

On cosine and sine functional equations

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Introduction. In this paper, the functional equations

$$(A) \quad f(xy) + f(xy^{-1}) = 2f(x)f(y)$$

and

$$(C) \quad f(xy)f(xy^{-1}) = f(x)^2 - f(y)^2,$$

where f is a complex-valued function on an arbitrary group G , are considered. Let g be a homomorphism of G into the multiplicative group of the complex numbers, K . Then, it is evident that

$$(B) \quad f(x) = \frac{g(x) + g^*(x)}{2}$$

and

$$(D) \quad f(x) = \frac{g(x) - g^*(x)}{2}$$

are solutions of (A) and (C) respectively, where $g^*(x) = g(x)^{-1}$.

1. First we shall consider the equation (A) and prove the following

THEOREM 1. *Let I be any arbitrary index set. Well order I . Let $\{G_\alpha\}$, ($\alpha \in I$) be a family of groups such that for every α in I , each solution of (A) on G_α is of the form (B) and $G_\alpha \subsetneq G_\beta$, $\alpha < \beta$, $\alpha, \beta \in I$. Then every solution of (A) on $G = \bigcup G_\alpha$, α in I , is of the form (B), provided $f(xyz) = f(xzy)$, for all x, y, z in G .*

Proof. Here we quote the following results, Lemma 1 and Lemma 2 (which are Lemma 1 and Theorem 3 of [2]) without proof, which will be used in the sequel.

LEMMA 1. *Let G be any group. Let f be a function on G with the properties that (1) f satisfies (A) on G , (2) $f(x)$ assumes the values ± 1 only on G and (3) $f(xyz) = f(xzy)$, for all x, y, z in G . Then f is of the form (B).*

LEMMA 2. *Let*

$$\frac{g_1(x) + g_1^*(x)}{2} = \frac{g_2(x) + g_2^*(x)}{2},$$

for all x in G , where g_1 and g_2 are homomorphisms of G into K . Then either $g_2 = g_1$ or $g_2^* = g_1$.

The proof of the theorem is based on transfinite induction. Let $G = \bigcup G_\alpha$, $\alpha \in I$. Let $f: G \rightarrow K$ be a solution of (A). Now we will prove that f is of the form (B). Let $f_\alpha = f|G_\alpha$ (f restricted to G_α) assume the values ± 1 , for every α in I . Then evidently $f = \pm 1$ on G and the proof follows by Lemma 1. Without loss of generality we can assume that

$$(1.1) \quad f|G_\alpha = f_\alpha \neq \pm 1, \quad \text{for each } \alpha \text{ in } I.$$

Otherwise there is a least index $\beta \in I$ such that $f_\beta \neq \pm 1$. Then $G = G_\beta \cup G_{\beta+1} \cup \dots$. Then we can consider the index set $J = \{\beta, \beta+1, \dots\}$ and we have $f_\gamma \neq \pm 1$, $\gamma \in J$. Let $\alpha < \beta$, $\alpha, \beta \in I$. By hypothesis, there are homomorphisms $g_\gamma: G_\gamma \rightarrow K$ ($\gamma = \alpha, \beta$) such that

$$(1.2) \quad f_\gamma = \frac{g_\gamma + g_\gamma^*}{2}.$$

Since $\alpha < \beta$ by Lemma 2, it follows that either g_β or g_β^* is the extension of g_α . If both are extensions of g_α , we would have $g_\alpha = \pm 1$ and so $f_\alpha = \pm 1$, a contradiction to (1.1). We call the unique extension of g_α to G_β as g_β . Now, we shall apply the transfinite induction. Let $\alpha \in I$ be such that the above result is true for the initial segment $s(\alpha)$, that is, for every β and $\gamma < \alpha$ (we may take $\beta < \gamma < \alpha$), $g_\beta: G_\beta \rightarrow K$ a homomorphism satisfying (B) on G_β has a unique extension g_γ to G_γ , also a homomorphism on G_γ into K , satisfying (1.2) on G_γ . Now, we will prove that there is a unique extension to G_α ; that is, there is a g_α on G_α such that g_α is the extension of g_β , $\beta < \alpha$. Let $H = \bigcup G_\beta$, $\beta < \alpha$. Let us define $g: H \rightarrow K$, as follows:

$$(1.3) \quad g(x) = g_\gamma(x), \quad x \in G_\gamma \setminus G_\beta, \quad \beta < \gamma < \alpha.$$

Now, g is well defined and further g is a homomorphism on H . In fact, let $x, y \in H$. Then there are $\beta, \gamma < \alpha$ such that $x \in G_\beta \setminus G_\mu$, $\mu < \beta < \alpha$ and $y \in G_\gamma \setminus G_\nu$, $\nu < \gamma < \alpha$. Let $\beta \leq \gamma$. Then $xy \in G_\gamma \setminus G_\nu$, $\nu < \gamma < \alpha$. By using the hypothesis and (1.3), we obtain

$$g(xy) = g_\gamma(xy) = g_\gamma(x)g_\gamma(y) = g_\beta(x)g_\gamma(y) = g(x)g(y).$$

Therefore, g is a homomorphism on H . Further

$$(1.4) \quad f|H = \frac{g + g^*}{2}.$$

Now consider G_α . By hypothesis, there is a $g_\alpha: G_\alpha \rightarrow K$, a homomorphism such that

$$(1.5) \quad f_\alpha = \frac{g_\alpha + g_\alpha^*}{2}.$$

So,

$$f|_H = \frac{g_\alpha + g_\alpha^*}{2} \Big|_H = \frac{g + g^*}{2} .$$

Hence, by Lemma 2, either g_α or g_α^* restricted to H gives g . Otherwise, we would have, $g_\beta = \pm 1$, for every $\beta < \alpha$, a contradiction to (1.1). We shall call the unique extension of g to G_α as g_α . Then g_α is the required unique extension to G_α , satisfying (1.5). Thus we have g_α on G_α is such that $g_\alpha|_{G_\beta} = g_\beta$, $\beta < \alpha$, $\alpha \in I$. Now, let us define $h: G \rightarrow K$, as follows:

$$(1.6) \quad h(x) = g_\alpha(x), \quad \text{where } x \in G_\alpha \setminus G_\beta, \alpha < \beta, \alpha, \beta \in I .$$

Then h is well defined. Further, proceeding as above we can show that h is a homomorphism on G such that $f = (h + h^*)/2$. This completes the proof of the theorem.

From Theorem 1, we can easily deduce the following two corollaries using the following Lemma 3 (which is Theorem (2.5) of [1]).

LEMMA 3. *Let G be any cyclic group finite or infinite. Then every complex valued function on G which is also a solution of (A) has the form (B).*

COROLLARY 1. *Consider the additive group of rationals Q . Let $G_n = \{k/n! : k \in Z\}$, where Z denotes the set of integers and n any positive integer greater than or equal to 1. Then evidently $Q = \bigcup G_n, n \geq 1$. G_n is a cyclic group for every n and $G_n \subsetneq G_m$, for $n < m$. Also it follows by Lemma 3 that every solution of (A) on G_n , for every n , has the form (B). Hence it follows from Theorem 1 that every solution of (A) on Q has the form (B).*

COROLLARY 2. *Consider the multiplicative group $Z(p^\infty)$, p a prime. Then*

$$Z(p^\infty) = \bigcup Z(p^n), \quad \text{for } n = 0, 1, 2, \dots$$

where $Z(p^n) = \{\exp(2k\pi i/p^n) : k \in Z\}$, Z being the set of integers. $Z(p^n)$ is a cyclic group for every n and $Z(p^n) \subsetneq Z(p^m)$, for $n < m$. Therefore by Lemma 3, it follows that every solution of (A) on $Z(p^n)$ for every n , has the form (B). Hence, again using Theorem 1, we see that every solution of (A) on $Z(p^\infty)$ has the form (B).

2. Now we take up the equation (C) and prove the following

THEOREM 2. *Let G be a cyclic group. Let f be a complex-valued function on G satisfying (C) with the properties that (1) f is not identically zero and (2) f is of the form (D). Then we assert that there is one and only one homomorphism g of G into K satisfying (D).*

Proof. Suppose that there are two homomorphisms $g_i: G \rightarrow K$ ($i = 1, 2$) such that

$$(2.1) \quad f(x) = \frac{g_i(x) - g_i^*(x)}{2},$$

for all x in G .

Now (2.1) gives that

$$g_1(x) - g_1^*(x) = g_2(x) - g_2^*(x),$$

for all x in G , equivalently,

$$g_1(x) - g_2(x) - g_1^*(x) + g_2^*(x) = 0,$$

hence

$$g_1(x) - g_2(x) + \frac{g_1(x) - g_2(x)}{g_1(x)g_2(x)} = 0,$$

which is the same as

$$[g_1(x) - g_2(x)][g_1(x)g_2(x) + 1] = 0.$$

From the above we conclude that either for each x ,

$$(2.2) \quad g_2(x) = g_1(x) \quad \text{or} \quad g_2(x) = -g_1^*(x).$$

Let G be generated by a . From (2.2), we would have either

$$(2.3) \quad g_2(a) = g_1(a)$$

which in turn implies that g_2 is the same as g_1 , since G is a cyclic group and there is nothing to prove, or

$$(2.4) \quad g_2(a) = -g_1^*(a).$$

If for every n in Z , $g_2(a^n) = -g_1^*(a^n)$, then g_2 cannot be a homomorphism. So, this case cannot happen. The only other possibility is that with (2.4), for some n in Z , we have

$$(2.5) \quad g_2(a^n) = g_1(a^n).$$

Let n_0 be the smallest positive integer satisfying (2.5). That is,

$$(2.6) \quad g_2(a^n) = \begin{cases} -g_1^*(a^n), & n < n_0, \\ g_1(a^n), & n = n_0. \end{cases}$$

Now let us consider $g_2(a^{n_0+1})$. From (2.2) we distinguish two cases according as $g_2(a^{n_0+1}) = g_1(a^{n_0+1})$ or $g_2(a^{n_0+1}) = -g_1^*(a^{n_0+1})$. First we consider the former case. Now,

$$g_2(a^{n_0+1}) = g_2(a^{n_0})g_2(a) = g_1(a^{n_0})[-g_1^*(a)] = -g_1(a^{n_0-1}).$$

Hence

$$g_1(a^{n_0+1}) = -g_1(a^{n_0-1}),$$

that is,

$$g_1(a^{n_0-1})[g_1(a^2) + 1] = 0.$$

Now $g_1(a^{n_0-1}) = 0$ implies $g_1(a) = 0$, which cannot be, leaving the other possibility that $g_1(a^2) = g_1(a)^2 = -1$. So, $g_1(a) = \pm i$. Now $g_1(a) = i$ implies $-g_1^*(a) = -1/i = i$. Hence $g_2(a) = -g_1^*(a) = g_1(a)$, whence g_2 is

identically equal to g_1 . Similarly we can establish in the case where $g_2(a) = -i$, that g_2 is the same as g_1 .

Finally, we are left with the remaining case, $g_2(a^{n_0+1}) = -g_1^*(a^{n_0+1})$. But, we have

$$g_2(a^{n_0+1}) = g_2(a^{n_0-1})g_2(a^2) = [-g_1^*(a^{n_0-1})][-g_1^*(a^2)] = g_1^*(a^{n_0+1}),$$

provided $n_0 \geq 3$.

So, we have, $g_1^*(a^{n_0+1}) = -g_1^*(a^{n_0+1})$. This means $g_1^*(a) = 0$, which is false. Hence this possibility cannot occur. If $n_0 = 2$, (2.6) becomes

$$(2.7) \quad g_2(a) = -g_1^*(a) \quad \text{and} \quad g_2(a^2) = g_1(a^2).$$

In case

$$g_2(a^3) = -g_1^*(a^3),$$

we have

$$g_2(a^3) = g_2(a^2)g_2(a) = g_1(a^2)[-g_1^*(a)] = -g_1(a) = -g_1^*(a^3).$$

So, $g_1(a^4) = 1$, giving $g_1(a) = \pm 1, \pm i$. $g_1(a) = \pm 1$, implies that f is a zero function, which contradicts the hypothesis (1). The equality $g_1(a) = \pm i$ as above shows that g_2 and g_1 coincide on G . Lastly we may have $g_2(a^3) = g_1(a^3)$. From (2.7), we have

$$g_2(a^3) = g_1(a^3) = g_1(a^2)g_1(a) = g_2(a^2)g_1(a).$$

Thus $g_2(a) = g_1(a)$. Hence g_2 and g_1 are one and the same on G . Thus we have established the uniqueness of the homomorphism g , when g satisfies (D). Hence the proof of the theorem is complete.

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References

- [1] Pl. Kannappan, *The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for Abelian groups*, Ph. D. thesis, University of Washington, Seattle 1964.
 [2] — *The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups*, Proc. Amer. Math. Soc. (to appear).

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