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On some symmetrical equations of the form $f_1(x_1+\ldots+x_n) = \sum_{(i_1,\ldots,i_n)} f_1(x_{i_1})\ldots f_n(x_{i_n})$

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The equations

$$f_1(x_1+\ldots+x_n) = \sum_{(i_1,\ldots,i_n)\in P_k} f_1(x_{i_1})\ldots f_n(x_{i_n}) \tag{1}$$

(where P_k is a set of k permutations of the numbers $1,\ldots,n$) were considered in [2].

According to [2], equation (1) is called *symmetrical* if its right side is symmetrical for all the variables. In other cases it is called *asymmetrical.*

All asymmetrical equations (1) were solved (by some assumptions of regularity) in [2]. We repeat here the essential results:

If the numbers of components of the right side of (1) with $f_1(x_1)$, $f_1(x_2),\ldots,f_1(x_n)$ are not all equal, then the only non-trivial differentiable solutions of equation (1) are the functions

$$\begin{gathered} f_1(x) = A_1e^{ax}, \quad f_2(x) = A_2e^{ax}, \quad \ldots, \quad f_{n-1}(x) = A_{n-1}e^{ax}, \\ f_n(x) = \frac{1}{kA_2\ldots A_{n-1}}e^{ax}, \end{gathered} \tag{2}$$

where $A_1,\ldots,A_{n-1},a$ $(A_1\neq 0,\ldots,A_{n-1}\neq 0)$ are arbitrary constants and k is the number of components of the right side of (1).

If the numbers of components with $f_1(x_1), f_1(x_2),\ldots,f_1(x_n)$ in asymmetrical equation (1) are all equal, then non-trivial differentiable solutions of equation (1) have form (2) and

$$\begin{gathered} f_1(x) = A_1xe^{ax}, \quad f_2(x) = A_2e^{ax}, \quad \ldots, \quad f_{n-1}(x) = A_{n-1}e^{ax}, \\ f_n(x) = \frac{1}{k_1A_2\ldots A_{n-1}}e^{ax}, \end{gathered} \tag{3}$$

where $A_1,\ldots,A_{n-1},a$ $(A_1\neq 0,\ldots,A_{n-1}\neq 0)$ are arbitrary constants, and k_1 is the number of components with $f_1(x_i)$.

(These assumptions of regularity can be weakened for some equations (1)).

It remains to consider symmetrical equations (1). All these equations are satisfied by functions (2) and (3) but they have also other solutions.

For $n=2$ there is one symmetrical equation:

$$f_1(x_1+x_2) = f_1(x_1)f_2(x_2)+f_1(x_2)f_2(x_1)\,.$$

This equation has solutions different from (2) and (3). All its non-trivial solutions such that $f_1(x)$ and $f_2(x)$ have at least one point of continuity in common were given in [1].

For $n=3$ we have two symmetrical equations:

$$(*)\qquad f_1(x_1+x_2+x_3) = f_1(x_1)f_2(x_2)f_3(x_3)+f_1(x_2)f_2(x_3)f_3(x_1)+f_1(x_3)f_2(x_1)f_3(x_2)$$

and

$$(**)\qquad \begin{aligned} f_1(x_1+x_2+x_3) = {} & f_1(x_1)[f_2(x_2)f_3(x_3)+f_2(x_3)f_3(x_2)]+ \\ & +f_1(x_2)[f_2(x_3)f_3(x_1)+f_2(x_1)f_3(x_3)]+ \\ & +f_1(x_3)[f_2(x_1)f_3(x_2)+f_2(x_2)f_3(x_1)]\,. \end{aligned}$$

In this paper we shall prove some theorems on the regularity of their solutions, and we shall solve them in order to show that they have solutions different from (2) and (3).

We introduce the following notation:

$$f_{ia} = f_i(a), \qquad f'_{ia} = f'_i(a), \qquad f''_{ia} = f''_i(a) \qquad (i = 1, 2, 3)\,.$$

Remark I. Every solution $f_1(x), f_2(x), f_3(x)$ of equation $(*)$ (or $(**)$) satisfying the condition $f_{10}\neq 0$ can be written in the form

$$f_1(x) = A_1 g_1(x)\,,$$
$$f_2(x) = A_2 g_2(x)\,,$$
$$f_3(x) = A_3 g_3(x)\,,$$

where $g_1(x), g_2(x), g_3(x)$ is a solution of equation $(*)$ (or $(**)$) satisfying the condition $g_1(0)=1$.

Thus in the case of $f_{10}\neq 0$ we may assume that $f_{10}=1$ without loss of generality.

On equation $(*)$.

LEMMA I. *If* $f_{10}=1$, *then*

$$(4)\qquad f_{20}f_{30} = \tfrac{1}{3}\,,$$

$$(5)\qquad f_3(x) = \frac{1}{f_{20}}\left[\tfrac{2}{3}f_1(x)-f_{30}f_2(x)\right],$$

$$(6)\qquad f_1(x_1+x_2)=\frac{1}{f_{20}}\big[\tfrac{1}{3}f_1(x_1)f_2(x_2)+\tfrac{1}{3}f_1(x_2)f_2(x_1)+ \\ +\tfrac{2}{3}f_{20}f_1(x_1)f_1(x_2)-f_{30}f_2(x_1)f_2(x_2)\big]\,,$$

$$(7)\qquad f_2(x)=\begin{cases}\dfrac{f_{20}f_1(x+c)-\frac{1}{3}(f_{2c}+2f_{20}f_{1c})f_1(x)}{\frac{1}{3}f_{1c}-f_{30}f_{2c}} & \text{if}\quad \frac{1}{3}f_{1c}-f_{30}f_{2c}\neq 0\,,\\ f_{20}f_1(x) & \text{if}\quad \frac{1}{3}f_1(x)-f_{30}f_2(x)\equiv 0\,.\end{cases}$$

Proof. Putting in $(*)$ $x_1=x_2=x_3=0$, we obtain (4). Putting in $(*)$ $x_1=x$, $x_2=x_3=0$ and making use of (4), we obtain after computations (5). Substituting (5) into $(*)$ and putting $x_3=0$, we obtain (6).

If $\frac{1}{3}f_{1c}-f_{30}f_{2c}\neq 0$ for some c, it follows from (6) that for $x_1=x$, $x_2=c$ we have

$$f_2(x)=\frac{f_{20}f_1(x+c)-\frac{1}{3}(f_{2c}+2f_{20}f_{1c})f_1(x)}{\frac{1}{3}f_{1c}-f_{30}f_{2c}}\,.$$

If $\frac{1}{3}f_1(x)-f_{30}f_2(x)\equiv 0$, then

$$f_2(x)=\frac{1}{3f_{30}}f_1(x)\,.$$

Thus (7) is also satisfied. Q. E. D.

LEMMA II. *If* $f_{10}=0$ *and* $f_{1d}\neq 0$ *for some* d, *then*

$$(8)\qquad f_{20}f_{30}=1\,,$$

$$(9)\qquad f_1(x_1+x_2)=f_{30}f_1(x_1)f_2(x_2)+f_{20}f_1(x_2)f_3(x_1)\,,$$

$$(10)\qquad f_2(x)=\frac{f_1(x+d)-f_{20}f_{3d}f_1(x)}{f_{30}f_{1d}}\,,$$

$$(11)\qquad f_3(x)=\frac{f_1(x+d)-f_{30}f_{3d}f_1(x)}{f_{20}f_{1d}}\,.$$

Proof. Putting in $(*)$ $x_1=d$, $x_2=x_3=0$, we conclude that condition (8) is satisfied. Putting in $(*)$ $x_3=0$, we obtain (9). Hence for $x_1=d$, $x_2=x$ (10) follows, and analogously for $x_1=x$, $x_2=d$ we obtain (11). Q. E. D.

THEOREM I. *If* $f_1(x), f_2(x), f_3(x)$ *satisfy equation* $(*)$, $f_1(x)\not\equiv 0$, $f_1(x)$ *and* $f_2(x)$ (*or* $f_1(x)$ *and* $f_3(x)$) *have at least one point of continuity in common, then* $f_1(x), f_2(x), f_3(x)$ *are continuous everywhere.*

Proof. Let a be the common point of continuity of the functions $f_1(x)$ and $f_2(x)$.

Fix an arbitrary point x and let $x_1=x-a$, $x_2\to a$. Then $x_1+x_2\to x$ and it follows from the continuity of the right sides of (6) and (9) that $f_1(x)$ is continuous everywhere. Now, it follows from (7), (5), (10) and (11) that the functions $f_2(x), f_3(x)$ are also continuous everywhere. Q. E. D.

THEOREM II. *If $f_1(x), f_2(x), f_3(x)$ satisfy equation* $(*)$, $f_1(x) \not\equiv 0$, $f_1(x)$ *and* $f_2(x)$ *(or* $f_1(x)$ *and* $f_3(x)$*) have a point of differentiability in common, then* $f_1(x), f_2(x), f_3(x)$ *have all the derivatives.*

Proof. The proof of the existence of the first derivatives is similar to that of Theorem I.

We shall now prove that the function $f_1(x)$ has a second derivative. In the case of $f_{10} \neq 0$ we make use of Lemma I. Differentiating equality (6) by x_1 and then putting $x_1 = 0$, $x_2 = x$, we obtain

$$f_1'(x) = \frac{1}{f_{20}}\left[\tfrac{1}{3}f_{10}'f_2(x) + \tfrac{1}{3}f_{20}'f_1(x) + \tfrac{2}{3}f_{10}'f_1(x) - f_{30}f_{20}'f_2(x)\right].$$

This function is obviously differentiable, i.e. $f_1(x)$ has a second derivative.

If $f_{10} = 0$, we can do the same with equality (9) of Lemma II and we conclude that the function $f_1(x)$ always has a second derivative.

From (7), (5), (10) and (11) it follows that the second derivatives of the functions $f_2(x)$ and $f_3(x)$ also exist.

We then prove analogously that there exist derivatives of higher orders. Q. E. D.

THEOREM III. *The only non-trivial real solutions of equation* $(*)$ *such that* $f_1(x)$ *and* $f_2(x)$ *(or* $f_1(x)$ *and* $f_3(x)$*) have a point of differentiability in common are:*

(i)
$$f_1(x) = A_1 e^{ax}, \quad f_2(x) = A_2 e^{ax}, \quad f_3(x) = \frac{1}{3A_2} e^{ax};$$

(ii)
$$f_1(x) = A_1 e^{ax}\cos bx, \quad f_2(x) = A_2 e^{ax}(\cos bx + \varepsilon\sqrt{3}\sin bx), \quad f_3(x) = \frac{1}{3A_2} e^{ax}(\cos bx - \varepsilon\sqrt{3}\sin bx);$$

(iii)
$$f_1(x) = A_1 x e^{ax}, \quad f_2(x) = A_2 e^{ax}, \quad f_3(x) = \frac{1}{A_2} e^{ax};$$

(iv)
$$f_1(x) = A_1 e^{ax}\sin bx, \quad f_2(x) = A_2 e^{ax}\left(\cos bx + \varepsilon\frac{1}{\sqrt{3}}\sin bx\right), \quad f_3(x) = \frac{1}{A_2} e^{ax}\left(\cos bx - \varepsilon\frac{1}{\sqrt{3}}\sin bx\right)$$

($A_1 \neq 0$, $A_2 \neq 0$, $b \neq 0$, a — *arbitrary constants*, $\varepsilon = const = \pm 1$).

Proof. By Theorem II $f_1(x), f_2(x), f_3(x)$ have all the derivatives.

1) Consider first the case of $f_{10} \neq 0$. If we put

$$a = \frac{f'_{10}}{f_{10}}$$

and

$$\varphi_i(x) = e^{-ax} f_i(x) \qquad (i = 1, 2, 3), \tag{12}$$

equation (*) can be written as

$$\begin{aligned} \varphi_1(x_1 + x_2 + x_3) \\ = \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) + \varphi_1(x_2)\varphi_2(x_3)\varphi_3(x_1) + \varphi_1(x_3)\varphi_2(x_1)\varphi_3(x_2), \end{aligned} \tag{13}$$

where $\varphi'_{10} = 0$. (Notice that for the functions $\varphi_1(x), \varphi_2(x), \varphi_3(x)$ we may apply Lemma I.)

Differentiating (13) by x_2, x_3 and then putting $x_1 = x$, $x_2 = x_3 = 0$, we obtain

$$\varphi''_1(x) = \Phi_{10}\varphi_1(x), \tag{14}$$

where $\Phi_{10} = \varphi'_{20}\varphi'_{30}$.

1.1) If $\Phi_{10} = 0$, then $\varphi_1(x) = B_1 x + A_1$ and, in view of $\varphi'_{10} = 0$, we obtain

$$\varphi_1(x) \equiv A_1. \tag{15}$$

Putting in (13) $x_1 = x_2 = x_3 = x$ and taking into account (15), we obtain

$$\varphi_2(x)\varphi_3(x) = \tfrac{1}{3}. \tag{16}$$

We are looking for non-trivial solutions and therefore $A_1 \neq 0$. Equation (13) has the form (*). Thus, in view of Remark I, it is enough to find its solutions with $\varphi_1(0) = 1$. We now assume that $A_1 = 1$ and we may apply Lemma I. It follows from (5) and (16) that

$$\frac{1}{3\varphi_2(x)} = \frac{1}{\varphi_{20}}\left[\tfrac{2}{3} - \varphi_{30}\varphi_2(x)\right].$$

In view of (4), the last equality can be written as

$$[\varphi_2(x)]^2 - 2\varphi_{20}\varphi_2(x) + \varphi_{20}^2 = 0.$$

Hence

$$\varphi_2(x) \equiv \varphi_{20} = A_2 \tag{17}$$

and it follows from (16) that

$$\varphi_3(x) \equiv \frac{1}{3A_2}. \tag{18}$$

Now, taking into account (12), we conclude on the basis of (15), (17), (18), and Remark I that in this case the solutions of equation $(*)$ have form (i).

1.2) In the case of $\Phi_{10} < 0$, we obtain from (14)

$$\varphi_1(x) = A_1 \cos bx + B_1 \sin bx \qquad (b \neq 0)$$

and, in view of $\varphi'_{10} = 0$, we have

$$\varphi_1(x) = A_1 \cos bx . \tag{19}$$

We now conclude, as in 1.1), that to find solutions $\varphi_1(x), \varphi_2(x), \varphi_3(x)$ for which $\varphi_1(0) \neq 0$ it is enough to find solutions for which $\varphi_1(0) = 1$. We assume that $\varphi_1(x) = \cos bx$ and we may apply Lemma I. Equality (5) of Lemma I can now be written as

$$\varphi_3(x) = \frac{1}{\varphi_{20}} [\tfrac{2}{3} \cos bx - \varphi_{30} \varphi_2(x)] . \tag{20}$$

Substituting in (13) $x_1 = x_2 = x_3 = x$ and taking into account (19) we obtain

$$\cos bx \varphi_2(x) \varphi_3(x) = \tfrac{1}{3} \cos 3bx .$$

Notice that $\cos bx \varphi_2(x)$ can vanish only if $\cos 3bx = 0$. Thus it follows from the last equality that

$$\varphi_3(x) = \frac{\cos 3bx}{3\varphi_2(x) \cos bx}$$

at all the points where $\cos 3bx \neq 0$.

Substituting this into (20), we obtain after computations

$$3\varphi_{20}\varphi_{30} \cos bx [\varphi_2(x)]^2 - 2\varphi_{20} \cos^2 bx \varphi_2(x) + \varphi_{20}^2 \cos 3bx = 0$$

and taking into account (4), we have

$$\cos bx [\varphi_2(x)]^2 - 2\varphi_{20} \cos^2 bx \varphi_2(x) + \varphi_{20}^2 \cos 3bx = 0$$

at all the points where $\cos 3bx \neq 0$. Hence

$$\varphi_2(x) = \varphi_{20} \left[\cos bx + \tilde{\varepsilon}(x) \sqrt{\cos^2 bx - \frac{\cos 3bx}{\cos bx}} \right]$$
$$= \varphi_{20} [\cos bx + \tilde{\tilde{\varepsilon}}(x) \sqrt{3} \sin bx] ,$$

i.e.

$$\varphi_2(x) = A_2 [\cos bx + \varepsilon(x) \sqrt{3} \sin bx] , \tag{21}$$

where $|\tilde{\tilde{\varepsilon}}(x)| = |\tilde{\varepsilon}(x)| = |\varepsilon(x)| = 1$.

Substituting (21) into (20), we obtain

$$\varphi_3(x) = \frac{1}{3A_2}[\cos bx - \varepsilon(x)\sqrt{3}\sin bx] \tag{22}$$

where $|\varepsilon(x)| = 1$.

We proved only that equalities (21) and (22) are satisfied at the points where $\cos 3bx \neq 0$. But, in view of (12), the functions $\varphi_2(x)$ and $\varphi_3(x)$ are continuous everywhere and therefore (21) and (22) must be satisfied also at the points where $\cos 3bx = 0$.

Now, putting in (13) $x_1 = x$, $x_2 = \pi/2b$, $x_3 = 0$ and taking into account (19), (21) and (22), we obtain after computations

$$\cos(bx + \tfrac{1}{2}\pi) = -\varepsilon(\pi/2b)\varepsilon(x)\sin bx\,.$$

Hence $\varepsilon(x) = \varepsilon(\pi/2b) = \text{const}$ at the points where $\sin bx \neq 0$. Since at the points where $\sin bx = 0$ $\varepsilon(x)$ in (21) and (22) may be arbitrary, equalities (21) and (22) can be written as

$$\varphi_2(x) = A_2(\cos bx + \varepsilon\sqrt{3}\sin bx) \tag{23}$$

and

$$\varphi_3(x) = \frac{1}{3A_2}(\cos bx - \varepsilon\sqrt{3}\sin bx)\,, \tag{24}$$

where $\varepsilon = \text{const}$, $|\varepsilon| = 1$.

Taking into account (12), we conclude from (19), (23) and (24) that in this case the solutions of equation (*) have form (ii).

1.3) If $\Phi_{10} > 0$, we obtain after similar considerations that $f_2(x)$ and $f_3(x)$ are complex.

2) Now we shall find non-trivial, real solutions with $f_{10} = 0$. Differentiating (9) by x_1 and then putting $x_1 = 0$, $x_2 = x$, we obtain

$$f_1'(x) = f_{20}f_{30}'f_1(x) + f_{30}f_{10}'f_2(x)\,. \tag{25}$$

We must have $f_{10}' \neq 0$. In fact, if $f_{10}' = 0$, it follows from (25) that $f_1'(x) = f_{20}f_{30}'f_1(x)$ and $f_1(x) = Ae^{ax}$. But then $f_1'(x) = Aae^{ax}$ and, by virtue of $f_{10}' = 0$, $A = 0$ or $a = 0$. For $A = 0$, we obtain $f_1(x) \equiv 0$ contrary to the assumption of the theorem. For $a = 0$, we have $f_1(x) = A$ and, in view of $f_{10} = 0$, also $f_1(x) \equiv 0$. Thus $f_{10}' \neq 0$ and it follows from (25) that

$$f_2(x) = \frac{f_1'(x) - f_{20}f_{30}'f_1(x)}{f_{30}f_{10}'}\,. \tag{26}$$

Similarly, differentiating (9) by x_2 and then putting $x_1 = x$, $x_2 = 0$, we obtain

$$f_3(x) = \frac{f_1'(x) - f_{30}f_{20}'f_1(x)}{f_{20}f_{10}'}\,. \tag{27}$$

In view of (26) and (27), we can write (9) as

$$f_1(x_1+x_2) = f_1(x_1)\frac{f_1'(x_2)-f_{20}f_{30}'f_1(x_2)}{f_{10}'}+f_1(x_2)\frac{f_1'(x_1)-f_{30}f_{20}'f_1(x_1)}{f_{10}'}.$$

Differentiating this equality twice by x_1 and then putting $x_1 = 0$, $x_2 = x$, we obtain

$$f_1''(x)-\frac{f_{10}''}{f_{10}'}f_1'(x)+\frac{f_{30}f_{20}'f_{10}''+f_{20}f_{30}'f_{10}''-f_{10}'''}{f_{10}'}f_1(x) = 0 . \tag{28}$$

It follows from (28) and from the condition $f_{10} = 0$ that $f_1(x)$ may have the form:

(a) $$f_1(x) = A_1 x e^{\alpha x} ,$$

(b) $$f_1(x) = A_1 e^{\alpha x}\sin bx ,$$

(c) $$f_1(x) = A_1(e^{\alpha x}-e^{\beta x}) \qquad (\alpha\neq\beta) .$$

Now, it is convenient to notice that substituting in equation (*) $x_1 = x_2 = x_3 = x$, we obtain

$$f_1(3x) = 3f_1(x)f_2(x)f_3(x) . \tag{29}$$

If $f_1(x)$ has form (a), it follows from (26) and (27) that

$$f_2(x) = (A_2+B_2x)e^{\alpha x} \tag{30}$$

and

$$f_3(x) = (A_3+B_3x)e^{\alpha x} . \tag{31}$$

Substituting (a), (30) and (31) into (29), we obtain

$$x = (A_2+B_2x)(A_3+B_3x)x .$$

Hence $A_2A_3 = 1$, $B_2B_3 = 0$, $A_2B_3+A_3B_2 = 0$ and therefore $A_3 = 1/A_2$, $B_2 = B_3 = 0$, i.e. we obtain solutions (iii).

If $f_1(x)$ has form (b), it follows from (26) and (27) that

$$f_2(x) = (A_2\cos bx+B_2\sin bx)e^{\alpha x} \tag{32}$$

and

$$f_3(x) = (A_3\cos bx+B_3\sin bx)e^{\alpha x} . \tag{33}$$

Substituting (b), (32) and (33) into (29), we obtain

$$\sin 3bx = 3\sin bx(A_2\cos bx+B_2\sin bx)(A_3\cos bx+B_3\sin bx)$$

or

$$3\sin bx\cos^2 bx-\sin^3 bx$$
$$=3A_2A_3\sin bx\cos^2 bx+3(A_2B_3+A_3B_2)\sin^2 bx\cos bx+3B_2B_3\sin^3 bx .$$

Hence $A_2A_3 = 1$, $B_2B_3 = -\frac{1}{3}$, $A_2B_3 + A_3B_2 = 0$ and therefore $A_3 = \dfrac{1}{A_2}$, $B_2 = \varepsilon \dfrac{1}{\sqrt{3}} A_2$, $B_3 = -\varepsilon \dfrac{1}{\sqrt{3}} A_3$, where $|\varepsilon| = 1$. Thus, we have proved that in case (b) the solutions of equation $(*)$ have form (iv).

If $f_1(x)$ has form (c), we obtain (after similar considerations) that $f_2(x)$ and $f_3(x)$ are complex. This completes the proof.

On equation $(**)$.

Lemma III. *If the functions $f_1(x), f_2(x), f_3(x)$ satisfy equation $(**)$, then*

$$f_1(3x) = 6f_1(x)f_2(x)f_3(x) \; . \tag{34}$$

Proof. Substituting in $(**)$ $x_1 = x_2 = x_3 = x$, we obtain (34). Q. E. D.

Lemma IV. *If $f_{10} = 1$, then*

$$f_{20}f_{30} = \tfrac{1}{6} \; , \tag{35}$$

$$f_1(x) = 3[f_{20}f_3(x) + f_{30}f_2(x)] \; , \tag{36}$$

$$f_2(x_1)f_3(x_2) + f_2(x_2)f_3(x_1) = f(x_1+x_2) - \tfrac{2}{3}f_1(x_1)f_1(x_2) \; , \tag{37}$$

$$\begin{aligned} f_1(x_1+x_2+x_3) = {} & f_1(x_1)f_1(x_2+x_3) + f_1(x_2)f_1(x_1+x_3) + \\ & + f_1(x_3)f_1(x_1+x_2) - 2f_1(x_1)f_1(x_2)f_1(x_3) \; . \end{aligned} \tag{38}$$

Proof. Putting in $(**)$ $x_1 = x_2 = x_3 = 0$, we obtain (35), and then putting $x_1 = x, x_2 = x_3 = 0$, we obtain after computations (36). Putting in $(**)$ $x_3 = 0$ and taking into account (36), we obtain (37). Equality (38) follows from $(**)$ in view of (37). Q. E. D.

Remark II. Every solution $f_1(x), f_2(x), f_3(x)$ of equation $(**)$ for which $f_{10} \neq 0$ can be written as

$$f_1(x) = A_1g_1(x) , \qquad f_2(x) = A_2g_2(x) , \qquad f_3(x) = A_3g_3(x) ,$$

where $g_1(x), g_2(x), g_3(x)$ is a solution of equation $(**)$ satisfying the conditions:

$$g_1(0) = g_2(0) = 1 , \qquad g_3(0) = \tfrac{1}{6} \, .$$

Lemma V. *If. $f_{10} = 1$, $f_{20} = 1$, $f_{30} = \frac{1}{6}$, then at all the points where $f_1(x) \neq 0$,*

$$f_2(x) = f_1(x) + \varepsilon(x)\sqrt{[f_1(x)]^2 - \frac{f_1(3x)}{f_1(x)}} \tag{39}$$

and

$$f_3(x) = \frac{1}{6}\left[f_1(x) - \varepsilon(x)\sqrt{[f_1(x)]^2 - \frac{f_1(3x)}{f_1(x)}}\right] \tag{40}$$

where $|\varepsilon(x)| = 1$.

Proof. It follows by our assumptions from (36) that

$$f_3(x) = \tfrac{1}{3} f_1(x) - \tfrac{1}{6} f_2(x) \tag{41}$$

and by virtue of Lemma III

$$f_1(x)[f_2(x)]^2 - 2[f_1(x)]^2 f_2(x) + f_1(3x) = 0 .$$

Hence we obtain (39). Substituting (39) into (41), we obtain (40). Q. E. D.

LEMMA VI. *If* $f_{10} = 0$ *and* $f_1(x) \not\equiv 0$, *then*

$$f_{20} f_{30} = \tfrac{1}{2} \tag{42}$$

and

$$f_1(x_1 + x_2) = f_1(x_1) g(x_2) + f_1(x_2) g(x_1) , \tag{43}$$

where

$$g(x) = \frac{1}{2f_{20}} f_2(x) + f_{20} f_3(x) . \tag{44}$$

Proof. Putting in (**) $x_1 = x, x_2 = x_3 = 0$, we obtain (by virtue of $f_1(x) \not\equiv 0$) condition (42). Now, putting in (**) $x_3 = 0$, we obtain (43), where $g(x)$ satisfies (44). Q. E. D.

Remark III. Every solution $f_1(x), f_2(x), f_3(x)$ of equation (**) for which $f_{10} = 0$ can be written as

$$f_1(x) = g_1(x), \qquad f_2(x) = A_2 g_2(x), \qquad f_3(x) = A_3 g_3(x) ,$$

where $g_1(x), g_2(x), g_3(x)$ is a solution of equation (**) satisfying the conditions

$$g_1(0) = 0, \qquad g_2(0) = 1, \qquad g_3(0) = \tfrac{1}{2} .$$

LEMMA VII. *If* $f_{10} = 0, f_{20} = 1, f_{30} = \tfrac{1}{2}$, *then at the points where* $f_1(x) \neq 0$, *we have*

$$f_2(x) = g(x) + \varepsilon(x) \frac{1}{\sqrt{3}} \sqrt{3[g(x)]^2 - \frac{f_1(3x)}{f_1(x)}} \tag{45}$$

and

$$f_3(x) = \frac{1}{2}\left[g(x) - \varepsilon(x) \frac{1}{\sqrt{3}} \sqrt{3[g(x)]^2 - \frac{f_1(3x)}{f_1(x)}}\right] \tag{46}$$

where $|\varepsilon(x)| = 1$.

Proof. In view of the conditions $f_{20} = 1, f_{30} = \tfrac{1}{2}$, it follows from (44) that

$$f_3(x) = g(x) - \tfrac{1}{2} f_2(x) . \tag{47}$$

Now, taking into account Lemma III, we obtain

$$3f_1(x)[f_2(x)]^2 - 6f_1(x) g(x) f_2(x) + f_1(3x) = 0 .$$

Hence we conclude that (45) must be satisfied at all points where $f_1(x) \neq 0$. Substituting (45) into (47), we obtain (46). Q. E. D.

LEMMA VIII. *If the function $f(x)$ is continuous everywhere, the functions*

$$\varphi(x) = \int_\gamma^\delta \int_\gamma^\delta f(x+u+v)\,du\,dv$$

and

$$\psi(x) = \int_\gamma^\delta \int_\gamma^\delta f(u)f(x+v)\,du\,dv$$

have continuous first derivatives.

Proof. Since the function $f(x)$ is continuous everywhere, the function

$$\Phi(u, v) = f(x+u+v)$$

is continuous in the square

$$Q: \quad \begin{array}{l} \gamma \leqslant u \leqslant \delta, \\ \gamma \leqslant v \leqslant \delta \end{array}$$

for every x.

Therefore

$$\begin{aligned}\varphi(x) &= \int_\gamma^\delta \int_\gamma^\delta f(x+u+v)\,du\,dv \\ &= \int_\gamma^\delta \Big[\int_\gamma^\delta f(x+u+v)\,du\Big]dv = \int_\gamma^\delta \Big[\int_{\gamma+x}^{\delta+x} f(t+v)\,dt\Big]dv\,.\end{aligned}$$

To prove that $\varphi(x)$ has a continuous first derivative it suffices to prove that the function

$$F(x, v) = \int_{\gamma+x}^{\delta+x} f(t+v)\,dt = \int_{\gamma+x+v}^{\delta+x+v} f(s)\,ds$$

has a continuous derivative $F'_x(x, v)$.

It follows from well-known theorem that

$$F'_x(x, v) = f(x+v+\delta) - f(x+v+\gamma)\,.$$

Hence the continuity of the function $F'_x(x, v)$ is obvious and therefore the function $\varphi(x)$ has a continuous first derivative

$$\varphi'(x) = \int_\gamma^\delta F'_x(x, v)\,dv = \int_\gamma^\delta [f(x+v+\delta) - f(x+v+\gamma)]\,dv\,.$$

The proof of the existence of a continuous first derivative of the function $\psi(x)$ is similar:

By the assumption of the lemma we can write

$$\psi(x) = \int_\gamma^\delta\int_\gamma^\delta f(u)f(x+v)\,du\,dv$$
$$= \Big(\int_\gamma^\delta f(u)\,du\Big)\int_\gamma^\delta f(x+v)\,dv = \Big(\int_\gamma^\delta f(u)\,du\Big)\int_{\gamma+x}^{\delta+x} f(t)\,dt$$

and hence

$$\psi'(x) = \Big(\int_\gamma^\delta f(u)\,du\Big)[f(x+\delta)-f(\gamma+x)]\,. \qquad \text{Q. E. D.}$$

THEOREM IV. *If $f_2(x)$ (or $f_3(x)$) is continuous everywhere and if $f_1(x)$ and $f_3(x)$ (or $f_2(x)$) have a point of continuity in common, they are continuous everywhere.*

Proof. Suppose that a is the common point of continuity of the functions $f_1(x), f_3(x)$.

In the case of $f_{10} \neq 0$ (e.g. $f_{10} = 1$) it follows from (37) that

$$f_1(x_1+x_2) = \tfrac{2}{3}f_1(x_1)f_1(x_2)+f_2(x_1)f_3(x_2)+f_2(x_2)f_3(x_1)\,.$$

If we fix an arbitrary x_2, the right side of the last equality is continuous at the point $x_1 = a$. Therefore $f_1(x)$ must be continuous everywhere. The continuity of the function $f_3(x)$ follows from equality (36) and from the fact that $f_{20} \neq 0$.

In the case of $f_{10} = 0$ it follows from (44) that $g(x)$ is continuous at the point $x = a$. Thus $f_1(x)$ and $g(x)$ have a point of continuity in common and, in view of the results of [1], they have all the derivatives. Now, it follows from (44) that $f_3(x)$ is continuous everywhere. Q. E. D.

THEOREM V. *All the non-trivial, real solutions of equation* (**) *which satisfy the conditions*:

1) $f_2(x)$ *is continuous everywhere,*

2) $f_1(x)$ *and* $f_3(x)$ *have a point of continuity in common*

are:

(i) $$f_1(x) = A_1e^{ax},\quad f_2(x) = A_2e^{ax},\quad f_3(x) = \frac{1}{6A_2}e^{ax};$$

(ii) $$f_1(x) = A_1e^{ax}\cos bx\,,\quad f_2(x) = A_2e^{ax}(\cos bx+\varepsilon\sqrt{3}\sin bx),\quad f_3(x) = \frac{1}{6A_2}e^{ax}(\cos bx-\varepsilon\sqrt{3}\sin bx)\,;$$

(iii) $f_1(x) = A_1 x e^{ax}$,

$$f_2(x) = A_2 e^{ax},$$

$$f_3(x) = \frac{1}{2A_2} e^{ax};$$

(iv) $f_1(x) = A_1 e^{ax} \sin bx$,

$$f_2(x) = A_2 e^{ax}\left(\cos bx + \frac{\varepsilon}{\sqrt{3}} \sin bx\right),$$

$$f_3(x) = \frac{1}{2A_2} e^{ax}\left(\cos bx - \frac{\varepsilon}{\sqrt{3}} \sin bx\right),$$

where A_1, A_2, b, a $(A_1 \neq 0, A_2 \neq 0, b \neq 0)$ *are arbitrary constants, and* $\varepsilon = const$, $|\varepsilon| = 1$.

Proof. It follows from Theorem IV that the functions $f_1(x), f_2(x)$, $f_3(x)$ are continuous everywhere.

1) If $f_{10} \neq 0$ (e.g. $f_{10} = 1$), we obtain from (38)

$$f_1(x_1) \int_\gamma^\delta \int_\gamma^\delta [f_1(x_2+x_3) - 2f_1(x_2)f_1(x_3)]\,dx_2\,dx_3$$
$$= \int_\gamma^\delta \int_\gamma^\delta [f_1(x_1+x_2+x_3) - f_1(x_2)f_1(x_1+x_3) - f_1(x_3)f_1(x_1+x_2)]\,dx_2\,dx_3 .$$

If $f_1(x_2+x_3) - 2f_1(x_2)f_1(x_3) \not\equiv 0$, it follows from the continuity of the function $f_1(x)$ that there exist γ, δ such that

$$\int_\gamma^\delta \int_\gamma^\delta [f_1(x_2+x_3) - 2f_1(x_2)f_1(x_3)]\,dx_2\,dx_3 = c \neq 0$$

and we conclude on the basis of Lemma VIII that the function

$$f_1(x_1) = \frac{1}{c}\int_\gamma^\delta \int_\gamma^\delta f_1(x_1+x_2+x_3)\,dx_2\,dx_3 - \tag{48}$$
$$-\frac{1}{c}\int_\gamma^\delta \int_\gamma^\delta f_1(x_2)f_1(x_1+x_3)\,dx_2\,dx_3 - \frac{1}{c}\int_\gamma^\delta \int_\gamma^\delta f_1(x_3)f_1(x_1+x_2)\,dx_2\,dx_3$$

has a continuous first derivative.

If $f_1(x_2+x_3) - 2f_1(x_2)f_1(x_3) \equiv 0$, we have

$$\int_\lambda^\mu f_1(x_2+x_3)\,dx_3 = 2f_1(x_2)\int_\lambda^\mu f_1(x_3)\,dx_3 . \tag{49}$$

Since $f_1(x)$ is continuous and $f_1(x) \not\equiv 0$, there exist λ, μ such that

$$\int_{\lambda}^{\mu} f_1(x_3)\, dx_3 = c^* \neq 0$$

and it follows from (49) and from the continuity of the function $f_1(x)$ that

$$f_1(x_2) = \frac{1}{2c^*} \int_{\lambda}^{\mu} f_1(x_2 + x_3)\, dx_3 = \frac{1}{2c^*} \int_{\lambda+x_2}^{\mu+x_2} f(t)\, dt . \tag{50}$$

The existence of a continuous first derivative of function (50) is obvious.

Now we differentiate (48) and (50) and we conclude that the function $f_1(x)$ has a continuous second derivative.

Differentiating (38) by x_2, x_3 and then putting $x_1 = x, x_2 = x_3 = 0$, we obtain

$$f_1''(x) - 2f_{10}'f_1'(x) + [2(f_{10}')^2 - f_{10}'']f_1(x) = 0 . \tag{51}$$

1.1) In the case of $f_{10}'' - (f_{10}')^2 = 0$ we find after computations that

$$f_1(x) = e^{ax}$$

and in view of (39) and (40)

$$f_2(x) = e^{ax}, \qquad f_3(x) = \tfrac{1}{6}e^{ax}.$$

We conclude hence that in this case the general form of solutions must be (i).

1.2) In the case of $f_{10}'' - (f_{10}')^2 < 0$ we must have

$$f_1(x) = e^{ax}\cos bx \tag{52}$$

and by (39) and (40)

$$f_2(x) = e^{ax}[\cos bx + \varepsilon(x)\sqrt{3}\sin bx] \tag{53}$$

and

$$f_3(x) = \tfrac{1}{6}e^{ax}[\cos bx - \varepsilon(x)\sqrt{3}\sin bx] \tag{54}$$

at all the points where $\cos bx \neq 0$. Since $f_2(x)$ and $f_3(x)$ are continuous, (53) and (54) are satisfied everywhere.

Substituting in (**) $x_1 = x, x_2 = \pi/2b, x_3 = 0$ and taking into account (52), (53), and (54), we obtain after computations

$$\cos\left(bx + \frac{\pi}{2}\right) = -\varepsilon(x)\varepsilon\left(\frac{\pi}{2b}\right)\sin bx .$$

Hence it follows that at the points where $\sin bx \neq 0$ we have $\varepsilon(x) = \varepsilon(\pi/2b) = \text{const}$. At the points where $\sin bx = 0$ $\varepsilon(x)$ in (53) and in (54) may be arbitrary, and therefore

$$f_2(x) = e^{ax}[\cos bx + \varepsilon\sqrt{3}\sin bx] \tag{55}$$

and

(56) $$f_3(x) = \tfrac{1}{6}e^{ax}[\cos bx - \varepsilon\sqrt{3}\sin bx]\,,$$

where $\varepsilon = \text{const}$, $|\varepsilon| = 1$.

Since (52), (55), and (56) are satisfied everywhere, we conclude (in view of Remark III) that in this case the general form of solutions is (ii).

1.3) In the case of $f_{10}'' - (f_{10}')^2 > 0$ it follows from (51) that

$$f_1(x) = \tfrac{1}{2}(e^{ax} + e^{\beta x}) \qquad (\alpha \neq \beta)$$

but then (by (39) and (40)) we obtain complex $f_2(x)$ and $f_3(x)$.

2) If $f_{10} = 0$, we can apply Lemma VI and it follows from (43) and (44) (on the basis of the results of [1]) that we can only have

(a) $$f_1(x) = A_1 x e^{ax}, \qquad g(x) = e^{ax},$$

(b) $$f_1(x) = A_1 e^{ax}\sin bx\,, \qquad g(x) = e^{ax}\cos bx\,,$$

(c) $$f_1(x) = A_1 e^{ax}\sinh bx, \qquad g(x) = e^{ax}\cosh bx\,.$$

2.1) In case (a) we obtain directly from (45) and (46)

$$f_2(x) = e^{ax}, \qquad f_3(x) = \tfrac{1}{2}e^{ax}\,.$$

Hence and from Remark III it follows that the general form of solutions is (iii).

2.2) In case (b) it follows from (45) and (46) that

$$f_2(x) = e^{ax}\left[\cos bx + \varepsilon(x)\frac{1}{\sqrt{3}}\sin bx\right]$$

and

$$f_3(x) = \tfrac{1}{2}e^{ax}\left[\cos bx - \varepsilon(x)\frac{1}{\sqrt{3}}\sin bx\right]$$

at all the points where $\sin bx \neq 0$.

Hence we conclude, as in 1.2), that

$$f_2(x) = e^{ax}\left[\cos bx + \varepsilon\frac{1}{\sqrt{3}}\sin bx\right]$$

and

$$f_3(x) = \tfrac{1}{2}e^{ax}\left[\cos bx - \varepsilon\frac{1}{\sqrt{3}}\sin bx\right]$$

($\varepsilon = \text{const}$, $|\varepsilon| = 1$) for every x.

Thus, in view of Remark III, we also have solutions (iv).

2.3) In case (c) $f_2(x)$ and $f_3(x)$ are complex. This completes the proof.

Similar methods can be applied to find the solutions of symmetrical equations (1) with $n > 3$ but it is more complicated than for $n = 2, 3$ and we shall not do it.

References

[1] H. Świątak, *On the equation* $[\varphi(x+y)]^2 = [\varphi(x)g(y)+\varphi(y)g(x)]^2$, Zeszyty Naukowe U. J., Prace Mat. 10 (1965).

[2] — *Connections between symmetry or asymmetry of the equations* $f_1(x_1+ \dots + x_n) = \sum_{(i_1,\dots,i_n)} f_1(x_{i_1}) \dots f_n(x_{i_n})$ *and their solutions*, Ann. Polon. Math., this volume, pp. 257-269.

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