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On absolute Nörlund summability factors for Fourier series

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1.1. Definitions. Let $\sum a_n$ be a given infinite series and let $\{s_n\}$ be the sequence of partial sums. Then the series $\sum a_n$ is said to be *summable* $|A|$ if $F(x) = \sum a_n x^n$ is convergent for $0 \leqslant x < 1$ and $F(x)$ is of bounded variation in $(0, 1)$ (see [11]).

Let s_n^α denote the nth Cesàro mean of order α $(\alpha > -1)$ of the sequence $\{s_n\}$ and $s_n^0 = s_n$. The series $\sum a_n$ is said to be *summable* $|C, \alpha|$, if the sequence $\{s_n^\alpha\}$ is of bounded variation, that is, if

$$\sum |s_n^\alpha - s_{n-1}^\alpha| < \infty .$$

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \ldots + p_n , \quad P_{-1} = p_{-1} = 0$$

(see [3] and [5]).

We define the sequence $\{t_n\}$ of Nörlund means by means of the transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{n-\nu} a_\nu \quad (P_n \neq 0) .$$

The series $\sum a_n$ is said to be *absolutely summable* (N, p_n), or *summable* $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation (see [6]).

1.2. We write $\tau = [1/t]$, where $[\theta]$ denotes the greatest integer in θ.

For any sequence $\{U_n\}$, we write

$$\Delta U_n = U_n - U_{n+1} , \quad \Delta^2 U_n = \Delta U_n - \Delta U_{n+1} .$$

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without loss of generality we can take the constant term of the Fourier series to be zero. Then we have

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \tag{1.2.1}$$

and we write

(1.2.2) $$\varphi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}.$$

We use the following notations:

(1.2.3) $$S_n(t) = \sum_{\nu=0}^{n} p_{n-\nu}\left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}}\right)\cos\nu t,$$

(1.2.4) $$S_{n\nu}(t) = \sum_{\mu=0}^{\nu} p_{n-\mu}\left(\frac{P_n}{p_n} - \frac{P_{n-\mu}}{p_{n-\mu}}\right)\cos\mu t,$$

(1.2.5) $$K_n(t) = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}\left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}}\right)\lambda_\nu\cos\nu t.$$

1.3. Generalizing a number of previous theorems due to Prasad ([10]), Izumi and Kawata ([4]) and Cheng ([1]), Pati proved in 1954 the following theorem concerning absolute Cesàro summability factors of Fourier series.

THEOREM A (see [7]) (1). *If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and*

(1.3.1) $$\int_0^t |\varphi(u)|\,du = o(t), \quad as \quad t\to 0\ (^2),$$

then $\sum \lambda_n A_n(x)$ is summable $|C, a|$ for every $a > 1$.

The purpose of the present paper is to obtain a result similar to that of Theorem A for absolute Nörlund summability factors of Fourier series.

2.1. We establish the following theorem:

THEOREM. *Let $\{\lambda_n\}$ be a sequence such that $\Delta\lambda_n \geqslant 0$ and the series $\sum n^{-1}\lambda_n$ is convergent and let $\{p_n\}$ be a positive monotonic increasing sequence such that*

(i) $$\frac{p_n}{P_n} = O\left(\frac{1}{n}\right),$$

(ii) $$\Delta\left(\frac{P_n}{p_n}\right) = O(1),$$

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(1) A more general form of Theorem A is contained in a later theorem of Pati and Sinha ([9]), where the condition of the convexity of λ_n is dispensed with and only '$\Delta\lambda_n \geqslant 0$' is assumed.

(2) It has been observed by Pati (see Pati [8]) that $o(1)$ in (1.3.1) can be replaced by $O(1)$.

and

(iii)
$$\sum_{n=2}^{\infty} \frac{\lambda_n}{n} \log n < \infty .$$

Then, if

$$\int_0^t |\varphi(u)|\,du = O(t)\,, \qquad as \qquad t \to 0\,,$$

$\sum \lambda_n A_n(x)$ *is summable* $|N, p_n|$.

2.2. We require the following lemmas:

LEMMA 1. *Let* t_n *be the* n*-th Nörlund mean of the series* $\sum U_n$. *Then*

$$\frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} \left(\frac{P_{n-1}}{p_n} - \frac{P_{n-\nu-1}}{p_{n-\nu}} \right) U_\nu = \frac{P_{n-1}}{p_n} (t_n - t_{n-1}) .$$

LEMMA 2. *Let* $0 < t < 2\pi$, *and*

$$S_n(t) = \sum_{\nu=0}^{n} p_{n-\nu} \left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}} \right) \cos \nu t .$$

Then

$$S_n(t) = \begin{cases} O(nP_n)\,, & for \quad 0 < t \leqslant 1/n; \\ O(np_n t^{-1})\,, & for \quad 1/n < t \leqslant \pi . \end{cases}$$

Proof. For $0 < t \leqslant 1/n$,

$$S_n(t) = O\left\{ \sum_{\nu=0}^{n} p_{n-\nu} \left| \frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}} \right| |\cos \nu t| \right\} = O\left(\frac{P_n}{p_n} \sum_{\nu=0}^{n} p_{n-\nu} \right) + O\left(\sum_{\nu=0}^{n} P_{n-\nu} \right)$$
$$= O(P_n^2/p_n) + O(nP_n) = O(nP_n) .$$

For $1/n < t \leqslant \pi$, applying Abel's transformation,

$$S_n(t) = \sum_{\nu=0}^{n-1} \Delta\left(- \frac{P_{n-\nu}}{p_{n-\nu}} \right) \sum_{\mu=0}^{\nu} p_{n-\mu} \cos \mu t + \left(\frac{P_n}{p_n} - \frac{P_0}{p_0} \right) \sum_{\nu=0}^{n} p_{n-\nu} \cos \nu t$$
$$= O\left\{ \sum_{\nu=0}^{n-1} \left| \Delta\left(- \frac{P_{n-\nu}}{p_{n-\nu}} \right) \right| p_n \max_{0 \leqslant \nu' \leqslant \nu} \left| \sum_{\mu=0}^{\nu'} \cos \mu t \right| \right\} +$$

$$+O\left\{\left|\left(\frac{P_n}{p_n}-\frac{P_{n-1}}{p_{n-1}}+\ldots-\frac{P_0}{p_0}\right)\right| p_n \max_{0\leqslant m\leqslant n}\left|\sum_{\nu=0}^{m}\cos\nu t\right|\right\}$$

(by Abel's Lemma)

$$=O\left(p_n\sum_{\nu=0}^{n}t^{-1}\right)+O(np_nt^{-1})=O(np_nt^{-1}).$$

LEMMA 3. *Let*

$$S_{n\nu}(t)=\sum_{\mu=0}^{\nu}p_{n-\mu}\left(\frac{P_n}{p_n}-\frac{P_{n-\mu}}{p_{n-\mu}}\right)\cos\mu t.$$

Then

$$S_{n\nu}(t)=\begin{cases}O(\nu^2p_n), & \text{for}\quad 0<t\leqslant 1/n;\\ O(\nu p_nt^{-1}), & \text{for}\quad 1/n<t\leqslant\pi.\end{cases}$$

Proof. By Abel's transformation,

$$S_{n\nu}(t)=\sum_{\mu=0}^{\nu-1}\Delta\left(-\frac{P_{n-\mu}}{p_{n-\mu}}\right)\sum_{k=0}^{\mu}p_{n-k}\cos kt+\left(\frac{P_n}{p_n}-\frac{P_{n-\nu}}{p_{n-\nu}}\right)\sum_{\mu=0}^{\nu}p_{n-\mu}\cos\mu t.$$

Using Abel's Lemma, we have

$$S_{n\nu}(t)=O\left\{\sum_{\mu=0}^{\nu-1}\left|\Delta\left(-\frac{P_{n-\mu}}{p_{n-\mu}}\right)\right|p_n\max_{0\leqslant\mu'\leqslant\mu}\left|\sum_{k=0}^{\mu'}\cos kt\right|\right\}+$$

$$+O\left\{\left|\left(\frac{P_n}{p_n}-\frac{P_{n-1}}{p_{n-1}}+\ldots-\frac{P_{n-\nu}}{p_{n-\nu}}\right)\right|p_n\max_{0\leqslant\nu'\leqslant\nu}\left|\sum_{\mu=0}^{\nu'}\cos\mu t\right|\right\}$$

$$=O\left(\sum_{\mu=0}^{\nu-1}p_n\mu\right)+O(\nu p_n\nu)=O(\nu^2p_n)$$

and for $1/n<t\leqslant\pi$, by Abel's Lemma,

$$S_{n\nu}(t)=O\left\{\sum_{\mu=0}^{\nu-1}\left|\Delta\left(-\frac{P_{n-\mu}}{p_{n-\mu}}\right)\right|p_n\max_{0\leqslant\mu'\leqslant\mu}\left|\sum_{k=0}^{\nu'}\cos kt\right|\right\}+$$

$$+O\left\{\left|\left(\frac{P_n}{p_n}-\frac{P_{n-1}}{p_{n-1}}+\ldots-\frac{P_{n-\nu}}{p_{n-\nu}}\right)\right|p_n\max_{0\leqslant\nu'\leqslant\nu}\left|\sum_{\mu=0}^{\nu'}\cos\mu t\right|\right\}$$

$$=O\left(p_n\sum_{\mu=0}^{\nu-1}t^{-1}\right)+O(\nu p_nt^{-1})=O(\nu p_nt^{-1}).$$

LEMMA 4. *Let*

$$K_n(t) = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}\left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}}\right)\lambda_\nu \cos\nu t\,.$$

Then

$$K_n(t) = \begin{cases} O\left(\dfrac{p_n}{P_n}\displaystyle\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\right) + O(n\lambda_n)\,, & \text{for} \quad 0 < t \leqslant 1/n; \\ O\left(\dfrac{p_n}{P_n}t^{-1}\displaystyle\sum_{\nu=1}^{n-1}\nu\Delta\lambda_\nu\right) + O\left(\lambda_\nu\dfrac{np_n}{P_n}t^{-1}\right), & \text{for} \quad 1/n < t \leqslant \pi\,. \end{cases}$$

Proof. By Abel's transformation,

$$K_n(t) = \frac{1}{P_n}\sum_{\nu=0}^{n-1}\Delta\lambda_\nu S_{n\nu}(t) + \frac{\lambda_n}{P_n}S_n(t) = O\left(\frac{p_n}{P_n}\sum_{\nu=1}^{n-1}\nu^2\Delta\lambda_\nu\right) + O(n\lambda_n)\,,$$

for $0 < t \leqslant 1/n$.

For $1/n < t \leqslant \pi$, we have

$$K_n(t) = O\left(\frac{p_n}{P_n}\sum_{\nu=0}^{n-1}\nu\Delta\lambda_\nu t^{-1}\right) + O\left(\frac{\lambda_n}{P_n}np_n t^{-1}\right)$$

$$= O\left(\frac{p_n}{P_n}t^{-1}\sum_{\nu=1}^{n-1}\nu\Delta\lambda_\nu\right) + O\left(\lambda_n\frac{np_n}{P_n}t^{-1}\right).$$

LEMMA 5. *If $\{\lambda_n\}$ is a monotonic non-increasing sequence such that*

$$\sum n^{-1}\lambda_n < \infty\,,$$

then

$$\sum_{1}^{\infty}\log(n+1)\Delta\lambda_n < \infty\,.$$

Proof. By Abel's transformation,

$$\sum_{n=1}^{m}\Delta\lambda_n\log\overline{n+1} = \sum_{n=1}^{m-1}\Delta\log\overline{n+1}\sum_{\nu=1}^{n}\Delta\lambda_\nu + \log(m+1)\sum_{n=1}^{m}\Delta\lambda_n$$

$$= \lambda_1\log 2 - \lambda_{m+1}\log(m+1) - \sum_{n=1}^{m-1}\Delta\log\overline{n+1}\,\lambda_{n+1}\,.$$

But, as $n\to\infty$,

$$\Delta\log(n+1) = O\left(\frac{1}{n+1}\right) \quad \text{and} \quad \lambda_n\log n = O(1)\,.$$

Hence,

$$\sum_{n=1}^{m}\Delta\lambda_n\log(n+1) = O(1)\,, \quad \text{as} \quad m\to\infty\,.$$

This proof is somewhat more general than the original lemma of Pati (see Pati [7], Lemma 3; see also Daniel [2]) where Dr. Pati suggested this.

3. Proof of the Theorem. By virtue of Lemma 1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \int_0^{\pi} |\varphi(t)|\,|K_n(t)|\,dt < \infty .$$

Now,

$$\sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \int_0^{\pi} |\varphi(t)|\,|K_n(t)|\,dt$$

$$= \sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \int_0^{1/n} |\varphi(t)|\,|K_n(t)|\,dt + \sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \int_{1/n}^{\pi} |\varphi(t)|\,|K_n(t)|\,dt$$

$$= \Sigma_1 + \Sigma_2 , \text{ say.}$$

Then, by hypothesis,

$$\Sigma_1 = O\left(\sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \frac{1}{n} \frac{p_n}{P_n} \sum_{\nu=1}^{n-1} \nu^2 \Delta\lambda_\nu\right) + O\left(\sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \frac{1}{n} \lambda_n n\right)$$

$$= O\left(\sum_{n=2}^{\infty} n^{-3} \sum_{\nu=1}^{n-1} \nu^2 \Delta\lambda_\nu\right) + O\left(\sum_{n=2}^{\infty} \frac{\lambda_n}{n}\right) = O\left(\sum_{\nu=1}^{\infty} \nu^2 \Delta\lambda_\nu \sum_{n=\nu+1}^{\infty} n^{-3}\right) + O(1)$$

$$= O\left(\sum_{\nu=1}^{\infty} \Delta\lambda_\nu\right) + O(1) = O(1) .$$

And

$$\Sigma_2 = O\left(\sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \frac{p_n}{P_n} \sum_{\nu=1}^{n-1} \nu \Delta\lambda_\nu \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\,dt\right) +$$

$$+ O\left(\sum_{n=2}^{\infty} \frac{p_n}{P_{n-1}} \frac{p_n}{P_n} n\lambda_n \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\,dt\right)$$

$$= O\left(\sum_{n=2}^{\infty} \frac{\log n}{n^2} \sum_{\nu=1}^{n-1} \nu\Delta\lambda_\nu\right) + O\left(\sum_{n=2}^{\infty} \frac{\lambda_n}{n} \log n\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} \nu\Delta\lambda_\nu \sum_{n=\nu+1}^{\infty} n^{-2} \log n\right) + O(1)$$

$$= O\left(\sum_{\nu=1}^{\infty} \log(\nu+1) \Delta\lambda_\nu\right) + O(1) = O(1) ,$$

by Lemma 4.

This completes the proof of the theorem.

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