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Green's formula for right invertible operators

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Dedicated to the memory of Jacek Szarski

1. Preliminaries. Let X be a linear space over a field $\mathscr{F}$ of scalars. In our further consideration we shall admit either $\mathscr{F} = \boldsymbol{R}$ or $\mathscr{F} = \boldsymbol{C}$. Let $L(X)$ be the set of all linear operators A such that the domain of A (denoted by $\operatorname{dom} A$) is a linear subset of X and $AX \subset X$. In particular, we write: $L_0(X) = \{A \in L(X)\colon \operatorname{dom} A = X\}$. Let $R(X)$ be the set of all right invertible operators belonging to $L(X)$. For a given $D \in R(X)$ we denote by $\mathscr{R}_D = \{R_\gamma\}_{\gamma\in\Gamma}$ the set of all right inverses of D. We shall assume that $\operatorname{dom} R_\gamma = X$ for $\gamma \in \Gamma$. Here and in the sequel we shall assume that $\dim\ker D > 0$, i.e. D is right invertible but not left invertible. Any element of $\ker D$ is a *constant* for D.

By $\mathscr{F}_D = \{F_\gamma\}_{\gamma\in\Gamma}$ we denote the set of all initial operators for D (cf. [7]). By definition, F is an initial operator for D if is a projection *onto* $\ker D$ such that $FR = 0$ for an $R \in \mathscr{R}_D$. This implies that F is an initial operator if and only if there is an $R \in \mathscr{R}_D$ such that $F = I - RD$ on $\operatorname{dom} D$.

One can also prove (cf. [7]) that *any* projection F onto $\ker D$ is an initial operator for D corresponding to a right inverse $R = R_0 - FR_0$ and that the definition of R does not depend on the choice of the right inverse R_0.

We recall the *Taylor–Gontcharov formula* for right invertible operators. Let $N \in \boldsymbol{N}$ and let $\{\gamma_n\} \subset \Gamma$ be arbitrarily fixed. Then the following identity holds on the domain of D^N:

$$I = F_{\gamma_0} + \sum_{k=1}^{N-1} R_{\gamma_0} \ldots R_{\gamma_{k-1}} F_{\gamma_k} D^k + R_{\gamma_0} \ldots R_{\gamma_{N-1}} D^N. \tag{1.1}$$

In particular, if $R_{\gamma_k} = R$, $F_{\gamma_k} = F$ for $k = 0, 1, \ldots, N-1$, then we obtain the *Taylor formula*:

$$I = \sum_{k=0}^{N-1} R^k F D^k + R^N D^N \quad \text{on } \operatorname{dom} D^N. \tag{1.2}$$

For an arbitrary $x \in X$ the element $R_\gamma x$ ($\gamma \in \Gamma$) is a *primitive* element for x. One can prove that the difference of primitive elements is a constant (cf. [7]). Consider the operator: $F_\beta R_\gamma - F_\alpha R_\gamma$ for arbitrarily fixed $\alpha, \beta, \gamma \in \Gamma$. This operator is independent of the choice of R and plays the role of a *definite integral* for D. We can prove that

$$F_\beta R_\gamma - F_\alpha R_\gamma = F_\beta R_\alpha \quad (\alpha, \beta, \gamma \in \Gamma) \tag{1.3}$$

and that

$$F_\beta R_\alpha D = F_\beta - F_\alpha \quad (\alpha, \beta \in \Gamma). \tag{1.4}$$

The last equality implies that a definite integral of an element is equal to the difference of initial values of a primitive element and that this constant is independent of the choice of a primitive element.

Let $A, B \in L(X)$ be arbitrary operators such that both superpositions AB and BA are well-defined. We shall write:

$$[A, B] = AB - BA.$$

Let X be a linear ring (an algebra) over $\mathscr{F}$. We shall admit the following convention:

$$[A, x] = Ax - xA \quad \text{for all } x \in \operatorname{dom} A, \tag{1.5}$$

i.e. $[A, x]y = (Ax)y - xAy$ for all $x, y \in \operatorname{dom} A$.

Let X be a commutative algebra (i.e. a commutative linear ring) over $\boldsymbol{R}$ and let $D \in R(X)$. X is said be a *D-algebra* if the following condition is satisfied:

$$x \in \operatorname{dom} D \text{ and } y \in \operatorname{dom} D \quad \text{implies} \quad xy \in \operatorname{dom} D. \tag{1.6}$$

Here and in the sequel we shall assume that X is a D-algebra. Write:

$$f_D(x, y) = D(xy) - c_D(xDy + yDx) \quad \text{for all } x, y \in \operatorname{dom} D, \tag{1.7}$$

where:

(i) c_D is a scalar dependent on D only;

(ii) f_D: $\operatorname{dom} D \operatorname{dom} D \to \operatorname{dom} D$ is a bilinear and symmetric mapping dependent on D only, i.e.

$$f_D(y, x) = f_D(x, y) \quad \text{for all } x, y \in \operatorname{dom} D. \tag{1.8}$$

Using the denotation (1.2) we can write:

$$D(xy) = c_D(xDy + yDx) + f_D(x, y) \quad \text{for } x, y \in \operatorname{dom} D. \tag{1.9}$$

The bilinear operator f_D will be called a *non-Leibniz* component. Non-Leibniz components for power of D are determined by the following recursion formula:

$$f_D^{(0)} = 0, \quad f_D^{(1)} = f_D \quad \text{and for } k = 2, 3, \ldots, x, y \in \operatorname{dom} D^k,$$

$$\begin{aligned} f_D^{(k)}(x, y) = {} & c_D^k[(Dx)(D^{k-1}y) + (D^{k-1}x)(Dy)] + \\ & + c_D^{k-1}[f_D^{(k-1)}(x, D^{k-1}y) + f_D^{(k-1)}(D^{k-1}x, y)] + Df_D^{(k-1)}(x, y). \end{aligned} \tag{1.10}$$

The proof is given in [16]. Similar formulae hold for a superposition of right invertible operators. If $p \neq 0$ is an arbitrary real then $c_{pD} = c_D$, $f_{pD} = pf_D$.

Other properties of non-Leibniz components and several examples of D-algebras can be found also in [16].

2. Lagrange and Green formulae for polynomials in a right invertible operators with operator coefficients. Here and in the sequel we shall assume for D-algebras under consideration the denotation used in formula (1.7). We also always assume that $c_D \neq 0$ (cf. Example 2.8 in [16]).

THEOREM 2.1. *Let X be a D-algebra. Write*

$$Q(D) = \sum_{k=0}^{N} Q_k D^k; \quad Q^+(D) = \sum_{k=0}^{N} (-1)^k D^k Q_k \tag{2.1}$$

where $Q_0, Q_1, \ldots, Q_N \in L_0(\operatorname{dom} D^N)$, $N \geqslant 1$ and Q_N is an invertible operator (or identity). Then for every $x, y \in \operatorname{dom} D^N$ the following identity holds:

$$\begin{aligned} xQ(D)y - yQ^+(D)x = \sum_{k=0}^{N} \{(-1)^{k+1} c_D^{-k} [D^k(yQ_k x) - f_D^{(k)}(Q_k x, y)] + \\ + [xQ_k - (Q_k x)] D^k y + [1 + (-1)^k](Q_k x) D^k y\}. \end{aligned} \tag{2.2}$$

Proof. Our assumptions imply that

$$\begin{aligned} xQ(D)y - yQ^+(D)x &= \sum_{k=0}^{N} [xQ_k D^k y - y(-1)^k D^k Q_k x] \\ &= \sum_{k=0}^{N} \{(-1)^{k+1} [(Q_k x) D^k y + y D^k(Q_k x)] + [xQ_k D^k y + (-1)^k (Q_k x) D^k y]\} \\ &= \sum_{k=0}^{N} \{(-1)^{k+1} c_D^{-k} [D^k(yQ_k x) - f_D^{(k)}(Q_k x, y)] + [xQ_k - (Q_k x)] D^k y + \\ &\qquad + [1 + (-1)^k](Q_k x) D^k y\}. \end{aligned}$$

Formula (2.2) is the *Lagrange formula* for polynomials in D with operator coefficients.

COROLLARY 2.1. *Let X be a D-algebra and let $Q(D)$ and $Q^+(D)$ be defined by formula* (2.1). *Then for all $F_\alpha, F_\beta \in \mathscr{F}_D$, $R_\alpha \in \mathscr{R}_D$ ($\alpha \neq \beta$) and for all $x, y \in \operatorname{dom} D^N$ the following identity holds:*

$$\begin{aligned} & F_\beta R_\alpha [xQ(D)y - yQ^+(D)x] \\ &\quad = \sum_{j=0}^{N-1} (-1)^j c_D^{-(j+1)} (F_\beta - F_\alpha) D^j (yQ_{j+1} x) + \\ &\qquad + F_\beta R_\alpha [g_{Q(D)}(x, y) + h_{Q(D)}(x, y)], \end{aligned} \tag{2.3}$$

where

$$(2.4)\qquad g_{Q(D)}(x,y) = \sum_{k=1}^{N} \{(-1)^k c_D^{-k} f_D^{(k)}(Q_k x, y) + [1+(-1)^k](Q_k x) D^k y\} + yQ_0 x,$$

$$(2.5)\qquad h_{Q(D)}(x,y) = \sum_{k=0}^{N} [xQ_k - (Q_k x)] D^k y.$$

Proof. Formulae (1.4), (2.2), (2.4), (2.5) together imply that for all $F_\alpha, F_\beta \in \mathscr{F}_D$, $R_\alpha \in \mathscr{R}_D$, $\alpha \neq \beta$ and $x, y \in \operatorname{dom} D^N$ we have

$$\begin{aligned}
&F_\beta R_\alpha[xQ(D)y - yQ^+(D)x] \\
&= F_\beta R_\alpha\Big\{-yQ_0x + \sum_{k=1}^{N}(-1)^{k+1}c_D^{-k}[D^k(yQ_kx)] + \sum_{k=0}^{N}(-1)^{k+2}c_D^{-k}f_D^{(k)}(x,y) + \\
&\qquad + \sum_{k=0}^{N}[xQ_k - (Q_kx)]D^ky + 2(Q_0x)y + \sum_{k=1}^{N}[1+(-1)^k](Q_kx)D^ky\Big\} \\
&= F_\beta R_\alpha\Big\{\sum_{j=0}^{N-1}(-1)^{j+2}c_D^{-(j+1)}[D^{j+1}(yQ_{j+1}x)]\sum_{k=0}^{N}c_D^{-k}f_D^{(k)}(x,y) + \\
&\qquad + \sum_{k=0}^{N}[xQ_k - (Q_kx)]D^ky + \sum_{k=1}^{N}[1+(-1)^k](Q_kx)D^ky + yQ_0x\Big\} \\
&= \sum_{j=0}^{N-1}(-1)^j c_D^{-(j+1)} F_\beta R_\alpha D[D^j(yQ_{j+1}x)] F_\beta R_\alpha[g_{Q(D)}(x,y) + h_{Q(D)}(x,y)] \\
&= \sum_{\beta=0}^{N-1}(-1)^j c_D^{-(j+1)}(F_\beta - F_\alpha)D^j(yQ_{j+1}x) + F_\beta R_\alpha[g_{Q(D)}(x,y) + h_{Q(D)}(x,y)]
\end{aligned}$$

since $f_D^{(0)} = 0$.

Formula (2.3) is the *Green formula* for polynomials in D with operator coefficients. Now we shall specify coefficients. Observe that bilinear operators appearing in this formula map the *space of constants into itself*.

COROLLARY 2.2. *Suppose that all assumptions of Corollary 2.1 are satisfied. If $Q_0, \ldots, Q_N$ are operators of multiplication by elements of X, i.e.*

$$(2.6)\qquad Q_k x = a_k x, \quad \text{where } a_k \in X \ (k = 0, 1, \ldots, N) \text{ for } x \in X,$$

then

$$Q(D)x = \sum_{k=0}^{N} a_k D^k x, \quad Q^+(D)x = \sum_{k=0}^{N}(-1)^k D^k(a_k x) \quad \text{for } x \in X, \quad h_{Q(D)} = 0,$$

where $h_{Q(D)}$ is defined by (2.5), and the Green formula is of the form

$$(2.7)\qquad F_\beta R_a[xQ(D)y - yQ^+(D)x]$$
$$= \sum_{j=0}^{N-1} (-1)^j c_D^{-(j+1)}(F_\beta - F_a)D^j(axy) + F_\beta R_a g_{Q(D)}(x, y)$$

where

$$g_{Q(D)} = \sum_{k=1}^{N} \{(-1)^k c_D^{-k} f_D^{(k)}(a_k x, y) + [1+(-1)^k]a_k x D^k y\} + a_0 xy.$$

Indeed, by our definition for all $x, y \in \operatorname{dom} D^N$

$$h_{Q(D)}(x, y) = \sum_{k=0}^{N} [xQ_k - (Q_k x)]D^k y = \sum_{k=0}^{N} [xa_k - a_k x]D^k y = 0$$

and $yQ_0 x = a_0 xy$.

Corollary 2.3. *Suppose that all assumptions of Corollary* 2.1 *are satisfied. If the coefficients $Q_0, \ldots, Q_N$ commute with D: $DQ_k = Q_k D$ $(k = 0, 1, \ldots, N)$, then*

$$(2.8)\qquad Q^+(D) = Q(-D) = Q(D^+), \quad \textit{where } D^+ = -D.$$

Indeed, by our assumptions we have

$$Q^+(D) = \sum_{k=0}^{N} (-1)^k D^k Q_k = \sum_{k=0}^{N} Q_k(-D)^k = Q(-D) = Q(D^+).$$

Corollary 2.4. *Suppose that all assumptions of Corollary* 2.1 *are satisfied. If the coefficients of $Q(D)$ are scalars, i.e. if $Q_k = q_k I$, where $q_k \in \mathbf{R}$ $(k = 0, 1, \ldots, N)$, $q_N \neq 0$, then*

$$Q^+(D) = Q(-D) = Q(D^+);$$

for all $x, y \in \operatorname{dom} D^N$

$$g_{Q(D)}(x, y) = \sum_{k=1}^{N} q_k\{(-1)^k c_D^{-k} f_D^{(k)}(x, y) + [1+(-1)^k]xD^k y\} + q_0 xy,$$

$$h_{Q(D)}(x, y) = 0,$$

and the Green formula is of the form

$$(2.9)\qquad F_\beta R_a[xQ(D)y - yQ(-D)x]$$
$$= \sum_{j=0}^{N-1} (-1)^j c_D^{-(j+1)} g_j (F_\beta - F_a) D^j(xy) + F_\beta R_a g_{Q(D)}(x, y).$$

Indeed, since $Q_0, \ldots, Q_N$ commute with D, Corollary 2.3 implies that $Q^+(D) = Q(-D)$. Moreover, we have

$$g_{Q(D)}(x, y) = \sum_{k=1}^{N} \{(-1)^k c_D^{-k} f_D^{(k)}(q_k x, y) + [1+(-1)^k] q_k x D^k y\} + q_0 xy$$
$$= \sum_{k=1}^{N} q_k \{(-1)^k c_D^{-k} f_D^{(k)}(x, y) + [1+(-1)^k] x D^k y + q_0 xy\}$$

since $f_D^{(k)}$ are bilinear operators. We also find

$$h_{Q(D)}(x, y) = \sum_{k=0}^{N} [xQ_k - Q_k x] D^k y = \sum_{k=0}^{N} q_k (x-x) D^k y = 0.$$

Therefore the Green formula in our case is of the form (2.8).

COROLLARY 2.5. *Suppose that all assumptions of Corollary* 2.4 *are satisfied. Then the Green formula* (2.9) *can be written as follows:*

$$(2.10)\quad F_\beta R_a [xQ(D)y - yQ(-D)x]$$
$$= c_D^{-1}(F_\beta - F_a)[Q(-c_D^{-1}D) - (-1)^N q_N D^N](xy) + F_\beta R_a g_{Q(D)}(x, y)$$

for all $x, y \in \operatorname{dom} D^N$, *where* $g_{Q(D)}$ *is as in Corollary* 2.4.

Indeed, formula (2.8) implies that for all $x, y \in \operatorname{dom} D^N$

$$F_\beta R_a [xQ(D)y - yQ(-D)x]$$
$$= \sum_{j=0}^{N-1} (-1)^j c_D^{-(j+1)} q_j (F_\beta - F_a) D^j (xy) + F_\beta R_a g_{Q(D)}(x, y)$$
$$= c_D^{-1}(F_\beta - F_a) \sum_{j=0}^{N-1} q_j (-1)^j c_D^{-j} D^j (xy) + F_\beta R_a g_{Q(D)}(x, y)$$
$$= c_D^{-1}(F_\beta - F_a)[Q(-c_D^{-1}D)(xy) - (-1)^N q_N D^N](xy) + F_\beta R_a g_{Q(D)}(x, y).$$

If $c_D = 1$, then formula (2.9) is of the form

$$(2.11)\quad F_\beta R_a [xQ(D)y - yQ(-D)x]$$
$$= (F_\beta - F_a) Q(-D) - (-1)^N q_N D^N (xy) + F_\beta R_a g_{Q(D)}(x, y),$$

where

$$g_{Q(D)}(x, y) = \sum_{k=1}^{N} q_k \{(-1)^k f_D^{(k)}(x, y) + [1+(-1)^k] x D^k y\} + q_0 xy.$$

COROLLARY 2.6. *Suppose that all assumptions of Corollary* 2.1 *are satisfied. Let* x, y *be solutions of equations*

$$(2.12)\qquad Q(D)y = v, \quad Q^+(D)x = u,$$

respectively, where u, $v \in X$ are given. Then

$$(2.13)\quad F_\beta R_a(xv-yu)$$
$$= \sum_{j=0}^{N-1}(-1)^j c_D^{-(j+1)}(F_\beta - F_a)D^j(yQ_{j+1}x) + F_\beta R_a[g_{Q(D)}(x, y) + h_{Q(D)}(x, y)].$$

The proof immediately follows from formula (2.3).

COROLLARY 2.7. *Let X be a D-algebra. Let N be an arbitrarily fixed positive integer and let $x, y \in \operatorname{dom} D^N$. Then the Lagrange formula for the operator D^N is of the form*

$$(2.14)\quad xD^N y - y(D^+)^N x = (-1)^{N+1}c_D^{-N}D^N(xy) - f_D^{(N)}(x, y) + \\ +[1+(-1)^N]xD^N y, \quad \textit{where } D^+ = -D$$

and the Green formula is of the form

$$(2.15)\quad F_\beta R_a[xD^N y - y(D^+)^N x] = (-1)^{N+1}c_D^{-N}(F_\beta - F_a)D^{N-1}(xy) + \\ +(-1)^N c_D^{-N} f_D^{(N)}(x, y) + [1+(-1)^N]xD^N y$$

where $F_a, F_\beta \in \mathcal{F}_D$, $R_a \in \mathcal{R}_D$, $\beta \neq a$.

Indeed, if we put in Theorem 2.1 and Corollary 2.1 $Q_0 = Q_1 = \ldots \ldots = Q_{N-1} = 0$, $Q_N = I$ *then we obtain*

$$Q(D) = D^N, \quad Q^+(D) = (-1)^N D^N = (D^+)^N.$$

COROLLARY 2.8. *Suppose that $D_1, D_2 \in R(X)$, $D = D_1D_2$ and X is a D-algebra. Then the Lagrange and Green formulae for the superposition $D = D_2D_1$ are of the form: for all $x, y \in \operatorname{dom} D$*

$$(2.16)\quad xD_2D_1y - yD_2^+D_1^+ \\ = c_{D_1}^{-1}c_{D_2}^{-1}\{D_2D_1(xy) - f_{D_2}(x, y) - D_2 f_{D_1}(x, y) + \\ + c_{D_1}c_{D_2}[(D_1x)(D_2y) + (D_2x)(D_1y)]\},$$

$$(2.17)\quad F_\beta R_a(xD_2D_1y - yD_2^+D_1^+) \\ = c_{D_1D_2}(F_\beta - F_a)(xy) - c_{D_1D_2}F_\beta R_a\{f_{D_2}(x, y) + D_2 f_{D_1}(x, y) + \\ + c_{D_1}c_{D_2}[(D_1x)(D_2y) + (D_2x)(D_1y)]\},$$

where we admit $D_1^+ = -D_1$, $D_2^+ = -D_2$, $D^+ = -D$ and

$$(2.18)\quad F_\beta R_a = [F_\beta^{(1)}R_a^{(1)} + R_\beta^{(1)}F^{(2)}]R_a^{(2)}, \\ F_a^{(i)}, F_\beta^{(i)} \in \mathcal{F}_{D_i};\ R_a^{(i)}, R_\beta^{(i)} \in \mathcal{R}_{D_i}\ (i = 1, 2) \textit{ and } \beta \neq a.$$

Indeed, $D^+ = -D_2^+D_1^+ = -(-D_2)(-D_1) = -D_2D_1 = -D$.
It is well known [8] that

$$F_\beta = F_\beta^{(1)} + R_{\beta\beta}^{(1)}F^{(2)}D_1 \in \mathcal{F}_D, \quad F_a = F_a^{(1)} + R_a^{(1)}F_a^{(2)}D_1 \in \mathcal{F}_D, \\ R_a = R_a^{(1)}R_a^{(2)} \in \mathcal{R}_D.$$

Hence

$$F_\beta R_a = (F_\beta^{(1)} + R_\beta^{(1)} F_\beta^{(2)} D_1) R_a^{(1)} R_a^{(2)} = [F_\beta^{(1)} R_a^{(1)} + R_\beta^{(1)} F_\beta^{(2)}] R_a^{(2)},$$

$$F_\beta - F_a = F_\beta^{(1)} - F_a^{(1)} + (R_\beta^{(1)} F_\beta^{(2)} - R_a^{(2)} F_a^{(2)}) D_1.$$

Since $D^+ = -D$, formulae (2.2) and (2.3) imply that for all $x, y \in \operatorname{dom} D$

$$xDy - yD^+x = xDy + yDx = c_D^{-1}[D(xy) - f_D(x, y)],$$

$$\begin{aligned} F_\beta R_a(xDy - yD^+x) &= F_\beta R_a\{c_D^{-1} D(xy) - f_D(x, y)\} \\ &= c_D^{-1} F_\beta R_a D(xy) - c_D^{-1} F_\beta R_a f_D(x, y) \\ &= c_D^{-1}(F_\beta - F_a)(xy) - c_D^{-1} F_\beta R_a f_D(x, y). \end{aligned}$$

This implies the required formulae (2.16) and (2.17).

COROLLARY 2.9. *Suppose that $D_1, D_2 \in R(X)$, $D = D_2^q D_1^p$, $p, q \in N$, are arbitrarily fixed and X is a D-algebra. Then we have*

(2.19) $$c_D = c_{D_1}^p c_{D_2}^q,$$

(2.20) $$f_D(x, y) = f_{D_2}^{(q)}(x, y) + D_2^q f_{D_1}^{(p)}(x, y) + c_{D_1}^p c_{D_2}^q [(D_1^p x)(D_2^q y) + (D_2^q x)(D_1^p y)] \quad \text{for } x, y \in \operatorname{dom} D,$$

(2.21) $$D^+ = (-1)^{p+q} D,$$

(2.22) $$F_\gamma = \sum_{k=0}^{p-1} R_\gamma^{(1)k} F^{(1)} D_1^R + (R_\gamma^{(1)})^p \sum_{j=0}^{q-1} (R_\gamma^{(2)})^j F_\gamma^{(2)} D_2^j D_1^p \qquad (\gamma = a \text{ or } \gamma = \beta),$$

(2.23) $$R_a = (R_a^{(1)})^p (R_a^{(2)})^q, \quad F_a^{(i)}, F_\beta^{(i)} \in \mathscr{F}_{D_i}, \; R_a^{(i)}, R_\beta^{(i)} \in \mathscr{R}_{D_i} \quad (i = 1, 2).$$

The Lagrange formula is of the form

$$xDy - yD^+x = c_D^{-1}[D(xy) - f_D(x, y)].$$

The Green formula is of the form

(2.24) $$F_\beta R_a(xDy - yD^+x) = c_D^{-1}(F_\beta - F_a)(xy) - F_\beta R_a c_D^{-1} f(x, y) \quad \text{for } x, y \in \operatorname{dom} D.$$

The proof follows from Corollary 2.8, Theorem 2.1 and Taylor formulae for operators D_1 and D_2 (cf. (1.2)). An immediate consequence is

COROLLARY 2.10. *Let X, D, c_D, f_D, D^+, F_a, F_β, R_a be defined as in Corollary 2.9. Let*

$$Q(D) = \sum_{k=0}^{N} Q_k D^k, \quad Q^+(D) = \sum_{k=0}^{N} (-1)^k D^k Q_k,$$

where $Q_k \in L_0(\operatorname{dom} D^N)$ $(k = 0, 1, \ldots, N)$ and Q_N is invertible. Then the Green formula (2.3) holds.

COROLLARY 2.11. *Let X, D, c_D, f_D, D^+, F_a, F_β, R_a be defined as in Corrollary 2.9 (with $p = q = 1$). Let*

$$(2.25)\quad \begin{aligned} P(D) &= D_2D_1 + AD_1 + BD_2 + C, \\ P^+(D) &= D_2D_1 - D_1A - D_2B + C. \end{aligned} \qquad A, B, C \in L_0(X),$$

Then the Green formula for all $x, y \in \operatorname{dom} D$ is of the form

$$(2.26)\quad F_\beta R_a[xP(D)y - yP^+(D)x]$$
$$= c_{D_1}^{-1}c_{D_2}^{-1}(F_\beta - F_a)(xy) + c_{D_1}^{-1}(F_\beta^{(1)} - F_a^{(1)})(yAx) + c_{D_2}^{-1}(F_\beta^{(2)} - F_a^{(2)})(yBx) +$$
$$+ F_\beta R_a\{c_{D_1}^{-1}c_{D_2}^{-1}[f_{D_2}(x, y) + D_2 f_D(x, y)] + [(D_1x)(D_2y) + (D_2x)(D_2y)] +$$
$$+ c_{D_1}^{-1}f_{D_1}(Ax, y) + c_{D_2}^{-1}f_{D_2}(Bx, y) + [x, A]D_1y + [x, B]D_2y + yCx - xCy\},$$

where $D = D_2D_1$, and

$$F_\gamma = F_\gamma^{(1)} + R_\gamma^{(1)}F_\gamma^{(2)}D_1 \in \mathscr{F}_D = \mathscr{F}_{D_2D_1} \quad (\gamma = a \text{ or } \gamma = \beta)$$
$$R_a = R_a^{(1)}R_a^{(2)} \in \mathscr{R}_D = \mathscr{R}_{D_1D_2}, \qquad F_a^{(i)}, F_\beta^{(i)} \in \mathscr{F}_{D_i}, R_a^{(i)} \in \mathscr{R}_{D_i} \qquad (i = 1, 2).$$

Indeed, observe that for $x, y \in \operatorname{dom} D$ we have

$$\begin{aligned} xP(D)y - yP^+(D)x &= xD_2D_1y - yD_2D_1x + \\ &\qquad + xAD_1y - yD_1Ax + xBD_2y - yD_2Bx + xCy - yCx \\ &= xDy - yDx + (Ax)D_1y - yD_1(Ax) + (Bx)(D_2y) - yD_2(Bx) + \\ &\qquad + xCy - yCx + [x, A]D_1y + [x, B]D_2y \\ &= c_D^{-1}[D(x, y) - f_D(x, y)] + c_{D_1}^{-1}[D_1(yAx) - f_{D_1}(Ax, y)] + \\ &\qquad + c_{D_2}^{-1}[D_2(yBx) - f_{D_2}(Bx, y)] + [x, A]D_1y + [x, B]D_2y + yCx - xCy. \end{aligned}$$

Further the proof is going in a similar way as in Corollaries 2.8 and 2.9.

EXAMPLE 2.1. Suppose that $X = C(\Omega)$, where

$$\Omega = \{(t, s)\colon 0 \leqslant t \leqslant a,\ 0 \leqslant s \leqslant b\}, \qquad D = \frac{\partial^2}{\partial t \partial s},$$
$$R_0 = \int_0^t \int_0^s, \qquad (F_0x)(t, s) = x(t, 0) + x(0, s) - x(0, 0).$$

The operator $D = D_2D_1$, where $D_2 = \partial/\partial t$, $D_1 = \partial/\partial s$ is right invertible, F_0 is an initial operator for D corresponding to its right inverse R_0. Moreover, F_0 is an initial operator induced by the classical Darboux problem for the operator D. To have Green formulae for the operator

$$(2.27)\qquad P(D) = \frac{\partial^2}{\partial t \partial s} + A\frac{\partial}{\partial t} + B\frac{\partial}{\partial s} + C, \qquad A, B, C \in X,$$

we have to find some more initial operators and right inverses for D and to apply Corollary 2.10. Suppose then that we are given functions $g, h \in C^1[0, a]$ such that $g'(t) > 0$, $h'(t) > 0$ of $0 \leqslant t \leqslant a$, $g(0) = h(0) = 0$, $g(a) = h(a) = b$.

Consider the following operators defined for $x \in C^1(\Omega)$

$$(F_1x)(t, s) = x\big(g^{-1}(s), s\big) + \int\limits_{g^{-1}(s)}^{t} x_t'\big(p, g(p)\big)dp,$$
$$(F_2x)(t, s) = x\big(g^{-1}(s), s\big) - x\big(g^{-1}(s), 0\big) + x(t, 0),$$
$$(F_3x)(t, s) = x\big(g(s), s\big) + \int\limits_{g^{-1}(s)}^{t} x_t'\big(p, h(p)\big)dp.$$

All these operators are initial operators for D because they are projections *onto* $\ker D$. The operator F_1 is induced by the Cauchy problem for D, the operator F_2 is induced by the Picard problem for D, the operator F_3 is induced by the generalized Cauchy problem for D (i.e. such a problem, where we are given values $x\big(t, g(t)\big)$ and $x_t'\big(t, h(t)\big)$, cf. [7], [10]).

It is easy to verify that for all $x \in X$ we have

$$(F_1R_0x)(t, s) = -\int\limits_0^{g^{-1}(s)} \Big[\int\limits_0^s x(p, q)dq\Big]dp - \int\limits_{g^{-1}(s)}^{t} \Big[\int\limits_0^{g(p)} x(p, q)dq\Big]dp,$$
$$(F_2R_0x)(t, s) = \int\limits_0^{g^{-1}(s)} \Big[\int\limits_0^s x(p, q)dq\Big]dp,$$
$$(F_3R_0x)(t, s) = \int\limits_{g^{-1}(s)}^{0} \Big[\int\limits_0^s x(p, q)dq\Big]dp - \int\limits_{g^{-1}(s)}^{t} \Big[\int\limits_0^{h(p)} x(p, q)dq\Big]dp.$$

Using these last expressions we can derive 3 different Green formulae for the operator $P(D)$ defined by formula (2.27).

Till now we have considered D-algebras over reals. Now we shall pass to D-algebras over the field C of complexes. Write: $Y = X \oplus iX$. The set Y is an algebra with the addition, multiplication by scalar and multiplication of elements defined as follows:

$$(a+ib)+(c+ib) = (a+c)+i(b+d),$$
$$\lambda(a+ib) = \lambda a + i\lambda b \quad \text{for } a, b, c, d \in X,\ \lambda \in C,$$
$$(a+ib)(c+id) = (ac-bd)+i(ad+bc).$$

Let $A = L(X)$ and let $u = x+iy$, where $x, y \in \operatorname{dom} A$. Write:

$$Au = Ax + iAy. \tag{2.28}$$

By this definition $A \in L(Y)$. Indeed, $A(\lambda u) = \lambda Au$ for all $\lambda \in C$, $u \in Y$. Observe that $\overline{Au} = A\bar{u}$, where we write $\bar{u} = \overline{x+iy} = x-iy$.

Indeed, we have $\overline{Au} = \overline{Ax+iAy} = Ax-iAy = A(x-iy) = A\overline{u}$.

It is easy to verify that $Y = X \oplus iX$ is a D-algebra over $\boldsymbol{C}$, provided that X is a D-algebra over $\boldsymbol{R}$. Initial operators and right inverses are extended according with formula (2.28). Write formula (1.7) in the complex case: For all $u, v \in \operatorname{dom} D$ (in Y) we have

$$f_D(u, v) = D(uv) - c_D(uDv + vDu)$$

where c_D is a coefficient defined by formula (1.7) in the D-algebra X. By this definition, since $c_D \in \boldsymbol{R}$, we have

$$\text{(2.29)} \qquad \overline{f_D(\overline{u}, \overline{v})} = f_D(u, v) = f_D(v, u) \quad \text{for all } u, v \in \operatorname{dom} D.$$

Indeed,

$$\begin{aligned} \overline{f_D(\overline{u}, \overline{v})} &= \overline{D(\overline{u}\overline{v})} - c_D(\overline{\overline{u}D\overline{v} + \overline{v}D\overline{u}}) = \overline{D(\overline{u}\overline{v})} - c_D(\overline{\overline{u}D\overline{v}} + \overline{\overline{v}D\overline{u}}) \\ &= D(uv) - c_D(uDv + vDu) = f_D(u, v) = f_D(v, u). \end{aligned}$$

Formula (2.29) implies that

$$\text{(2.30)} \qquad \overline{u}Dv - \overline{vD^+u} = c_D^{-1}D(\overline{u}v) - f_D(\overline{u}, v) \quad \text{for all } u, v \in \operatorname{dom} D.$$

Indeed, since $D^+ = -D$, we find

$$\overline{u}Dv - \overline{vD^+u} = uDv - vD^+u = \overline{u}Dv + vD\overline{u} = c_D^{-1}[D(\overline{u}v) - f_D(\overline{u}, v)].$$

We also have

$$\text{(2.31)} \qquad (\lambda D)^+ = \overline{\lambda} D^+ \quad \text{for all } \lambda \in \boldsymbol{C},$$

i.e. the operator D^+ is antilinear in the complex case.

Indeed, by formulae (2.30), (2.31) we have for $\hat{D} = \lambda D$

$$\begin{aligned} \overline{u}(\lambda D)v - \overline{v(\lambda D)^+u} &= \overline{u}\hat{D}v - v\hat{D}^+u = c_{\hat{D}}^{-1}[\hat{D}(\overline{u}v) - f_{\hat{D}}(\overline{u}, v)] \\ &= c_D^{-1}[\lambda D(uv) - f_{\lambda D}(\overline{u}, v)] = \lambda c_D^{-1}[D(\overline{u}v) - f_D(\overline{u}, v)] \\ &= \lambda(\overline{u}Dv - vD^+u). \end{aligned}$$

This implies that for all $\lambda \in \boldsymbol{C}$, $u \in \operatorname{dom} D$ we have

$$\overline{\lambda}D^+\overline{u} = \overline{\lambda D^+u} = \overline{(\lambda D)^+u} = (\lambda D)^+u.$$

The arbitrariness of $u \in \operatorname{dom} D$ implies (2.31).

In particular, we have

$$\text{(2.32)} \qquad (iD)^+ = iD.$$

All further considerations are going in similar as in the real case. In particular, the Green formula (2.3) in the complex case is of the form

$$\begin{aligned} \text{(2.33)} \quad & F_\beta R_\alpha[\overline{x}Q(D)y - \overline{y\overline{Q}^+(D)x}] \\ &= \sum_{j=0}^{N-1}(-1)^j c_D^{-(j+1)}(F_\beta - F_\alpha)(yQ_{j+1}\overline{x}) + F_\beta R_\alpha[g_{Q(D)}(\overline{x}, y) + h_{Q(D)}(\overline{x}, y)], \end{aligned}$$

where $g_{Q(D)}$, $h_{Q(D)}$ are defined by formulae (2.4), (2.5).

Indeed, if we put x instead of $\bar{x}$ and we shall make use of (2.29), we obtain that

$$\overline{Q^+(D)x} = \overline{\sum_{k=0}^{N} (-1)^k D^k Q_k x} = \sum_{k=0}^{N} (-1)^k D^k Q_k \bar{x},$$

which implies formula (2.33).

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