ANNALES
POLONICI MATHEMATICI
XXI (1969)

Topology of Laplace transformable functions

by T. K. MUKHERJEE (Jadavpur) and S. GANGULY (Barasat)

Introduction. We consider here the set of all functions that are Laplace transformable with regard to their structure both algebraic and topological. Dutta [2] has revealed some very interesting features in his recent work but, for the sake of convenience, he has considered both the object and the image spaces of transformable function as L_2-spaces. But Doetsch [1] has pointed out that they are not L_2-spaces Sneddon [3] remarked that the spaces of transformable functions are Banach spaces. We shall study certain topological properties of the set of Laplace transformable functions with the help of a metric which is natural in a sense to be made clear subsequently.

Let s_f denote the abscissa of convergence (see [1]) for a Laplace transformable function f, as such for all $s > s_f$, $\int_0^\infty e^{-sx} f dx$ exists in the Lebesgue sense and is finite, i.e. $e^{-sx} f \in L_1[0, \infty) = L$. Evidently we cannot assert that $\int_0^\infty e^{-s_f x} f dx$ will exist and will be finite. In fact, s_f can be obtained by a Dedekind cut and as such the behaviour at s_f cannot be ensured. It is easy to see that there is no loss of generality if we restrict s_f in $[0, \infty)$ since a function is already in a L_1-space if its abscissa of convergence is less than zero. For our convenience we use L. T. for Laplace transformable functions throughout this paper.

Algebraic Structure.

THEOREM. *The L.T.-set is an Abelian group with respect to the operation of addition (in the usual sense).*

Proof. Let f_1 and f_2 be two functions and let us suppose that they belong to L.T.-set, i.e. there exist s_1 and s_2 such that $\int_0^\infty e^{-s_1 x} f_1 dx$ and $\int_0^\infty e^{-s_2 x} f_2 dx$ exist. Evidently, if we take $s = \max^m(s_1, s_2)$, $\int_0^\infty e^{-sx}(f_1+f_2) dx$ exists. Thus the set is closed for addition. The associative property is evident. The null element and the additive inverse are respectively the ordinary zero and $-f(x)$. Since $f(x)$ is L.T., $-f(x)$ is also so. The commutative property is obvious. Hence the theorem.

It becomes now very easy to verify that our L.T.-set is a linear system.

Note 1. If we consider only the positive L.T.-set, the set becomes only an Abelian semi-group.

We now define certain symbols which we use throughout. C_s — the class of all functions in the L.T.-set such that $s_f = s$. Then $C_s = C_s^1 \cup C_s^2$, where

$$\{f : e^{-sx} f \in L_1[0, \infty)\} = C_s^1 \quad \text{and} \quad \{f : e^{-sx} f \notin L_1[0, \infty)\} = C_s^2 .$$

If $s = 0$, then $C_0 = C_{0-} \cup C_{0+}$, where $C_{0-} = \{f : s_f < 0\}$ and $C_{0+} = \{f : s_f = 0\} = C_0^1 \cup C_0^2$. Hence $C_0 = C_{0-} \cup C_0^1 \cup C_0^2$. Thus the L.T.-set $= \bigcup_r C_r = (\bigcup_r C_r^1) \cup (C_r^2) = C^1 \cup C^2$ (r being real and $\geqslant 0$).

Again evidently for $s > 0$ and $t > 0$, $C_t = e^{(t-s)x} C_s$; for $s = 0$, $C_t = e^{tx} C_{0+}$, i.e. $C_t^1 = e^{tx} C_{0+}^1$ and $C_t^2 = e^{tx} C_{0+}^2$. We define $L^s[0, \infty)$ as a Banach space with the norm

$$\|f\| = \int_0^\infty e^{-sx} |f(x)|\, dx < \infty .$$

The norm introduces the metric

$$\varrho(f, g) = \int_0^\infty e^{-sx} |f(x) - g(x)|\, dx < \infty .$$

Definition of metric. Let f and g be two functions belonging to the L.T.-set. We define the metric as follows:

$$\varrho(f, g) = |s_f - s_g| + \frac{\int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|\, dx}{1 + \int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|\, dx} .$$

Note 2. s_f can be looked upon as a functional on the L.T.-set. It may be seen that in this topology s_f is a continuous functional.

We may now show that the metric satisfies all the required conditions:

(i) *If $f = g$, evidently $\varrho(f, g) = 0$; conversely, if $\varrho(f, g) = 0$,*

$$|s_f - s_g| + \frac{\int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|\, dx}{1 + \int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|\, dx} = 0 .$$

The two portions being separately positive, they must vanish separately, i.e. $|s_f - s_g| = 0$ giving

$$s_f = s_g \quad \text{and} \quad \int_0^\infty |e^{-s_f x} f - e^{-s_g x} g|\, dx = 0 ,$$

i.e. $e^{-s_f x} f = e^{-s_g x} g$.

But $s_f = s_g$; hence $f = g$.

The property of symmetry and transitivity being very obvious it follows that ϱ is a metric.

Note 3. $s_{f_n} \to s_f$ if

$$\varrho(f_n, f) \to 0 \quad \text{and} \quad \begin{Bmatrix} f_n \to f \\ s_{f_n} \to s_f \end{Bmatrix} \Leftrightarrow g_n \to f$$

where $g_n \in C_{s_f}$, g_n being equal to $\exp\{(s_{f_n} - s_f)x\} f_n(x)$.

This shows that if $f_n \to f$, then there is a sequence $g_n \in C_{s_f}$ such that $g_n \to f$ so that any convergent sequence can always be taken to be confined in a given class C_s.

Note 4. s_f being a continuous linear functional on the L.T.-space and C_s being equal to $\{f : s_f = s\}$ it follows that every C_s is closed.

Note 5. In C_s^1 the metric becomes $\varrho(f, g) = \int_0^\infty e^{-sx} |f(x) - g(x)|\, dx$ so that the relative topology in C_s^1 induced by the L.T.-space is the same as the one induced by $L_s[0, \infty)$. Suppose now that C_s is given with its topology as induced by $L_s[0, \infty)$; then one way of metrising $\bigcup C_s^1$ so that each subspace C_s^1 has the same relative topology as above, is given by our metric.

Note 6. The usual uniform convergence in L.T.-space does not imply convergence as induced by the above metric.

We now study C_0 alone, but these considerations can easily be extended to C_s.

In C_0 we define a relation between f and g:

$$fRg \quad \text{iff} \quad \int_0^\infty |f(x) - g(x)|\, dx < \infty .$$

Evidently R is an equivalence relation. Obviously since $g(x) = f(x) + [g(x) - f(x)]$, it follows that $|g(x)| \leqslant |f(x) - g(x)| + |f(x)|$, where $\big(g(x) - f(x)\big) \in L_1$. It follows that C_0 is partitioned into disjoint classes and each class is of the form $(f + L_1)$, where $\int_0^\infty |f(x)|\, dx = \infty$. The distance between any two elements of two classes is 1. In fact, these are elements of the factor space of C_0 with respect to L_1 in C_0. Thus the factor space in its quotient topology is discrete. This is not very unnatural since our metric has not made any use of the crucial property of a function $f \in C_0^2$, i.e. $\int_0^\infty e^{-\varepsilon x}|f(x)|\,dx < \infty$ for every $\varepsilon > 0$.

It appears that our metric is not sensitive enough for studying the L.T.-set. Perhaps we can consider a different metric in this way:

$$\varrho(f, g) = |s_f - s_g| + \sum_{n=1}^{a} \frac{1}{2^n} \frac{\{\int_0^n |f-g|^p dx\}^{1/p}}{1+\{\int_0^n |f-g|^p dx\}^{1/p}} .$$

The metric introduces the L_p-convergence on each compact subset of reals. Actually, by this metric the distance between two classes does not become unity and the factor space topology will not be discrete.

THEOREM. *The L.T.-space is complete with our metric.*

Let $\{f_n\}$ be a Cauchy sequence in the L.T.-space.

Let $N_1 \subseteq N$, N being the set of positive integers, be defined as $N_1 = \{n : g_n(x) \in C_0^1\}$ and $N_2 \subseteq N = \{n : g_n(x) \in C_0^2\}$.

Then neither N_1 nor N_2 can be infinite for that would contradict the fact that $\{f_n\}$ is a Cauchy sequence.

Now

$$\varrho(f_n, f_{n+m}) = |s_{f_n} - s_{f_{n+m}}| + \frac{\int_0^\infty |\exp\{-s_{f_n}x\}f_n - \exp\{-s_{f_{n+m}}x\}f_{n+m}|\,dx}{1+\int_0^\infty |\exp\{-s_{f_n}x\}f_n - \exp\{-s_{f_{n+m}}x\}f_{n+m}|\,dx} .$$

But since the real number space is complete and $C_{0-} \cup C_0^1 = L_1$ is complete, then $g_n \to g$, i.e. $f_n(x) \to \{e^{sx}g(x)\}$. If g_n, $n \geqslant N_0$, belongs to C_0^2, then $g_n \to (f)$ for $n \geqslant N_0$, where $f \in C_0^2$. Then $g_n \to f + L_1$, i.e.

$$\int_0^a |(g_n - f) - (g_{n+m} - f)|\,dx = \int_0^a |g_n' - g_{n+m}'|\,dx \to 0 .$$

L_1 being complete, $g_n' \to g'$, $g' \in L_1$ and $g_n \to f + g'$, i.e. $f_n \to e^{sx}(f+g') \in$ $\in$ L.T.-space.

Hence the L.T.-space is complete.

THEOREM. *The L.T.-space is disconnected.*

Proof. We know that the L.T.-space $= C^1 \cup C^2$. We have just shown that C^1 is complete, and so evidently it is closed. Similarly C^2 is closed.

Hence the L.T.-space is disconnected.

Note 7. Every C_s thus becomes disconnected in its relative topology and $C_s = C_s^1 \cup C_s^2$, where C_s^1 and C_s^2 are relatively closed in C_s.

SEPARABILITY. *Theorem* $-C^1$ *is separable in the relative topology.*

Proof. In fact, let $\{s_n\}$ be a countable dense subset of $[0, \infty)$. Then let $\{f_n\}$ be a dense subset of $[C_{0-} \cup C^1_{0+}]$ which in its relative topology is identical with $L_1[0, \infty)$. Then $\{e^{s_n x} f_m(x)\}$ is a dense subset in C^1 as can easily be seen.

Hence the proof.

We shall now consider only the positive functions. The continuity of the additive operation can be proved in the following manner:

Let $s_f, s_g, s_{f_n}, s_{g_n}$ be the abscissas of convergence for f, g, f_n and g_n respectively.

Case I. Let $s_g > s_f$; then there is a neighbourhood of s_g in which there is no element of the sequence s_{f_n}; and, since s_g is the limit of s_{g_n}, this neighbourhood contains all s_{g_n} for $n \geqslant n_0$.

Hence for all $n \geqslant n_0$ the abscissa of convergence of $f_n + g_n$ is s_{g_n}:

$$
\begin{aligned}
\varrho(f_n+g_n, f+g) = {} & |s_{g_n} - s_g| + \\
& + \int_0^\infty |\exp\{-s_{g_n}x\}(f_n+g_n) - \exp\{-s_g x\}(f+g)|\,dx \\
\leqslant {} & |s_{g_n} - s_g| + \int_0^\infty |\exp\{-s_{g_n}x\}g_n - \exp\{-s_g x\}g|\,dx + \\
& + \int_0^\infty |\exp\{-s_{g_n}x\}f_n - \exp\{-s_g x\}f|\,dx \\
\leqslant {} & \varrho(g_n, g) + \int_0^\infty |\exp\{-s_{g_n}x + s_{f_n}x\}\exp\{-s_{f_n}x\}f_n - \\
& - \exp\{-s_{g_n}x + s_{f_n}x\}\exp\{-s_f x\}f|\,dx + \\
& + \int_0^\infty |\exp\{-s_{g_n}x + s_{f_n}x\}\exp\{-s_f x\}f - \\
& - \exp\{-s_g x + s_f x\}\exp\{-s_f x\}f|\,dx \\
= {} & \varrho(g_n, g) + \int_0^\infty |\exp\{-s_{g_n}x + s_{f_n}x\}| \times \\
& \times |\exp\{-s_{f_n}x\}f_n - \exp\{-s_f x\}f|\,dx + \\
& + \int_0^\infty |\exp\{-s_f x\}f|\,|\exp\{-s_{g_n}x + s_{f_n}x\} - \exp\{-s_g x + s_f x\}|\,dx \\
\leqslant {} & \varrho(g_n, g) + \varrho(f_n, f) + \int_0^\infty |\exp\{-s_f x\}f|\,|\exp\{-s_{g_n}x + s_{f_n}x\} - \\
& - \exp\{-s_g x + s_f x\}|\,dx \to 0
\end{aligned}
$$

as $n \to a$, i.e. $(f_n + g_n) \to (f + g)$ as $n \to a$.

Case II $s_f = s_g$. In this case in every neighbourhood of s_f and similarly in every neighbourhood of s_g there are elements of s_{f_n} also. But in a certain case $s_{f_n+g_n} = s_{f_n}$, and in other cases s_{g_n}. But in all cases we have

$$\varrho(f_n+g_n, f+g) = |s_{f_n} - s_f| + \int_0^\infty \exp\{-s_{f_n}x\}(f_n+g_n) - \exp\{-s_f x\}(f+g)\,dx$$

or

$$|s_{g_n} - s_g| + \int_0^\infty |\exp\{-s_{g_n}x\}(f_n+g_n) - \exp\{-s_g x\}(f+g)|\,dx$$

$$\leq \varrho(g_n, g) + \varrho(f_n, f) + \left\{\begin{matrix}\int_0^\infty |\exp\{-s_g x\}g|\,|\exp\{s_{f_n}x + s_{g_n}x\} - \exp\{-s_f x + s_g x\}|\,dx \\ + \\ \int_0^\infty |\exp\{-s_f x\}f|\,|\exp\{-s_{g_n}x + s_{f_n}x\} - \exp\{-s_g x + s_f x\}|\,dx\end{matrix}\right\} \to 0$$

as $n \to \infty$, i.e. $(f_n+g_n) \to (f+g)$ as $n \to \infty$.

This shows that $(f_n+g_n) \to (f+g)$ for all $f_n \to f$ and $g_n \to g$. Now the L.T.-space can be shown not to be a linear metric space with the metric as introduced above. The property that $\alpha_n f \to \alpha f$ is not valid for all funnctions f in the space. In fact, if we consider only the set of all positive functions, then that set will form a topological semi-group.

We express our thanks and gratitude to Prof. Dr. M. Dutta for his kind help and assistance in the preparation of the paper.

References

[1] G. Doetsch, *Theorie und Anwendung der Laplace transformation,* 1943.

[2] M. Dutta, *On abstract structure of theories of integral transforms,* Proc. Nat. Inst. Sci. India 32, A (4) (1966).

[3] I. N. Sneddon, *Fourier Transforms,* 1951.

Reçu par la Rédaction le 7. 8. 1967