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On some functional equations

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§ 1. Introduction. Let us consider the following system of p functional equations

$$\varphi_i(x) = G_i\big(x, \Phi(x), \Phi[f_1(x)], \Phi[f_2(x)], \ldots, \Phi[f_q(x)]\big), \quad i = 1, 2, \ldots, p, \tag{1}$$

with p unknown functions $\Phi(x) = \big(\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)\big)$. The functions $x = f_0(x), f_1(x), \ldots, f_q(x)$ and $G_i(x, Y)$, where

$$Y = (y_{10}, y_{20}, \ldots, y_{p0}, y_{11}, y_{21}, \ldots, y_{p1}, \ldots, y_{1q}, y_{2q}, \ldots, y_{pq})$$

are known.

For $p = 1$ we obtain one equation of the form

$$\varphi(x) = G\big(x, \varphi(x), \varphi[f_1(x)], \varphi[f_2(x)], \ldots, \varphi[f_q(x)]\big) \tag{2}$$

(we omit the index $i = 1$).

This equation has been dealt with by M. Bajraktarević ([1]) and by B. Choczewski ([2]), but on the right side of the equation a function $\varphi(x)$ has not appeared. J. Kordylewski and M. Kuczma have also analysed equation (2), in the implicit form, with regard to $\varphi(x)$ ([6]). The present paper is an attempt to obtain a solution of system (1) by using weaker assumptions about $f_k(x)$ than in [2] and [6].

With certain assumptions, the existence of a solution of the system (1) results from Schauder's fixed-point-theorem (§ 2). In § 3 we prove that under suitable assumptions system (1) has exactly one solution which is continuous. We obtain it by successive approximations. § 4 gives several examples. The solutions obtained are applied to the solution of a generalized problem of Goursat ([9]). § 5 deals with a linear functional equation for which a uniqueness theorem has been given for a solution in a set of functions of class C^r. § 5 was written on the suggestion of dr M. Kuczma, who encouraged me to investigate also the solutions of class C^r. I take this opportunity to thank him cordially for his kind interest in my work and his valuable remarks.

§ 2. Existence of a solution of system (1). We assume that:

(H_1) *The functions* $f_k(x)$, $k = 1, 2, \ldots, q$, *are defined and continuous in the closed interval* $\langle a, b\rangle$, $b > a$.

(H_2) $a \leqslant f_k(x) \leqslant b$ *for* $x \in \langle a, b\rangle$, $k = 1, 2, \ldots, q$.

(H_3) *The functions* $G_i(x, Y)$, $Y = (\ldots, y_{ij}, \ldots)$ (see § 1) *are defined and continuous in a* $[p(q+1)+1]$*-dimensional region* V:

$$V: \{a \leqslant x \leqslant b,\ |y_{ij}| \leqslant R\},$$

the constant $R \geqslant 0$, $i = 1, 2, \ldots, p$, $j = 0, 1, 2, \ldots, q$.

(H_4) $|G_i(x, Y)| \leqslant R$ *for* $i = 1, 2, \ldots, p$, $(x, Y) \in V$.

(H_5) *There exist functions* $\eta = \omega_i(\xi)$, $i = 1, 2, \ldots, p$, *continuous and strictly increasing in the interval* $\langle 0, b-a\rangle$, *for which* $\omega_i(0) = 0$, *such that*

$$|G_i(x_1, \widetilde{Y}) - G_i(x_2, \overline{Y})| \leqslant \omega_i(|x_1 - x_2|)$$

for every $x_1, x_2 \in \langle a, b\rangle$ *and* $\widetilde{Y}$ *and* $\overline{Y}$ *fulfilling the inequalities*:

$$|\tilde{y}_{ij}| \leqslant R, \quad |\bar{y}_{ij}| \leqslant R, \quad |\tilde{y}_{ij} - \bar{y}_{ij}| \leqslant \omega_i\big(|f_j(x_1) - f_j(x_2)|\big),$$
$$i = 1, 2, \ldots, p,\ j = 0, 1, 2, \ldots, q.$$

THEOREM 1. *If the assumptions* (H_1)-(H_5) *are fulfilled, then system* (1) *possesses a solution* $\Phi(x)$ *continuous in* $\langle a, b\rangle$.

Proof. Theorem 1 comes out from the fixed-point-theorem of Schauder ([10]).

Let us consider the space E, the points of which are the systems $\Phi(x) = \big(\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)\big)$ of p functions continuous in $\langle a, b\rangle$. We define the sum of its two points, and the product of a point by a number as we usually do with vectors. We introduce in E the metric

$$\varrho(\Phi, \widetilde{\Phi}) = \sum_{i=1}^{p} \sup_{\langle a, b\rangle} |\varphi_i(x) - \tilde{\varphi}_i(x)|, \tag{3}$$

where $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_p)$ and $\widetilde{\Phi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_p)$ belong to E. Now

$$\|\Phi\| = \varrho(\Phi, \theta) = \sum_{i=1}^{p} \sup_{\langle a, b\rangle} |\varphi_i(x)|,$$

where $\theta = (0, 0, \ldots, 0)$. Of course $\varrho(\Phi, \widetilde{\Phi}) = \|\Phi - \widetilde{\Phi}\|$. It is easy to prove that E is complete and hence it is a Banach space.

Let us take the convex set $T \subset E$ of points Φ, for which

$$\begin{aligned} &|\varphi_i(x)| \leqslant R, \quad x \in \langle a, b\rangle, \\ &|\varphi_i(x_1) - \varphi_i(x_2)| \leqslant \omega_i(|x_1 - x_2|), \quad x_1, x_2 \in \langle a, b\rangle, \quad i = 1, 2, \ldots, p. \end{aligned} \tag{4}$$

The constant R is the same as in assumptions (H_3) and (H_4). Now we consider the operation

(5) $\quad \psi_i(x) = A\big(\varphi_i(x)\big) = G_i\big(x, \Phi(x), \Phi[f_1(x)], \ldots, \Phi[f_q(x)]\big), \qquad i = 1, 2, \ldots, p$

applied to the functions of T.

It follows from assumptions (H_4) and (H_5) that if $\Phi \in T$ then $A(\Phi) = \big(A(\varphi_1), A(\varphi_2), \ldots, A(\varphi_p)\big) \in T$. Besides, operation (5) is continuous and the set $A(T)$ is compact. This last property results from a well-known theorem of Arzela. Thanks to (H_4), the functions $\psi_i = A(\varphi_i)$ are equibounded and thanks to (H_5), they are also equicontinuous. In fact, let us take $0 < \varepsilon < \min\limits_i \omega_i(b-a)$ and two points $x, x_0 \in \langle a, b\rangle$. Then for $i = 1, 2, \ldots, p$ and for every $\Phi \in T$

$$\begin{aligned}|\psi_i(x)-\psi_i(x_0)| &= \big|A\big(\varphi_i(x)\big)-A\big(\varphi_i(x_0)\big)\big| \\ &= \big|G_i\big(x, \Phi(x), \Phi[f_1(x)], \ldots, \Phi[f_q(x)]\big) - \\ &\qquad -G_i\big(x_0, \Phi(x_0), \Phi[f_1(x_0)], \ldots, \Phi[f_q(x_0)]\big)\big| \\ &\leqslant \omega_i(|x-x_0|) < \varepsilon,\end{aligned}$$

if $|x-x_0| < \delta(\varepsilon) = \min\limits_i \omega_i^{-1}(\varepsilon)$.

Hence and on account of the theorem of Schauder cited above operation (5) has a fixed point in T, i.e. $\varphi_i = A(\varphi_i)$, for $i = 1, 2, \ldots, p$, which proves Theorem 1.

§ 3. Uniqueness of the solution of system (1). In this section we make the assumptions (H_1)-(H_4) and

(H_6) *There exist constant numbers* $K_{ijk} > 0$, $i, j = 1, 2, \ldots, p$, $k = 0, 1, 2, \ldots, q$, *such that for* $i = 1, 2, \ldots, p$ *and for every two points* (x, Y), $(x, \overline{Y})$ *of the set* V

$$|G_i(x, Y)-G_i(x, \overline{Y})| \leqslant \sum_{j=1}^{p}\sum_{k=0}^{q} K_{ijk}|y_{jk}-\overline{y}_{jk}|, \tag{6}$$

where, if we put

$$\overline{K}_i = \max_j \Big(\sum_{k=0}^{q} K_{ijk}\Big), \qquad j = 1, 2, \ldots, p,$$

we assume that the numbers $\overline{K}_i$ *fulfil the inequality*

$$\sum_{i=1}^{p} \overline{K}_i < 1. \tag{7}$$

Now we can prove the following

THEOREM 2. *If the assumptions* (H_1)-(H_4) *and* (H_6) *are fulfilled, then system* (1) *has a unique solution* $\Phi(x)$ *such that the functions* $\varphi_i(x)$, $i = 1, 2, \ldots, p$, *are continuous in the interval* $\langle a, b\rangle$ *and satisfy the inequalities* $|\varphi_i(x)| \leqslant R$ *in* $\langle a, b\rangle$.

Proof. We consider the space E the same as in the proof of Theorem 1. Suppose we are given, in E, the set T_1 of systems of p functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)$ continuous in $\langle a, b\rangle$ and satisfying in this interval the inequalities $|\varphi_i(x)| \leqslant R$, $i = 1, 2, \ldots, p$. Operation (5) transforms T_1 into itself; moreover, for every two arbitrary points Φ and $\bar{\Phi}$ of T_1, we have on account of (5), (3) and (H_6)

$$\begin{aligned}\varrho\big(A(\Phi), A(\bar{\Phi})\big) &= \sum_{i=1}^{p} \sup_{\langle a,b\rangle} |A(\varphi_i) - A(\bar{\varphi}_i)| \\ &= \sum_{i=1}^{p} \sup_{\langle a,b\rangle} \Big|G_i\big(x, \Phi(x), \Phi[f_1(x)], \ldots, \Phi[f_q(x)]\big) - \\ &\qquad\qquad - G_i\big(x, \bar{\Phi}(x), \bar{\Phi}[f_1(x)], \ldots, \bar{\Phi}[f_q(x)]\big)\Big| \\ &\leqslant \sum_{i=1}^{p}\Big[\sum_{j=1}^{p}\Big(\sum_{k=0}^{q} K_{ijk}\Big) \sup_{\langle a,b\rangle} |\varphi_j(x) - \bar{\varphi}_j(x)|\Big] \\ &\leqslant \Big(\sum_{i=1}^{p} \bar{K}_i\Big)\varrho(\Phi, \bar{\Phi}).\end{aligned}$$

Consequently, according to the theorem of Banach-Cacciopoli, the operation $A(\varphi_i)$ has a unique fixed point in T_1. This proves Theorem 2.

Remark 1. We receive the solution of system (1) by the method of successive approximations. This solution is also given by formula (8)

$$\varphi_i(x) = \varphi_i^{(0)} + \sum_{n=0}^{\infty} [\varphi_i^{(n+1)}(x) - \varphi_i^{(n)}(x)], \tag{8}$$

where $\varphi_i^{(0)}(x)$ are arbitrary functions continuous in $\langle a, b\rangle$, and such that

$$|\varphi_i^{(0)}(x)| \leqslant R,$$

$$\varphi_i^{(n+1)}(x) = G_i\big(x, \Phi_n(x), \Phi_n[f_1(x)], \ldots, \Phi_n[f_q(x)]\big),$$

$$n = 1, 2, 3, \ldots, \quad i = 1, 2, \ldots, p, \quad x \in \langle a, b\rangle,$$

$$\Phi_n(x) = \big(\varphi_1^{(n)}(x), \varphi_2^{(n)}(x), \ldots, \varphi_p^{(n)}(x)\big).$$

Remark 2. Let V_1 be a set of the points (x, Y) such that

$$V_1\colon \{a \leqslant x \leqslant b,\ y_{ij} \text{ arbitrary}\}, \quad i = 1, 2, \ldots, p,\ j = 0, 1, 2, \ldots, q.$$

Let us denote by (H_3') and (H_6') the assumptions which we obtain by replacing in (H_3) and in (H_6) the set V by the set V_1.

We now have the following

THEOREM 3. *If the assumptions* (H_1), (H_2), (H_3') *and* (H_6') *are fulfilled, then system* (1) *possesses a unique solution* $\Phi(x)$ *continuous in* $\langle a, b\rangle$.

It is so, because the inequality

$$\varrho\big(A(\Phi), A(\bar{\Phi})\big) \leqslant \Big(\sum_{i=1}^{p} \bar{K}_i\Big) \varrho(\Phi, \bar{\Phi})$$

is now true in the whole space E, i.e. for every two points Φ and $\bar{\Phi}$ of this space.

Remark 3. In Theorem 3 we can replace assumption (H_6') by the following one:

(H_6'') *There exist constant numbers $L_{ij} > 0$, $i, j = 1, 2, \ldots, p$, such that for $i = 1, 2, \ldots, p$ and for every two points $\Phi(x)$ and $\bar{\Phi}(x)$ of the space E conditions* (6′) *are fulfilled*:

$$(6') \qquad \Big|G_i\big(x, \Phi(x), \Phi[f_1(x)], \ldots, \Phi[f_q(x)]\big) - G_i\big(x, \bar{\Phi}(x), \bar{\Phi}[f_1(x)], \ldots, \bar{\Phi}[f_q(x)]\big)\Big| \leqslant \sum_{j=1}^{p} L_{ij} \sup_{\langle a,b\rangle} |\varphi_j(x) - \bar{\varphi}_j(x)|\,,$$

where for $j = 1, 2, \ldots, p$,

$$(7') \qquad 0 < \sum_{i=1}^{p} L_{ij} \leqslant L < 1\,.$$

L is a positive number, $0 < L < 1$.

Remark 4. Theorems 1-3 are true for the system of equations

$$\varphi_i(x) = G_i\big(x, \Phi(x), \Phi[f_{i1}(x)], \Phi[f_{i2}(x)], \ldots, \Phi[f_{iq}(x)]\big)\,, \qquad i = 1, 2, \ldots, p\,,$$

where the functions f_{ij}, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, are defined and continuous in $\langle a, b\rangle$ and, moreover, if $a \leqslant f_{ij}(x) \leqslant b$, for $x \in \langle a, b\rangle$.

Remark 5. It sometimes happens that the assumptions of Theorem 2 or 3 are not fulfilled in the whole interval $\langle a, b\rangle$, but only in an interval $\langle a, \beta\rangle \subset \langle a, b\rangle$. In this case Theorems 2 and 3 give exactly one solution $\tilde{\Phi}(x)$ of system (1) continuous in $\langle a, \beta\rangle$. Under the supplementary assumptions $\tilde{\Phi}(x)$ can be uniquely extended to the whole $\langle a, b\rangle$ in such a manner that it will satisfy system (1) and be continuous in $\langle a, b\rangle$ (for $p = 1$, see [6], [5], and [2], lemma 2).

§ 4. Examples. Theorem 2 easily gives a solution of the following equation (if the assumptions of Theorem 2 are fulfilled):

$$(9) \qquad \varphi(x) = G\big(x, \varphi(x), \varphi[f(x)], \varphi[f^2(x)], \ldots, \varphi[f^q(x)]\big)\,,$$

where $f^k(x)$ denotes the kth iteration of the function $f(x)$, i.e.

$$(10)\qquad \begin{aligned} f^0(x) &= x\,,\\ f^{k+1}(x) &= f\big(f^k(x)\big)\,, \qquad k = 0, 1, 2, \ldots \end{aligned}$$

(see J. Kordylewski [5]).

Let us take, in particular, a linear equation of the order q,

$$(11)\qquad \varphi(x) - A_1(x)\varphi[f(x)] - A_2(x)\varphi[f^2(x)] - \ldots - A_q(x)\varphi[f^q(x)] = F(x)\,,$$

treated in some particular cases by J. Kordylewski and M. Kuczma ([7]). We have for it the

THEOREM 4. *If the functions $A_j(x)$, $j = 1, 2, \ldots, q$, $f(x)$ and $F(x)$ are defined and continuous in the interval $\langle a, b\rangle$, $f(x) \in \langle a, b\rangle$, if moreover. $A_q(x) \neq 0$ in $\langle a, b\rangle$ and if there exists a number L, $0 < L < 1$, such that*

$$(12)\qquad \sum_{j=1}^{q} |A_j(x)| \leqslant L < 1\,, \qquad \textit{for} \qquad x \in \langle a, b\rangle\,,$$

then equation (11) *possesses exactly one solution continuous in $\langle a, b\rangle$. This solution is given by the formula*

$$(13)\qquad \varphi(x) = F(x) + \sum_{n=0}^{\infty} \lambda_n(x) F[f^{n+1}(x)]\,,$$

where

$$(13')\qquad \begin{aligned} \lambda_0 &= A_1\,,\\ \lambda_n &= \begin{cases} A_1(f^n)\lambda_{n-1} + A_2(f^{n-1})\lambda_{n-2} + \ldots + A_n(f)\lambda_0 + A_{n+1}\,, & \textit{for} \quad n < q\,,\\ A_1(f^n)\lambda_{n-1} + A_2(f^{n-1})\lambda_{n-2} + \ldots + A_q(f^{n-q+1})\lambda_{n-q}\,, & \textit{for} \quad n \geqslant q \end{cases} \end{aligned}$$

(in order to simplify we omit x).

This theorem follows from Theorem 3 and from Remark 3. We obtain a solution of form (13), (13′) from formula (8), (8′) after an elementary modification.

In particular, for $q = 2$ ([9]) we have the equation of the second order

$$\varphi(x) - A_1(x)\varphi[f(x)] - A_2(x)\varphi[f^2(x)] = F(x)$$

and the solution in form (13) in which

$$\begin{aligned} \lambda_0(x) &= A_1(x)\,,\\ \lambda_1(x) &= A_1[f(x)]\lambda_0(x) + A_2(x)\,,\\ \lambda_n(x) &= A_1[f^n(x)]\lambda_{n-1}(x) + A_2[f^{n-1}(x)]\lambda_{n-2}(x)\,, \qquad \text{for} \qquad n \geqslant 2\,. \end{aligned}$$

At last, for $q = 1$, equation (11) will be of the first order, namely

$$\varphi(x) - A(x)\varphi[f(x)] = F(x)\,,$$

and in the solution (13)

$$\lambda_0(x) = A(x),$$
$$\lambda_n(x) = A[f^n(x)]\lambda_{n-1}(x), \quad \text{for} \quad n \geqslant 1.$$

In this last example the solution can be represented in the form

$$\varphi(x) = F(x) + \sum_{n=1}^{\infty}\Big(\prod_{i=0}^{n} A[f^i(x)]\Big)F[f^{n+1}(x)], \qquad x \in \langle a, b\rangle.$$

§ 5. Solution of class C^r of a linear functional equation. For the sake of simplicity, we do not deal in this section with the system but only with one linear functional equation of the form

$$\varphi(x) - \sum_{i=1}^{q} A_i(x)\varphi[f_i(x)] = F(x), \qquad x \in \langle a, b\rangle. \tag{14}$$

In further considerations we assume $A_q \neq 0$ in $\langle a, b\rangle$. $A_i(x)$, $f_i(x)$ and $F(x)$ denote the known functions and $\varphi(x)$ is the function to be found.

We profit once more by the above-mentioned theorem of Banach-Cacciopoli and prove that with certain assumptions equation (14) has exactly one solution of class C^r, $1 \leqslant r < \infty$, in the interval $\langle a, b\rangle$. M. Kuczma has dealt with this problem in the case of $q = 1$ ([8]) (1). However, his results differ (also for $q = 1$) from ours here because we make different assumptions from those in [8].

Let us write

$$H(x) \equiv \sum_{i=1}^{q} A_i(x)\varphi[f_i(x)] + F(x), \tag{15}$$

and assume that

(H_7) *The functions $A_i(x)$, $f_i(x)$, for $i = 1, 2, \ldots, q$, and the function $F(x)$ are of class C^r in $\langle a, b\rangle$, $1 \leqslant r < \infty$.*

(H_8) $f_i(a) = a$, *for* $i = 1, 2, \ldots, q$.

Lemma 1. (See an analogous lemma in [8]). *If* (H_2) *and* (H_7) *are fulfilled and the function $\varphi(x)$ is of class C^r in $\langle a, b\rangle$, then we have for $s = 1, 2, \ldots, r$,*

$$\frac{d^sH(x)}{dx^s} \tag{16}$$
$$= \sum_{i=1}^{q} A_i(x)[f_i'(x)]^s \frac{d^s\varphi[f_i(x)]}{dx^s} + \sum_{j=0}^{s-1}\sum_{i=1}^{q} W_{sji}(x)\frac{d^j\varphi[f_i(x)]}{dx^j} + \frac{d^sF(x)}{dx^s}\ (2),$$

where $W_{sji}(x)$ are functions of the class C^m, $r-s \leqslant m \leqslant r-1$ in $\langle a, b\rangle$.

(1) For the non-linear equation, see [3].

(2) $\dfrac{d^j\varphi[f_i(x)]}{dx^j}$ denotes here $\left.\dfrac{d^j\varphi(u)}{du^j}\right|_{u=f_i(x)}$, $j = 1, 2, \ldots, s$.

We easily obtain formula (16) from (15).

It follows from the assumptions of Lemma 1 that there exist two numbers $M > 0$ and $K > 0$ such that in $\langle a, b\rangle$

$$\sum_{i=1}^{q} |A_i(x)[f_i'(x)]^r| \leqslant M\,, \tag{17}$$
$$\sum_{i=1}^{q} |W_{rji}(x)| \leqslant K, \quad \text{for} \quad j = 0, 1, \ldots, r-1\,.$$

In the sequel we make the following assumptions:

(H_9) *If* $b-a \neq 1$ $(b > a)$, *then*

$$M + K(b-a)[1-(b-a)^r] : [1-(b-a)] \leqslant \alpha < 1\,.$$

But if $b-a = 1$, *then*

$$M + Kr \leqslant \alpha < 1, \quad \textit{where} \quad 0 < \alpha < 1\,.$$

(H_{10}) *The system of equations*

$$a_0 = \Big[\sum_{j=1}^{q} A_j(a)\Big] a_0 + F(a)\,, \tag{18}$$
$$a_s = \Big(\sum_{j=1}^{q} A_j(a)[f_j'(a)]^s\Big) a_s + \sum_{j=0}^{s-1}\Big(\sum_{i=1}^{q} W_{sji}(a)\Big)\, a_j + \frac{d^s F(a)}{dx^s}, \quad s = 1, 2, \ldots, r\,,$$

with unknown numbers $a_0, a_1, \ldots, a_r$, *has solutions* (*one or more*).

THEOREM 5. *Let* (H_2) *and* (H_7)-(H_{10}) *be fulfilled. Then for every system of values* $a_0, a_1, \ldots, a_r$, *fulfilling* (18) *there exists a unique function* $\varphi(x)$ *of class* C^r *in* $\langle a, b\rangle$ *satisfying equation* (14) *and the conditions*

$$\frac{d^s\varphi(a)}{dx^s} = a_s, \quad s = 0, 1, \ldots, r\,. \tag{19}$$

Proof. We define $\mathfrak{R}$ as the space of all functions $\varphi(x)$ that are of class C^r in $\langle a, b\rangle$ and fulfil conditions (19). We introduce in $\mathfrak{R}$ the metric

$$\varrho(\varphi, \bar{\varphi}) = \sup_{\langle a,b\rangle}\left|\frac{d^r\varphi(x)}{dx^r} - \frac{d^r\bar{\varphi}(x)}{dx^r}\right|.$$

Now we have

$$\|\varphi\| = \varrho(\varphi, 0) = \sup_{\langle a,b\rangle}\left|\frac{d^r\varphi}{dx^r}\right|.$$

From Taylor's formula, we can easily get the inequality

$$\sup_{\langle a,b\rangle}\left|\frac{d^j\varphi}{dx^j}-\frac{d^j\bar\varphi}{dx^j}\right| \leqslant (b-a)^{r-j}\sup_{\langle a,b\rangle}\left|\frac{d^r\varphi}{dx^r}-\frac{d^r\bar\varphi}{dx^r}\right|, \quad \text{for} \quad j=0,1,\dots,r-1, \tag{20}$$

valid for any two $\varphi,\bar\varphi$ belonging to $\mathfrak{K}$. Hence it follows that $\mathfrak{K}$ is complete. The operation

$$A(\varphi)=\sum_{i=1}^{q}A_i(x)\varphi[f_i(x)]+F(x)$$

(which leaves numbers a_s unchanged) maps $\mathfrak{K}$ into itself. Moreover, we have for every two functions $\varphi(x)$, $\bar\varphi(x)$ of $\mathfrak{K}$

$$\begin{aligned}
\varrho\big(A(\varphi),A(\bar\varphi)\big) &= \sup_{\langle a,b\rangle}\left|\frac{d^rA(\varphi)}{dx^r}-\frac{d^rA(\bar\varphi)}{dx^r}\right| \\
&\leqslant \sup_{\langle a,b\rangle}\left\{\sum_{i=1}^{q}|A_i(x)|\,|f_i'(x)|^r\left|\frac{d^r\varphi[f_i(x)]}{dx^r}-\frac{d^r\bar\varphi[f_i(x)]}{dx^r}\right|+\right. \\
&\qquad \left.+\sum_{j=0}^{r-1}\sum_{i=1}^{q}|W_{rji}(x)|\left|\frac{d^j\varphi[f_i(x)]}{dx^j}-\frac{d^j\bar\varphi[f_i(x)]}{dx^j}\right|\right\} \\
&\leqslant \sup_{\langle a,b\rangle}\left[\left(\sum_{i=1}^{q}|A_i(x)|\,|f_i'(x)|^r\right)\sup_{\langle a,b\rangle}\left|\frac{d^r\varphi(x)}{dx^r}-\frac{d^r\bar\varphi(x)}{dx^r}\right|+\right. \\
&\qquad \left.+\sum_{j=0}^{r-1}\left(\sum_{i=1}^{q}|W_{rji}(x)|\right)\sup_{\langle a,b\rangle}\left|\frac{d^j\varphi(x)}{dx^j}-\frac{d^j\bar\varphi(x)}{dx^j}\right|\right] (^3).
\end{aligned}$$

Using inequality (20) and hypothesis (H_9), we obtain hence

$$\varrho\big(A(\varphi),A(\bar\varphi)\big)\leqslant\left[M+K\sum_{j=0}^{r-1}(b-a)^{r-j}\right]\sup_{\langle a,b\rangle}\left|\frac{d^r\varphi}{dx^r}-\frac{d^r\bar\varphi}{dx^r}\right|\leqslant a\varrho(\varphi,\bar\varphi),$$

which proves Theorem 5, on account of the theorem of Banach-Cacciopoli.

Here we can also make a remark analogous to Remark 5 in § 3, (see [6] and [4]).

(3) About the meaning of $\frac{d^j\varphi[f_i(x)]}{dx^j}$ see footnote (2).

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