$, $x \notin B_i$. By rearrangement, we may assume $x \notin B_1$. By the previous Lemma, $L(x) \cap B_1 \neq \emptyset$. Let $x_1 \in L(x) \cap B_1$. If $x_1 \notin \mathscr{D}$, we may assume $x_1 \notin B_2$; hence, $L(x_1) \cap B_2 \neq \emptyset$ (the previous Lemma). But $L(x_1) \subset L(x)$, for $L(L(x)) \subset L(x)$, and $L(x_1) \subset B_1$. Hence: $L(x) \cap B_2 \cap B_1 \neq \emptyset$. Repeating the procedure at most $n+1$ times, we arrive at: $L(x) \cap \bigcap_{i=1}^{n+1} B_i = L(x) \cap \mathscr{D} \neq \emptyset$.

We see that any finite collection of sets in $\{L(x) \cap B_i\}$ has a non-empty intersection. Hence, with the same reasoning as that used in Theorem 1.2, we get $L(x) \cap \mathscr{D} \neq \emptyset$.

Until now, the use of the attractivity condition has prescribed $F(T) \neq \emptyset$. We ask: when does $\mathscr{D} \neq \emptyset$ imply $F(T) \neq \emptyset$?

THEOREM 1.3. *Let X be a Banach space with the the metric induced by its norm, and T a continuous self-mapping of X, $\mathscr{D} \neq \emptyset$, such that, for some $n \geqslant 0$, $T^n(\mathscr{D})$ is convex. Then, $F(T) \neq \emptyset$.*

Proof. Let $Y_n = T^n(\mathscr{D})$, with $Y_0 = \mathscr{D}$. It is clear that $Y_n \supset Y_{n+1} = T(Y_n)$. Hence, by Schauder's Theorem, we get our result.

Of course, $T^n(\mathscr{D})$ convex for some n is by no means necessary. For, let $X = \{0, 1, 2, 3\}$, $T(0) = 1$, $T(1) = 1$, $T(2) = 2$, $T(3) = 2$. Then, $\mathscr{D} = F(T) = \{1, 2\}$, but $T^n(\mathscr{D})$ is not convex for any n.

2. In this Section we will show the relationships between $F(T)$, $\mathscr{A}$ and $\mathscr{D}$ for some well-known classes of operators.

We recall the following definitions for T, a self-mapping of the metric space (X, d) and x, y arbitrary elements in X.

DEFINITION 2.1. T is called:

(a) *Strictly contractive* if there exists an $r \in [0, 1)$ such that $d(Tx, Ty) \leqslant r d(x, y)$.

(b) *Contractive* if $d(Tx, Ty) < d(x, y)$, $x \neq y$.

(c) *Non-expansive* if $d(Tx, Ty) \leqslant d(x, y)$.

(d) *Isometric* if $d(Tx, Ty) = d(x, y)$.

(e) *Non-contractive* if $d(Tx, Ty) \geqslant d(x, y)$.

(f) *Expansive* if $d(Tx, Ty) > d(x, y)$, $x \neq y$.

(g) *Strictly expansive* if there exists an $s > 1$ such that $d(Tx, Ty) \geqslant s d(x, y)$.

LEMMA 2.1. *T contractive, $F(T) \neq \emptyset$ implies $F(T) \in \mathscr{A}$* (we observe that no restrictions were made on X, so that we have to prescribe $F(T) \neq \emptyset$).

Proof. Clearly, $F(T)$ consists of exactly one element, say u. Let $y \notin F(T)$, i.e. $y \neq u$. Then: $d(Ty, F(T)) = d(Ty, u) = d(Ty, Tu) < d(y, u) = d(y, F(T))$.

COROLLARY 2.1. *T strictly contractive, $F(T) \neq \emptyset$, imply $F(T) \in \mathscr{A}$.*

COROLLARY 2.2. *T contractive (strictly contractive), $F(T) \neq \emptyset$, imply $F(T) = \mathscr{D}$.*

If T is merely non-expansive, the above results are not necessarily true.

EXAMPLE 2.1. Let $X = [-1, 1] \times [-1, 1]$, with l_1 norm. If $u = (x, y)$, let $Tu = (y, |x|)$. Hence, T is non-expansive, and $F(T) = \{(x, x)/0 \leqslant x \leqslant 1\}$. Let $v = (-1, -1)$. Then $Tv = (-1, 1)$, and $d(Tv, F(T)) = d(v, F(T))$, so that $F(T) \notin \mathscr{A}$, and Lemma 2.1 does not hold.

We now show that $\mathscr{A} \neq \emptyset$. Let B be the union of the first and second quadrant in X. Clearly, $T(B) \subset B$. Furthermore, if $z \notin B$, $Tz \in B$, so that $B \in \mathscr{A}$.

Any suitable $B' \in \mathscr{A}$ has to contain $F(T)$. Now, if it contains any other point in the first quadrant, since its immage is symmetric with respect to $F(T)$, and B must be T-invariant, it will be symmetric with respect to $F(T)$. If it does not contain some point u in that quadrant it will not contain its symmetric Tu with respect to $F(T)$ either, by the same argument, and hence $d(u, B') = d(Tu, B')$, so that $B' \notin \mathscr{A}$. Hence, any set in $\mathscr{A}$ contains the first quadrant, i.e. $F(T)$ will be a proper subset of $\mathscr{D}$, so that Corollary 2.2 does not hold for non-expansive operators.

LEMMA 2.2. *T expansive, $A \subset X$, A: closed and bounded, implies $T(A) \not\subset A$.*

Proof. Assume $T(A) \subset A$. Let $D = \operatorname{diam}(A) = \max_A d(x, y)$.

Then, there exist $u, v \in A$ such that $d(u, v) = D$. Hence: $D = d(u, v) < d(Tu, Tv) \leqslant \max_A d(Tx, Ty) = \operatorname{diam}(T(A))$, i.e. $\operatorname{diam}(A) < \operatorname{diam}(T(A)) \leqslant \operatorname{diam}(A)$, which is clearly impossible.

COROLLARY 2.3. *T expansive (strictly expansive) implies $\mathscr{A} = \emptyset$.*

LEMMA 2.3. *T non-contractive implies $F(T) \in \mathscr{C}\mathscr{A}$ ($\mathscr{C}\mathscr{A}$ = complement of $\mathscr{A}$).*

Proof. Assume $F(T) \neq \emptyset$. Otherwise, the result is obvious.

Let $x \in F(T)$. Then:

$$d(Tx, F(T)) = \inf_{F(T)} d(Tx, u) = \inf_{F(T)} d(Tx, Tu) \geqslant \inf_{F(T)} d(x, u) = d(x, F(T)).$$

COROLLARY 2.4. *T isometric implies $F(T) \in \mathscr{C}\mathscr{A}$.*

COROLLARY 2.5. *T non-contractive, isometric, expansive, or strictly expansive, $F(T) \neq \emptyset$, imply $F(T) \notin \mathscr{A}$.*

We now ask; do Lemma 2.1 and or Corollaries 2.1, 2.2 characterize contractive or strictly contractive operators? The answer is no.

EXAMPLE 2.2. Let $X = [0, 2]$ with the usual metric, and define T as follows:

$$Tx = \begin{cases} \frac{1}{2}x & \text{if } 0 \leqslant x \leqslant 1, \\ \frac{3}{2}x - 1 & \text{if } 1 \leqslant x \leqslant \frac{3}{2}, \\ \frac{5}{4} & \text{if } \frac{3}{2} \leqslant x \leqslant 2. \end{cases}$$

Clearly, $F(T) = \{0\}$. Hence, $d(x, F(T)) = x$, $d(Tx, F(T)) = Tx$, $0 < x \leqslant 1 \Rightarrow Tx = \frac{1}{2}x < x$; $1 \leqslant x \leqslant \frac{3}{2} \Rightarrow Tx = \frac{3}{2}x - 1 = x + \frac{1}{2}x - 1 = x + \frac{1}{2}(x-2) < x$; $\frac{3}{2} \leqslant x \leqslant 2 \Rightarrow Tx = \frac{5}{4} < \frac{3}{2} < x$.

Hence, $x \notin F(T) \Rightarrow d(Tx, F(T)) < d(x, F(T))$, i.e. $F(T) \in \mathscr{A}$. However, T is not even non-expansive. This also shows that Corollary 2.2 does not characterize contractive and/or strictly contractive operators. The same example shows that we could not even hope that Lemma 2.1 characterizes contractive operators, if one prescribes $F(T)$ to be exactly one point.

Examples 2.1 and 2.2 show that attractiveness is indeed a new property, independent of non-expansiveness.

To summarize, our results could be visualized as follows, with the basic assumption $F(T) \neq \emptyset$.

$$\begin{array}{ccccccccccccc} \underset{\text{str. contr.}}{T} & \Rightarrow & \underset{\text{contr.}}{T} & \Rightarrow & \underset{\text{non-exp.}}{T} & \Leftarrow & \underset{\text{isom.}}{T} & \Rightarrow & \underset{\text{non-contr.}}{T} & \Leftarrow & \underset{\text{exp.}}{T} & \Leftarrow & \underset{\text{str. exp.}}{T} \\ & & \Downarrow & & \not\Downarrow & & & & \Downarrow & & \Downarrow & & \\ & & F(T) \in \mathscr{A} & & & & & & F(T) \in \mathscr{C}\mathscr{A} & & \mathscr{A} = \emptyset & & \\ & & \Downarrow & & & & & & \Downarrow & & & & \\ & & F(T) = \mathscr{D} & & & & & & F(T) \notin \mathscr{A} & & & & \end{array}$$

3. Remark 3.1. In [4], Kirk introduced the following concepts: let X be a Banach space, $G \subset K \subset X$, $T: K \to X$ with the property that for each $x \in K$ there is an $a(x) < 1$ such that $\|Tu - Tx\| \leqslant a(x)\|u - x\|$ for each $u \in G$. Then T is said to be *uniformly strict contractive on G relative to K*.

Define $d(x, y) = \|x - y\|$, and assume $T(G) \subset G$, $K = X$; if $x \notin G$, then $x \notin T(G)$ and

$$d(Tx, T(G)) = \inf_G d(Tx, Tu) \leqslant a(x) \inf_G d(x, u) = a(x)\, d(x, G),$$

$T(G) \subset G \Rightarrow d(Tx, T(G)) \geqslant d(Tx, G)$. Hence, $d(Tx, G) \leqslant a(x)\, d(x, G) < d(x, G)$, so that if G is T-invariant, and T is uniformly strict contractive on G relative to X, then T is G-attractive.

References

[1] F. E. Browder and W. V. Petryshyn, *The solution by iteration of non-linear functional equations in Banach spaces*, Bull. Amer. Math. Soc. 72, 3 (1966), p. 571-575.

[2] J. B. Díaz and F. T. Metcalf, *On the set of subsequential limit points of successive approximations*, Trans. Amer. Math. Soc. 135 (1969), p. 459-485.

[3] W. G. Dotson, Jr., *Fixed points of quasi-non-expansive mappings*, Notices Amer. Math. Soc. 16, 1 (1969), p. 231.

[4] W. A. Kirk, *On non-linear mappings of strongly semicontractive type*, J. Math. Anal. Appl. 27 (1969), p. 409-412.

[5] J. Reinermann, *On a fixed-point theorem of Banach type*, Ann. Polon. Math. 23 (1970), p. 105-107.

[6] W. Sierpiński, *General topology*, University of Toronto Press, Toronto 1956.

Reçu par la Rédaction le 2. 1. 1971