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On the Chaplygin method for partial differential-functional equations of the first order

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Abstract. Suppose that a solution $\bar{z}(x, Y)$ of the first order partial differential-functional equation

$$z_x(x, Y) = f(x, Y, z(x, Y), z(\cdot)) + \sum_{i=1}^{n} g^{(i)}(x, Y) z_{y_i}(x, Y)$$

is defined on

$$E = \{(x, Y):\ 0 \leqslant x - x_0 < a,\ \tilde{r}_i(x) \leqslant y_i \leqslant \tilde{s}_i(x),\ i = 1, \ldots, n\}.$$

Let

$$G(x, Y, z, Q; u) = f(x, Y, u(x, Y), u(\cdot)) + f_z(x, Y, u(x, Y), u(\cdot))(z - u(x, Y)) + \sum_{i=1}^{n} g^{(i)}(x, Y) q_i,$$

$$H(x, Y, z, Q; u, v) = f(x, Y, v(x, Y), v(\cdot)) + \frac{f(x, Y, u(x, Y), v(\cdot)) - f(x, Y, v(x, Y), v(\cdot))}{u(x, Y) - v(x, Y)}(z - v(x, Y)) + \sum_{i=1}^{n} g^{(i)}(x, Y) q_i.$$

Suppose that a sequence $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$ is such that

$$u_x^{(m)}(x, Y) = G(x, Y, u^{(m)}(x, Y), u_Y^{(m)}(x, Y); u^{(m-1)}),$$

$$v_x^{(m)}(x, Y) = H(x, Y, v^{(m)}(x, Y), v_Y^{(m)}(x, Y); u^{(m-1)}, v^{(m-1)}).$$

In this paper we prove that under certain assumptions concerning the functions $f, g^{(i)}, u^{(m)}, v^{(m)}$ the conditions

$$u^{(m-1)}(x, Y) \leqslant u^{(m)}(x, Y) \leqslant \bar{z}(x, Y) \leqslant v^{(m)}(x, Y) \leqslant v^{(m-1)}(x, Y), \quad (x, Y) \in E,\ m = 1, 2, \ldots,$$

and

$$\lim_{m\to\infty} (v^{(m)}(x, Y) - u^{(m)}(x, Y)) = 0, \qquad (x, Y) \in E,$$

are satisfied. We also give estimates of the difference between the exact and the approximate solutions.

Let $C(E_0 \cup E, R)$ be the class of continuous functions from $E_0 \cup E$ into R, where $R = (-\infty, +\infty)$ and

$$E_0 = \{(x, Y)\colon x_0 - \tau_0 \leqslant x \leqslant x_0,\ \tau_0 \geqslant 0,\ Y = (y_1, \ldots, y_n),$$
$$\bar{r}_i(x) \leqslant y_i \leqslant \bar{s}_i(x),\ i = 1, \ldots, n\},$$

$$E = \{(x, Y)\colon 0 \leqslant x - x_0 < a,\ a > 0,\ \tilde{r}_i(x) \leqslant y_i \leqslant \tilde{s}_i(x),\ i = 1, \ldots, n\}.$$

Assume that

(a) $\bar{r}_i$ and $\bar{s}_i$, $i = 1, \ldots, n$, are continuous functions on $[x_0 - \tau_0, x_0]$ and $\bar{r}_i(x) \leqslant \bar{s}_i(x)$, $x \in [x_0 - \tau_0, x_0]$,

(b) $\tilde{r}_i$ and $\tilde{s}_i$, $i = 1, \ldots, n$, are of class C^1 on $[x_0, x_0 + a)$ and $\tilde{r}_i(x) < \tilde{s}_i(x)$ for $x \in [x_0, x_0 + a)$, $\tilde{r}_i(x_0) = \bar{r}_i(x_0)$, $\tilde{s}_i(x_0) = \bar{s}_i(x_0)$.

Elements of $C(E_0 \cup E, R)$ will be denoted by $z(\cdot)$, $u(\cdot)$, $v(\cdot)$ and the like.

Let

$$H_x = \{(\xi, \Theta)\colon (\xi, \Theta) \in E_0 \cup E,\ \xi \leqslant x\}.$$

Suppose that the function $f(x, Y, z, u(\cdot))$ is defined on $E \times R \times C(E_0 \cup E, R)$. We assume that f satisfies the Volterra condition, i.e., if $(x, Y, z, u_i(\cdot)) \in E \times R \times C(E_0 \cup E, R)$ for $i = 1, 2$ and $u_1(\xi, \Theta) = u_2(\xi, \Theta)$ for $(\xi, \Theta) \in H_x$, then $f(x, Y, z, u_1(\cdot)) = f(x, Y, z, u_2(\cdot))$. Suppose that the functions $g^{(i)}(x, Y)$ are defined for $(x, Y) \in E$.

We shall consider here the initial problem for the first order partial differential-functional equation

$$\begin{aligned} &z_x(x, Y) = f(x, Y, z(x, Y), z(\cdot)) + \sum_{i=1}^{n} g^{(i)}(x, Y) z_{y_i}(x, Y), \qquad (x, Y) \in E, \\ (1)\quad & \\ &z(x, Y) = a(x, Y) \qquad \text{for } (x, Y) \in E_0, \end{aligned}$$

where a is a given initial function.

We are interested in working out a method of approximation of a solution of the Cauchy problem for equation (1) by solutions of an associated linear equation and in estimating the difference between the exact and approximate solutions. This is precisely what the Chaplygin method accomplishes (see [3], [6], [8], p. 90–96).

The basic tool in our investigations are theorems on partial differential-functional inequalities of the first order.

The paper is divided into two parts. The first part deals with partial differential-functional inequalities. The second part contains theorems on the Chaplygin method for the Cauchy problem (1).

Our results are generalizations of some results of paper [7], where the Chaplygin method for partial differential equations of the first order was considered.

I. Partial differential-functional inequalities. Let

$$\bar{K} = \{(x, Y): x = x_0, \tilde{r}_i(x_0) \leqslant y_i \leqslant \tilde{s}_i(x_0), i = 1, \ldots, n\},$$

$$S = \{(x, Y): (x, Y) \in E - \bar{K} \text{ and there exists } j, 1 \leqslant j \leqslant n, \text{ such that } y_j = \tilde{s}_j(x) \text{ or } y_j = \tilde{r}_j(x)\}.$$

We introduce

ASSUMPTION H_1. Suppose that

1° the real function $F(x, Y, z, u(\cdot), Q)$ is defined for $(x, Y, z, u(\cdot), Q) \in E \times R \times C(E_0 \cup E, R) \times \Omega$, where Ω is a domain in the n-dimensional Euclidean space,

2° F satisfies the Volterra condition, i.e., if $(x, Y, z, u_i(\cdot), Q) \in E \times R \times C(E_0 \cup E, R) \times \Omega$ for $i = 1, 2$ and $u_1(\xi, \Theta) = u_2(\xi, \Theta)$ for $(\xi, \Theta) \in H_x$, then $F(x, Y, z, u_1(\cdot), Q) = F(x, Y, z, u_2(\cdot), Q)$,

3° $F(x, Y, z, u(\cdot), Q)$ is non-decreasing with respect to the functional argument $u(\cdot)$,

4° for each point $(\bar{x}, \bar{Y}) \in S$, $\bar{Y} = (\bar{y}_1, \ldots, \bar{y}_n)$, there exist sets of integers I_1, I_2, I_3 such that $I_1 \cup I_2 \cup I_3 = \{1, 2, \ldots, n\}$ and

$$\bar{y}_i = \tilde{r}_i(\bar{x}) \quad \text{for } i \in I_1,$$

$$\bar{y}_i = \tilde{s}_i(\bar{x}) \quad \text{for } i \in I_2,$$

$$\tilde{r}_i(\bar{x}) < \bar{y}_i < \tilde{s}_i(\bar{x}) \quad \text{for } i \in I_3.$$

We assume that

$$(2) \qquad F(\bar{x}, \bar{Y}, z, u(\cdot), Q) - F(\bar{x}, \bar{Y}, z, u(\cdot), \bar{Q}) + \sum_{i \in I_1} \tilde{r}_i'(\bar{x})(q_i - \bar{q}_i) + \sum_{i \in I_2} \tilde{s}_i'(\bar{x})(q_i - \bar{q}_i) \leqslant 0,$$

where $Q = (q_1, \ldots, q_n)$, $\bar{Q} = (\bar{q}_1, \ldots, \bar{q}_n)$ and $q_i \leqslant \bar{q}_i$ for $i \in I_1$, $q_i \geqslant \bar{q}_i$ for $i \in I_2$, $q_1 = \bar{q}_i$ for $i \in I_3$,

5° $\bar{u}(\cdot), \bar{v}(\cdot) \in C(E_0 \cup E, R)$ possess the derivatives $\bar{u}_x, \bar{v}_x, \bar{u}_Y = (\bar{u}_{y_1}, \ldots, \bar{u}_{y_n})$, $\bar{v}_Y = (\bar{v}_{y_1}, \ldots, \bar{v}_{y_n})$ on $E - \bar{K}$ (not necessarily continuous), and the total derivative on S,

6° $\bar{u}_Y(x, Y), \bar{v}_Y(x, Y) \in \Omega$ for $(x, Y) \in E - \bar{K}$.

Remark 1. If F satisfied the Lipschitz condition

$$|F(x, Y, z, u(\cdot), Q) - F(x, Y, z, u(\cdot), Q)| \leqslant \sum_{k=1}^{n} M_k |q_k - \bar{q}_k|$$

and

$$\tilde{r}_k(x) = y_k^{(0)} - b_k + M_k(x - x_0), \quad k = 1, \ldots, n,$$

$$\tilde{s}_k(x) = y_k^{(0)} + b_k - M_k(x - x_0), \quad k = 1, \ldots, n,$$

where $b_k > 0$, $y_k^{(0)} - b_k = \tilde{r}_k(x_0)$, $y_k^{(0)} + b_k = \tilde{s}_k(x_0)$, $a < \min_i (b_i/M_i)$, then condition (2) is satisfied (see [2]).

THEOREM 1.1. *Suppose that*

1° *Assumption* H_1 *is satisfied,*

2° *the functions* $\bar{u}(\cdot)$ *and* $\bar{v}(\cdot)$ *fulfil the initial inequalities*

$$\begin{aligned} \bar{u}(x, Y) &\leqslant \bar{v}(x, Y) \quad \text{for } (x, Y) \in E_0, \\ \bar{u}(x_0, Y) &< \bar{v}(x_0, Y) \quad \text{for } (x_0, Y) \in \bar{K}, \end{aligned} \tag{3}$$

and the differential inequalities

$$\bar{u}_x(x, Y) < F\big(x, Y, \bar{u}(x, Y), \bar{u}(\cdot), \bar{u}_Y(x, Y)\big), \quad (x, Y) \in E - \bar{K}, \tag{4}$$

$$\bar{v}_x(x, Y) \geqslant F\big(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)\big), \quad (x, Y) \in E - \bar{K}. \tag{5}$$

Under these assumptions

$$\bar{u}(x, Y) < \bar{v}(x, Y) \tag{6}$$

for $(x, Y) \in E$.

Proof. If assertion (6) is false, then the set

$$Z = \{x\colon x \in [x_0, x_0 + a),\ \bar{u}(x, Y) \geqslant \bar{v}(x, Y) \text{ for some } (x, Y) \in E\}$$

is non-empty. Write $x^* = \inf Z$; it is clear from (3) that $x^* > x_0$ and $\bar{u}(x^*, Y^*) = \bar{v}(x^*, Y^*)$ for some $(x^*, Y^*) \in E$, $Y^* = (y_1^*, \ldots, y_n^*)$.

Now, there are two cases to be distinguished.

(a) If (x^*, Y^*) is an interior point of E, then $\bar{u}_Y(x^*, Y^*) = \bar{v}_Y(x^*, Y^*)$ and

$$\bar{u}_x(x^*, Y^*) - \bar{v}_x(x^*, Y^*) \geqslant 0. \tag{7}$$

Since $\bar{u}(\xi, \Theta) \leqslant \bar{v}(\xi, \Theta)$ for $(\xi, \Theta) \in H_{x^*}$, it follows from conditions 2°, 3° of Assumption H_1 and from (4), (5) that

$$\begin{aligned} &\bar{u}_x(x^*, Y^*) - \bar{v}_x(x^*, Y^*) \\ &\quad < F\big(x^*, Y^*, \bar{u}(x^*, Y^*), \bar{u}(\cdot), \bar{u}_Y(x^*, Y^*)\big) - F\big(x^*, Y^*, \bar{v}(x, Y^*), \bar{v}(\cdot), \\ &\qquad \bar{v}_Y(x^*, Y^*)\big) \leqslant 0, \end{aligned}$$

which contradicts (7).

(b) Suppose that $(x^*, Y^*) \in S$. We may assume (rearranging the indices if necessary) that

$$\begin{aligned} y_i^* &= \tilde{s}_i(x^*) \quad \text{for } i = 1, \ldots, s, \\ y_i^* &= \tilde{r}_i(x^*) \quad \text{for } i = s+1, \ldots, p, \\ \tilde{r}_i(x^*) < y_i^* &< \tilde{s}_i(x^*) \quad \text{for } i = p+1, \ldots, n. \end{aligned} \tag{8}$$

Then

$$
(9) \qquad \begin{aligned}
&\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*) \geqslant 0 \quad &&\text{for } i = 1, \ldots, s,\\
&\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*) \leqslant 0 \quad &&\text{for } i = s+1, \ldots, p,\\
&\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*) = 0 \quad &&\text{for } i = p+1, \ldots, n.
\end{aligned}
$$

Now, for $x_0 \leqslant x \leqslant x^*$ put

$$\tilde{Y}(x) = \big(\tilde{s}_1(x), \ldots, \tilde{s}_s(x), \tilde{r}_{s+1}(x), \ldots, \tilde{r}_p(x), y^*_{p+1}, \ldots, y^*_n\big)$$

and consider the composite function $\bar{u}\big(x, \tilde{Y}(x)\big) - \bar{v}\big(x, \tilde{Y}(x)\big)$. It attains maximum at x^*, and hence

$$
(10) \qquad \bar{u}_x(x^*, Y^*) - \bar{v}_x(x^*, Y^*) + \sum_{i=1}^{s} [\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*)]\tilde{s}'_i(x^*) + \\
+ \sum_{i=s+1}^{p} [\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*)]\tilde{r}'_i(x^*) \geqslant 0.
$$

Since $\bar{u}(\xi, \Theta) \leqslant \bar{v}(\xi, \Theta)$ on H_{x^*}, we obtain by conditions 2°, 3° from Assumption $\mathbf{H}_1$ and by (2), (4), (5), (8), (9) that

$$
\begin{aligned}
&\bar{u}_x(x^*, Y^*) - \bar{v}_x(x^*, Y^*)\\
&< F\big(x^*, Y^*, \bar{u}(x^*, Y^*), \bar{u}(\cdot), \bar{u}_Y(x^*, Y^*)\big) - F\big(x^*, Y^*, \bar{v}(x^*, Y^*),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \bar{v}(\cdot), \bar{v}_Y(x^*, Y^*)\big)\\
&\leqslant -\sum_{i=1}^{s} [\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*)]\tilde{s}'_i(x^*) -\\
&\qquad\qquad - \sum_{i=s+1}^{p} [\bar{u}_{y_i}(x^*, Y^*) - \bar{v}_{y_i}(x^*, Y^*)]\tilde{r}'_i(X^*),
\end{aligned}
$$

which contradicts (10).

Hence Z is empty, and the statement (6) follows.

Remark 2. If $\tau_0 = 0$ and F does not contain the functional argument, then from Theorem 1.1 we obtain the well-known theorem on strong first order partial differential inequalities (see [8], p. 169–171, [5], vol. II, p. 113–116, [9], Theorem 1.1).

Remark 3. In Theorem 1.1 we might assume instead of (4), (5) that

$$
\begin{aligned}
&\bar{u}_x(x, Y) \leqslant F\big(x, Y, \bar{u}(x, Y), \bar{u}(\cdot), \bar{u}_Y(x, Y)\big), \quad &&(x, Y) \in E - \bar{K},\\
&\bar{v}_x(x, Y) \geqslant F\big(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)\big), \quad &&(x, Y) \in E - \bar{K},
\end{aligned}
$$

where for each $(x, Y) \in E - \bar{K}$ equality holds in at most one place.

We introduce

ASSUMPTION H_2. Suppose that the condition

$$F(x, Y, z, u(\cdot), Q) - F(x, Y, \bar{z}, u(\cdot), Q) \leqslant \sigma(x-x_0, z-\bar{z})$$

is satisfied for $(x, Y, z, u(\cdot), Q), (x, Y, \bar{z}, u(\cdot), Q) \in E \times R \times C(E_0 \cup E, R) \times \Omega$ and $z \leqslant \bar{z}$, where the function $\sigma(t, z)$ has the following properties (see [1]):

1° $\sigma(t, z)$ is continuous and non-negative for $t \in [0, a)$, $z \leqslant 0$, and $\sigma(t, 0) = 0$,

2° the left-hand minimum solution of the equation

$$z' = \sigma(t, z)$$

satisfying the condition $\lim_{t \to a-} z(t) = 0$ is $z(t) \equiv 0$.

THEOREM 2.1 (see [1]). *Suppose that*

1° *Assumptions* H_1 *and* H_2 *are satisfied,*

2° $\bar{u}(\cdot)$ *and* $\bar{v}(\cdot)$ *satisfy the differential inequalities*

$$\begin{aligned} \bar{u}_x(x, Y) &\leqslant F(x, Y, \bar{u}(x, Y), \bar{u}(\cdot), \bar{u}_Y(x, Y)), \quad (x, Y) \in E - \bar{K}, \\ \bar{v}_x(x, Y) &\geqslant F(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)), \quad (x, Y) \in E - \bar{K} \end{aligned} \tag{11}$$

and the initial inequalities

$$\begin{aligned} \bar{u}(x, Y) &\leqslant \bar{v}(x, Y) \quad \text{for } (x, Y) \in E_0, \\ \bar{u}(x_0, Y) &< \bar{v}(x_0, Y) \quad \text{for } (x_0, Y) \in \bar{K}. \end{aligned} \tag{12}$$

Under these assumptions the inequality

$$\bar{u}(x, Y) < \bar{v}(x, Y) \tag{13}$$

is satisfied for $(x, Y) \in E$.

Proof. At first we prove (13) for $(x, Y) \in E$ and $x_0 \leqslant x < x_0 + a - \varepsilon$, where $0 < \varepsilon < a$.

Let $0 < z_0 < \min_{(x,Y) \in \bar{K}} [\bar{v}(x, Y) - \bar{u}(x, Y)]$. For $\delta > 0$ denote by $\omega(t)$ the right-hand minimum solution of the equation

$$z' = -\sigma(t, -z) - \delta$$

through $(0, z_0)$. If $z_0 > 0$ is fixed, then to every $\varepsilon > 0$ there corresponds $\delta_0(\varepsilon) > 0$ such that for $0 < \delta < \delta_0(\varepsilon)$ the solution $\omega(t)$ exists and is positive in the interval $[0, a-\varepsilon)$ (see [1]). Let $\delta > 0$ be such a small constant that the function $\omega(t)$ satisfies the above conditions.

Denote by $\tilde{z}(x, Y)$ a continuous function defined in E_0 such that

$$\bar{u}(x, Y) \leqslant \tilde{z}(x, Y) \leqslant \bar{v}(x, Y), \quad (x, Y) \in E_0, \tag{14}$$

and

$$\tilde{z}(x_0, Y) = \bar{u}(x_0, Y) + z_0, \quad (x_0, Y) \in \bar{K}.$$

Let

$$\tilde{u}(x, Y) = \begin{cases} \tilde{z}(x, Y) & \text{for } (x, Y) \in E_0, \\ \bar{u}(x, Y) + \omega(x - x_0) & \text{for } (x, Y) \in E,\ x_0 \leqslant x < x_0 + a - \varepsilon. \end{cases}$$

We shall prove that

(15) $$\tilde{u}(x, Y) < \bar{v}(x, Y) \quad \text{for } (x, Y) \in E,\ x \in [x_0, x_0 + a - \varepsilon).$$

By Assumption H_2 and (11) we get

$$\begin{aligned} \tilde{u}_x(x, Y) &= \bar{u}_x(x, Y) + \omega'(x - x_0) \\ &\leqslant F\big(x, Y, \bar{u}(x, Y), \bar{u}(\cdot), \bar{u}_Y(x, Y)\big) - \sigma\big(x - x_0, -\omega(x - x_0)\big) - \delta \\ &\leqslant \big[F\big(x, Y, \bar{u}(x, Y), \tilde{u}(\cdot), \tilde{u}_Y(x, Y)\big) - F\big(x, Y, \tilde{u}(x, Y), \tilde{u}(\cdot), \tilde{u}_Y(x, Y)\big)\big] + \\ &\qquad + F\big(x, Y, \tilde{u}(x, Y), \tilde{u}(\cdot), \tilde{u}_Y(x, Y)\big) - \sigma\big(x - x_0, -\omega(x - x_0)\big) - \delta \\ &\leqslant F\big(x, Y, \tilde{u}(x, Y), \tilde{u}(\cdot), \tilde{u}_Y(x, Y)\big) - \delta. \end{aligned}$$

It follows from the above estimations that

(16) $$\tilde{u}_x(x, Y) < F\big(x, Y, \tilde{u}(x, Y), \tilde{u}(\cdot), \tilde{u}_Y(x, Y)\big)$$

for $(x, Y) \in E - \bar{K}$ and $x \in [x_0, x_0 + a - \varepsilon)$.

By the definition of $\tilde{u}(x, Y)$ we have

$$\begin{aligned} \tilde{u}(x, Y) &\leqslant \bar{v}(x, Y) && \text{for } (x, Y) \in E_0, \\ \tilde{u}(x_0, Y) &< \bar{v}(x_0, Y) && \text{for } (x_0, Y) \in \bar{K}, \end{aligned}$$

and hence, by the second inequality of (11) and by (16), we get from Theorem 1.1 that

$$\tilde{u}(x, Y) < \bar{v}(x, Y) \quad \text{for } (x, Y) \in E,\ x \in [x_0, x_0 + a - \varepsilon).$$

Since ε is arbitrary, inequality (13) holds true in E.

We introduce

ASSUMPTION H_3. Suppose that

1° a function $\sigma(t, z_1, z_2)$ is continuous and non-negative for $t \geqslant 0$, $z_1 \geqslant 0, z_2 \geqslant 0$, and $\sigma(t, 0, 0) = 0$,

2° the right-hand maximum solution of the initial problem

$$z' = \sigma(t, z, z), \qquad z(0) = 0$$

is $z(t) \equiv 0$,

3° $$F\big(x, Y, z_1, u_1(\cdot), Q\big) - F\big(x, Y, z_2, u_2(\cdot), Q\big) \geqslant -\sigma\Big(x - x_0 + \tau_0, z_2 - z_1, \sup_{(\xi, \Theta) \in H_x} [u_2(\xi, \Theta) - u_1(\xi, \Theta)]\Big)$$

for $z_1 \leqslant z_2$, $u_1(\xi, \Theta) \leqslant u_2(\xi, \Theta)$ in H_x.

THEOREM 3.1. *Suppose that*

1° *Assumptions* H_1 *and* H_3 *are satisfied,* 2° *the differential inequalities*

$$(17)\qquad \begin{aligned} &\bar{u}_x(x, Y) \leqslant F\big(x, Y, \bar{u}(x, Y), \bar{u}(\cdot), \bar{u}_Y(x, Y)\big), \qquad (x, Y) \in E - \bar{K},\\ &\bar{v}_x(x, Y) \geqslant F\big(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)\big), \qquad (x, Y) \in E - \bar{K}, \end{aligned}$$

and the initial inequalities

$$(18)\qquad \bar{u}(x, Y) \leqslant \bar{v}(x, Y), \qquad (x, Y) \in E_0,$$

are satisfied.

Under these assumptions the inequality

$$(19)\qquad \bar{u}(x, Y) \leqslant \bar{v}(x, Y)$$

is satisfied for $(x, Y) \in E$.

Proof. For $\varepsilon > 0$ denote by $\omega(t; \varepsilon)$ the right-hand maximum solution through the point $(0, \varepsilon)$ of the equation

$$z' = \sigma(t, z, z) + \varepsilon.$$

For $\varepsilon > 0$ sufficiently small $\omega(t; \varepsilon)$ is defined on $[0, a+\tau_0)$ and

$$(20)\qquad \lim_{\varepsilon\to 0} \omega(t; \varepsilon) = 0 \qquad \text{on } [0, a+\tau_0).$$

Consider the function

$$(21)\qquad \tilde{v}(x, Y) = \bar{v}(x, Y) + \omega(x - x_0 + \tau_0, \varepsilon), \qquad (x, Y) \in E_0 \cup E.$$

By Assumption H_3 and (17), (21) we get

$$\begin{aligned} &\tilde{v}_x(x, Y) = \bar{v}_x(x, Y) + \omega'(x - x_0 + \tau_0, \varepsilon)\\ &\geqslant F\big(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)\big) + \omega'(x - x_0 + \tau_0, \varepsilon)\\ &= F\big(x, Y, \tilde{v}(x, Y), \tilde{v}(\cdot), \tilde{v}_Y(x, Y)\big) +\\ &+ \big[F\big(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)\big) - F\big(x, Y, \tilde{v}(x, Y), \tilde{v}(\cdot), \bar{v}_Y(x, Y)\big)\big] +\\ &+ \sigma\big(x - x_0 + \tau_0,\ \omega(x - x_0 + \tau_0, \varepsilon), \omega(x - x_0 + \tau_0, \varepsilon)\big) + \varepsilon\\ &\geqslant -\sigma\big(x - x_0 + \tau_0, \tilde{v}(x, Y) - \bar{v}(x, Y), \sup_{(\xi, \Theta)\in H_x} [\tilde{v}(\xi, \Theta) - \bar{v}(\xi, \Theta)]\big) +\\ &+ \sigma\big(x - x_0 + \tau_0, \omega(x - x_0 + \tau_0, \varepsilon), \omega(x - x_0 + \tau_0, \varepsilon)\big) + \varepsilon +\\ &+ F\big(x, Y, \tilde{v}(x, Y), \tilde{v}(\cdot), \tilde{v}_Y(x, Y)\big) = F\big(x, Y, \tilde{v}(x, Y), \tilde{v}(\cdot), \tilde{v}_Y(x, Y)\big) + \varepsilon. \end{aligned}$$

Thus we see that the strong differential inequality

$$(22)\qquad \tilde{v}_x(x, Y) > F\big(x, Y, \tilde{v}(x, Y), \tilde{v}(\cdot), \tilde{v}_Y(x, Y)\big), \qquad (x, Y) \in E - \bar{K},$$

is satisfied.

By (21) we have

$$\tilde{v}(x_0, Y) = \bar{v}(x_0, Y) + \omega(\tau_0, \varepsilon) > \bar{u}(x_0, Y) \qquad \text{on } \bar{K},$$

and hence, by the first inequality of (17) and by (22), we get from Theorem 1.1 (see also Remark 1) that

$$\bar{u}(x, Y) < \tilde{v}(x, Y) = \bar{v}(x, Y) + \omega(x - x_0 + \tau_0, \varepsilon)$$

for $(x, Y) \in E$. From the above inequality and from (20) we obtain in the limit (letting ε tend to 0) inequality (18).

THEOREM 4.1. *Suppose that*

1° *Assumptions* H_1 *and* H_3 *are satisfied,*

2° *for* $(x, Y) \in E - \bar{K}$

$$\bar{u}_x(x, Y) \leqslant F\big(x, Y, \bar{u}(x, Y), \bar{u}(\cdot), \bar{u}_Y(x, Y)\big),$$
$$\bar{v}_x(x, Y) \geqslant F\big(x, Y, \bar{v}(x, Y), \bar{v}(\cdot), \bar{v}_Y(x, Y)\big),$$

where for each $(x, Y) \in E - \bar{K}$ *equality holds in at most one place,*

3° *for* $(x, Y) \in E_0$ *we have*

$$\bar{u}(x, Y) \leqslant \bar{v}(x, Y).$$

Under these assumptions

$$\bar{u}(x, Y) < \bar{v}(x, Y) \quad \textit{for } (x, Y) \in E - \bar{K}.$$

This theorem can be proved by applying the weak differential-functional inequalities theorem (Theorem 3.1) and then repeating the argument used in the proof of the theorem on strong differential-functional inequalities (Theorem 1.1).

II. Chaplygin method. In order to simplify the formulation of subsequent theorems we first introduce the following definitions.

A function $u(x, Y)$ is said to be of *class D* in $E_0 \cup E$ if $u(x, Y)$ is continuous in $E_0 \cup E$, has the first order derivatives $u_x(x, Y)$, $u_Y(x, Y)$ for $(x, Y) \in E$ and has the total derivative on S.

In this section we shall consider the almost linear differential-functional equation (1). For given functions $u(x, Y)$, $v(x, Y)$ of class D in $E_0 \cup E$ we define

$$(23) \quad G(x, Y, z, Q; u) = f\big(x, Y, u(x, Y), u(\cdot)\big) + f_z\big(x, Y, u(x, Y), u(\cdot)\big)\big(z - u(x, Y)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) q_i$$

and

$$(24) \quad H(x, Y, z, Q; u, v) = f\big(x, Y, v(x, Y), v(\cdot)\big) + \frac{f\big(x, Y, u(x, Y), v(\cdot)\big) - f\big(x, Y, v(x, Y), v(\cdot)\big)}{u(x, Y) - v(x, Y)}\big(z - v(x, Y)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) q_i \quad \text{if } u(x, Y) - v(x, Y) \neq 0$$

and

$$H(x, Y, z, Q; u, v) = f(x, Y, v(x, Y), v(\cdot)) + \tag{25}$$
$$+ f_z(x, Y, v(x, Y), v(\cdot))(z - v(x, Y)) + \sum_{i=1}^{n} g^{(i)}(x, Y) q_i$$
$$\text{if } u(x, Y) - v(x, Y) = 0,$$

where $Q = (q_1, \ldots, q_n)$.

Suppose that we are given a sequence $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$, where $u^{(m)}(x, Y)$ and $v^{(m)}(x, Y)$ are of class D in $E_0 \cup E$ and

$$u_x^{(m)}(x, Y) = G(x, Y, u^{(m)}(x, Y), u_Y^{(m)}(x, Y); u^{(m-1)}), \tag{26}$$
$$(x, Y) \in E, \ m = 1, 2, \ldots,$$

$$v_x^{(m)}(x, Y) = H(x, Y, v^{(m)}(x, Y), v_Y^{(m)}(x, Y); u^{(m-1)}, v^{(m-1)}), \tag{27}$$
$$(x, Y) \in E, \ m = 1, 2, \ldots$$

A sequence $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$ which satisfies (26), (27) will be called the *Chaplygin sequence.*

Remark 4. If $u^{(m-1)}, v^{(m-1)}$ are known functions, then $u^{(m)}, v^{(m)}$ are solutions of linear partial differential equations. Sufficient conditions for the global existence of solutions of linear partial equations can be found in [4].

We introduce

ASSUMPTION H_4. Suppose that

1° the real function $f(x, Y, z, u(\cdot))$ is defined for $(x, Y, z, u(\cdot)) \in E \times R \times C(E_0 \cup E, R)$ and satisfies the Volterra condition, i.e. if $(x, Y, z, u_i(\cdot)) \in E \times R \times C(E_0 \cup E, R)$ for $i = 1, 2$ and $u_1(\xi, \Theta) = u_2(\xi, \Theta)$ for $(\xi, \Theta) \in H_x$, then $f(x, Y, z, u_1(\cdot)) = f(x, Y, z, u_2(\cdot))$,

2° $f(x, Y, z, u(\cdot))$ is non-decreasing with respect to the functional argument $u(\cdot)$,

3° the functions $g^{(i)}(x, Y)$, $i = 1, \ldots, n$, are defined on E and for each $(x, Y) \in S$ such that

$$y_i = \tilde{r}_i(x) \quad \text{for } i \in I_1,$$
$$y_i = \tilde{s}_i(x) \quad \text{for } i \in I_2,$$
$$\tilde{r}_i(x) < y_i < \tilde{s}_i(x) \quad \text{for } i \in I_3,$$

where $I_1 \cup I_2 \cup I_3 = \{1, \ldots, n\}$, the inequality

$$\sum_{i=1}^{n} g^{(i)}(x, Y)(q_i - \bar{q}_i) + \sum_{i \in I_1} \tilde{r}_i'(x)(q_i - \bar{q}_i) + \sum_{i \in I_2} \tilde{s}_i'(x)(q_i - \bar{q}_i) \leqslant 0$$

is satisfied for $q_i \leqslant \bar{q}_i$ for $i \in I_1$, $q_i \geqslant \bar{q}_i$ for $i \in I_2$, $q_i = \bar{q}_i$ for $i \in I_3$, 4° the derivative f_z exists on $E\times R\times C(E_0\cup E, R)$,

5° $\bar{z}(x, Y)$ is a solution of (1) and is of class D in $E_0\cup E$.

THEOREM 1.2. *Suppose that*

1° *Assumption* H_4 *is satisfied,*

2° *the derivative* f_z *is increasing with respect to* z,

3° $u^{(0)}(x, Y)$ *and* $v^{(0)}(x, Y)$ *are functions of class* D *in* $E_0\cup E$ *and for* $(x, Y)\in E-\bar{K}$

$$(28)\quad u_x^{(0)}(x, Y) < f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big)+\sum_{i=1}^{n} g^{(i)}(x, Y)u_{y_i}^{(0)}(x, Y),$$

$$(29)\quad v_x^{(0)}(x, Y) > f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big)+\sum_{i=1}^{n} g^{(i)}(x, Y)v_{y_i}^{(0)}(x, Y),$$

4° *the Chaplygin sequence* $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$ *of functions of class* D *in* $E_0\cup E$ *satisfies the initial conditions*

$$(30)\quad u^{(m-1)}(x, Y)\leqslant u^{(m)}(x, Y)\leqslant \bar{z}(x, Y)\leqslant v^{(m)}(x, Y)\leqslant v^{(m-1)}(x, Y),$$
$$(x, Y)\in E_0 \text{ and } m=1, 2, \ldots$$

and

$$(31)\quad u^{(m-1)}(x_0, Y) < u^{(m)}(x_0, Y) < \bar{z}(x_0, Y) < v^{(m)}(x_0, Y) < v^{(m-1)}(x_0, Y),$$
$$(x_0, Y)\in \bar{K},\ m=1, 2, \ldots$$

Under these assumptions

$$(32)\quad u^{(m-1)}(x, Y) < u^{(m)}(x, Y) < \bar{z}(x, y) < v^{(m)}(x, Y) < v^{(m-1)}(x, Y),$$
$$(x, Y)\in E,\ m=1, 2, \ldots,$$

and

$$(33)\quad u_x^{(m)}(x, Y) < f\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big)+\sum_{i=1}^{n} g^{(i)}(x, Y)u_{y_i}^{(m)}(x, Y),$$
$$(x, Y)\in E-\bar{K},\ m=1, 2, \ldots,$$

$$(34)\quad v_x^{(m)}(x, Y) > f\big(x, Y, v^{(m)}(x, Y), v^{(m)}(\cdot)\big)+\sum_{i=1}^{n} g^{(i)}(x, Y)v_{y_i}^{(m)}(x, Y),$$
$$(x, Y)\in E-\bar{K},\ m=1, 2, \ldots$$

Proof. From (23), (26), (28) it follows that

$$(35)\quad u_x^{(0)}(x, Y) < G\big(x, Y, u^{(0)}(x, Y), u_Y^{(0)}(x, Y); u^{(0)}\big),\qquad (x, Y)\in E-\bar{K},$$

and

$$(36)\quad u_x^{(1)}(x, Y) = G\big(x, Y, u^{(1)}(x, Y), u_Y^{(1)}(x, Y); u^{(0)}\big),\qquad (x, Y)\in E.$$

Since $u^{(0)}(x, Y) < u^{(1)}(x, Y)$ for $(x, Y)\in \bar{K}$, we obtain by theorems on strong differential inequalities (see [5], [8]) that

$$(37)\qquad u^{(0)}(x, Y) < u^{(1)}(x, Y)\quad \text{for } (x, Y)\in E.$$

In a similar way we prove that

$$v^{(1)}(x, Y) < v^{(0)}(x, Y) \quad \text{for } (x, Y) \in E. \tag{38}$$

Now, we prove that for $(x, Y) \in E - \bar{K}$

$$u_x^{(1)}(x, Y) < f\big(x, Y, u^{(1)}(x, Y), u^{(1)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(1)}(x, Y). \tag{39}$$

Inequality (37) together with the monotonicity condition of f and assumption 2° of our theorem, imply

$$f\big(x, Y, u^{(1)}(x, Y), u^{(1)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(1)}(x, Y) - u_x^{(1)}(x, Y)$$
$$\geqslant f\big(x, Y, u^{(1)}(x, Y), u^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(1)}(x, Y) -$$
$$-f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big) - f_z\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big)\big(u^{(1)}(x, Y) -$$
$$-u^{(0)}(x, Y)\big) - \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(1)}(x, Y) > 0.$$

Thus we see that differential-functional inequality (39) is true.

Since the functions $u^{(1)}(x, Y)$ and $\bar{z}(x, Y)$ fulfil the assumptions of Theorem 1.1, we obtain

$$u^{(1)}(x, Y) < \bar{z}(x, Y) \quad \text{for } (x, Y) \in E. \tag{40}$$

From (28)–(31) and from Theorem 1.1 it follows that

$$u^{(0)}(x, Y) < v^{(0)}(x, Y) \quad \text{for } (x, Y) \in E. \tag{41}$$

Hence,

$$u_x^{(0)}(x, Y) < f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(0)}(x, Y)$$
$$\leqslant f\big(x, Y, u^{(0)}(x, Y), v^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(0)}(x, Y)$$
$$= H\big(x, Y, u^{(0)}(x, Y), u_Y^{(0)}(x, Y); u^{(0)}, v^{(0)}\big),$$

and consequently, we have, applying theorems on partial differential inequalities (see [8], Theorem 57.1, [5], Theorem 9.5.1)

$$u^{(0)}(x, Y) < v^{(1)}(x, Y) \quad \text{for } (x, Y) \in E. \tag{42}$$

The function f is convex in z and non-decreasing with respect to the functional argument; hence for $(x, Y) \in E - \bar{K}$ we have

$$v^{(1)}(x, Y) = f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big) +$$
$$+ \frac{f\big(x, Y, u^{(0)}(x, Y), v^{(0)}(\cdot)\big) - f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big)}{u^{(0)}(x, Y) - v^{(0)}(x, Y)} \times$$
$$\times \big(v^{(1)}(x, Y) - v^{(0)}(x, Y)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v^{(1)}_{y_i}(x, Y)$$
$$> f\big(x, Y, v^{(1)}(x, Y), v^{(1)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v^{(1)}_{y_i}(x, Y).$$

Thus, we have for $(x, Y) \in E - \bar{K}$

$$v^{(1)}_x(x, Y) > f\big(x, Y, v^{(1)}(x, Y), v^{(1)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v^{(1)}_{y_i}(x, Y). \tag{43}$$

From the initial inequalities (30), (31) for $m = 1$ and from (43) we obtain by Theorem 1.1

$$\bar{z}(x, Y) < v^{(1)}(x, Y) \quad \text{for } (x, Y) \in E. \tag{44}$$

It follows from (37)–(40), (43), (44) that assertions (32)–(34) are satisfied for $m = 1$. The proof of (32)–(34) for $m \geqslant 2$ is simple, it runs by induction.

Now, we consider the case when the derivative f_z is decreasing with respect to z.

THEOREM 2.2. *Suppose that*

1° *Assumption* H_4 *is satisfied,*

2° *the derivative* f_z *is decreasing with respect to* z,

3° $u^{(0)}(x, Y)$ *and* $v^{(0)}(x, Y)$ *are functions of class* D *in* $E_0 \cup E$ *and for* $(x, Y) \in E - K$

$$u^{(0)}_x(x, Y) < f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u^{(0)}_{y_i}(x, Y),$$

$$v^{(0)}_x(x, Y) > f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v^{(0)}_{y_i}(x, Y),$$

4° *there exists a sequence* $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$, *where* $u^{(m)}(x, Y)$ *and* $v^{(m)}(x, Y)$ *are of class* D *in* $E_0 \cup E$ *and*

$$u^{(m)}_x(x, Y) = H\big(x, Y, u^{(m)}(x, Y), u^{(m)}_Y(x, Y); v^{(m-1)}, u^{(m-1)}\big),$$
$$(x, Y) \in E, \; m = 1, 2, \ldots,$$

$$v^{(m)}_x(x, Y) = G\big(x, Y, v^{(m)}(x, Y), v^{(m)}_Y(x, Y); v^{(m-1)}\big),$$
$$(x, Y) \in E, \; m = 1, 2, \ldots,$$

5° *the initial inequalities*

$$u^{(m-1)}(x, Y) \leqslant u^{(m)}(x, Y) \leqslant \bar{z}(x, Y) \leqslant v^{(m)}(x, Y) \leqslant v^{(m-1)}(x, Y),$$
$$(x, Y) \in E_0, \ m = 1, 2, \ldots,$$
$$u^{(m-1)}(x_0, Y) < u^{(m)}(x_0, Y) < \bar{z}(x_0, Y) < v^{(m)}(x_0, Y) < v^{(m-1)}(x_0, Y),$$
$$(x_0, Y) \in \bar{K}, \ m = 1, 2, \ldots,$$

are satisfied.

Under these assumptions

$$u^{(m-1)}(x, Y) < u^{(m)}(x, Y) < \bar{z}(x, Y) < v^{(m)}(x, Y) < v^{(m-1)}(x, Y),$$
$$(x, Y) \in E, \ m = 1, 2, \ldots,$$

and

$$u_x^{(m)}(x, Y) < f\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(m)}(x, Y),$$

$$v_x^{(m)}(x, Y) > f\big(x, Y, v^{(m)}(x, Y), v^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v_{y_i}^{(m)}(x, Y),$$
$$(x, Y) \in E - \bar{K}, \ m = 1, 2, \ldots$$

We omit the proof of this theorem.

THEOREM 3.2. *Suppose that*

1° *Assumption* H_4 *is satisfied,*

2° *the derivative* f_z *is non-decreasing with respect to* z,

3° *there exists a constant* $N \geqslant 0$ *such that*

$$\big|f_z\big(x, Y, z, u(\cdot)\big)\big| \leqslant N$$

on $E \times R \times C(E_0 \cup E, R)$,

4° $u^{(0)}(x, Y)$ *and* $v^{(0)}(x, Y)$ *are functions of class* D *in* $E_0 \cup E$ *and for* $(x, Y) \in E - \bar{K}$

$$u_x^{(0)}(x, Y) \leqslant f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(0)}(x, Y),$$

$$v_x^{(0)}(x, Y) \geqslant f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v_{y_i}^{(0)}(x, Y),$$

5° *the Chaplygin sequence* $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$ *of functions of class* D *in* $E_0 \cup E$ *satisfies the initial conditions* (30) *and* (31).

Under these assumptions

$$u^{(m-1)}(x, Y) < u^{(m)}(x, Y) < \bar{z}(x, Y) < v^{(m)}(x, Y) < v^{(m-1)}(x, Y),$$
$$(x, Y) \in E, \ m = 1, 2, \ldots,$$

and

$$u_x^{(m)}(x, Y) \leqslant f\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y)\, u_{y_i}^{(m)}(x, Y),$$

$$v_x^{(m)}(x, Y) \geqslant f\big(x, Y, v^{(m)}(x, Y), v^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y)\, v_{y_i}^{(m)}(x, Y),$$

$$(x, Y) \in E - \bar{K}, \quad m = 1, 2, \ldots$$

This theorem can be proved by applying Theorem 2.1 and by an argument similar that used in the proof of Theorem 1.2.

It is easy to formulate a theorem analogous to Theorem 3.2 in thef case when f_z is non-increasing with respect to z (see Theorem 2.2).

The above investigations were based on Theorem 1.1 and Theorem 1.2. We have been dealing with strict inequalities between $u^{(m-1)}(x, Y)$, $u^{(m)}(x, Y)$, $\bar{z}(x, Y)$, $v^{(m)}(x, Y)$ and $v^{(m-1)}(x, Y)$ on E.

The next theorem concerns weak inequalities.

THEOREM 4.2. *Suppose that*

1° *Assumption* H_4 *is satisfied,*

2° *the derivative* f_z *is non-decreasing with respect to* z,

3° *there exist constants* $M, N \geqslant 0$ *such that*

$$\big|f_z(x, Y, z, u(\cdot))\big| \leqslant N$$

and

$$\big|f(x, Y, z, u(\cdot)) - f(x, Y, z, v(\cdot))\big| \leqslant M \sup_{(\xi,\Theta)\in H_x} |u(\xi, \Theta) - v(\xi, \Theta)|$$

for $(x, Y, z, u(\cdot))$, $(x, Y, z, v(\cdot)) \in E \times R \times C(E_0 \cup E, R)$,

4° $u^{(0)}(x, Y)$ *and* $v^{(0)}(x, Y)$ *are functions of class* D *in* $E_0 \cup E$ *and for* $(x, Y) \in E - \bar{K}$

$$(45)\quad \begin{aligned} u_x^{(0)}(x, Y) &\leqslant f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y)\, u_{y_i}^{(0)}(x, Y), \\ v_x^{(0)}(x, Y) &\geqslant f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y)\, v_{y_i}^{(0)}(x, Y), \end{aligned}$$

5° *the Chaplygin sequence* $\{u^{(m)}(x, Y), v^{(m)}(x, Y)\}$ *of functions of class* D *in* $E_0 \cup E$ *satisfies the initial conditions*

$$(46)\quad u^{(m-1)}(x, Y) \leqslant u^{(m)}(x, Y) \leqslant \bar{z}(x, Y) \leqslant v^{(m)}(x, Y) \leqslant v^{(m-1)}(x, Y)$$

for $(x, Y) \in E_0$, $m = 1, 2, \ldots$

Under these assumptions

$$u^{(m-1)}(x, Y) \leqslant u^{(m)}(x, Y) \leqslant \bar{z}(x, Y) \leqslant v^{(m)}(x, Y) \leqslant v^{(m-1)}(x, Y)$$

for $(x, Y) \in E$, $m = 1, 2, \ldots$ *and*

$$u_x^{(m)}(x, Y) \leqslant f\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(m)}(x, Y),$$

$$v_x^{(m)}(x, Y) \geqslant f\big(x, Y, v^{(m)}(x, Y), v^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v_{y_i}^{(m)}(x, Y),$$

for $(x, Y) \in E - \bar{K}$, $m = 1, 2, \ldots$

This theorem can be proved by applying the weak differential-functional inequalities theorem and by the argument used in the proof of Theorem 1.2.

Remark 5. It follows from Theorem 4.1 that if assumptions 1°, 3°, 5° of Theorem 4.2 are satisfied and if

1° f_z is increasing with respect to z,

2° for $(x, Y) \in E - \bar{K}$

$$u_x^{(0)}(x, Y) < f\big(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(0)}(x, Y),$$

$$v_x^{(0)}(x, Y) > f\big(x, Y, v^{(0)}(x, Y), v^{(0)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v_{y_i}^{(0)}(x, Y),$$

then

$$u^{(m-1)}(x, Y) < u^{(m)}(x, Y) < \bar{z}(x, Y) < v^{(m)}(x, Y) < v^{(m-1)}(x, Y),$$
$$(x, Y) \in E - \bar{K}, \quad m = 1, 2, \ldots,$$

and

$$u_x^{(m)}(x, Y) < f\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) u_{y_i}^{(m)}(x, Y),$$

$$v_x^{(m)}(x, Y) > f\big(x, Y, v^{(m)}(x, Y), v^{(m)}(\cdot)\big) + \sum_{i=1}^{n} g^{(i)}(x, Y) v_{y_i}^{(m)}(x, Y),$$
$$(x, Y) \in E - \bar{K}, \quad m = 1, 2, \ldots$$

Remark 6. It is easy to formulate a theorem analogous to Theorem 4.2 in the case when f_z is non-increasing with respect to z (see Theorem 2.2).

In the next theorem we take up the problem of convergence of the Chaplygin sequence to the exact solution. We also give estimates of the difference between the exact and the approximate solutions.

THEOREM 5.2. *Suppose that*

1° *assumptions of Theorem* 4.2 *are satisfied,* 2° *for* $(x, Y) \in E$ *we have*

(47) $$\bar{z}(x, Y) - u^{(0)}(x, Y) \leqslant A,$$

(48) $$v^{(0)}(x, Y) - \bar{z}(x, Y) \leqslant \bar{A},$$

and

(49) $$\bar{z}(x, Y) - u^{(m)}(x, Y) \leqslant \varepsilon_m \quad \text{for } (x, Y) \in E_0,\ m = 1, 2, \ldots,$$

(50) $$v^{(m)}(x, Y) - \bar{z}(x, Y) \leqslant \bar{\varepsilon}_m \quad \text{for } (x, Y) \in E_0,\ m = 1, 2, \ldots,$$

where

(51) $$\lim_{m\to\infty} \varepsilon_m = \lim_{m\to\infty} \bar{\varepsilon}_m = 0.$$

Under these assumptions

$$\lim_{m\to\infty} u^{(m)}(x, Y) = \bar{z}(x, Y), \qquad \lim_{m\to\infty} v^{(m)}(x, Y) = \bar{z}(x, Y)$$

uniformly with respect to $(x, Y) \in E$ *and*

(52) $$\bar{z}(x, Y) - u^{(m)}(x, Y) \leqslant h_m(x), \quad (x, Y) \in E,\ m = 1, 2, \ldots,$$

(53) $$v^{(m)}(x, Y) - \bar{z}(x, Y) \leqslant \bar{h}_m(x), \quad (x, Y) \in E,\ m = 1, 2, \ldots,$$

where

$$h_m(x) = e^{N(x-x_0)}\Big[\varepsilon_m + \varepsilon_{m-1}\frac{(2N+M)(x-x_0)}{1!} + \varepsilon_{m-2}\frac{(2N+M)^2(x-x_0)^2}{2!} + \ldots + \varepsilon_1\frac{(2N+M)^{m-1}(x-x_0)^{m-1}}{(m-1)!} + A\frac{(2N+M)^m(x-x_0)^m}{m!}\Big]$$

and

$$\bar{h}_m(x) = e^{N(x-x_0)}\Big[\bar{\varepsilon}_m + \bar{\varepsilon}_{m-1}\frac{(2N+M)(x-x_0)}{1!} + \bar{\varepsilon}_{m-2}\frac{(2N+M)^2(x-x_0)^2}{2!} + \ldots + \bar{\varepsilon}_1\frac{(2N+M)^{m-1}(x-x_0)^{m-1}}{(m-1)!} + \bar{A}\frac{(2N+M)^m(x-x_0)^m}{m!}\Big].$$

Proof. At first we prove (52).
Since

$$\begin{aligned}\bar{z}_x(x, Y) - u_x^{(1)}(x, Y) &\leqslant [f(x, Y, \bar{z}(x, Y), \bar{z}(\cdot)) - f(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot))] + \\ &\quad + f_z(x, Y, u^{(0)}(x, Y), u^{(0)}(\cdot))[u^{(1)}(x, Y) - u^{(0)}(x, Y)] + \\ &\quad + \sum_{i=1}^{n} g^{(i)}(x, Y)[\bar{z}_{y_i}(x, Y) - u_{y_i}^{(1)}(x, Y)],\end{aligned}$$

it follows that the function $\bar{z}(x, Y)-u^{(1)}(x, Y)$ satisfies the differential inequality

$$\bar{z}_x(x, Y)-u_x^{(1)}(x, Y)\leqslant N[\bar{z}(x, Y)-u^{(1)}(x, Y)]+A(2N+M)e^{N(x-x_0)}+ \\ +\sum_{i=1}^{n} g^{(i)}(x, Y)[\bar{z}_{y_i}(x, Y)-u_{y_i}^{(1)}(x, Y)], \quad (x, Y)\in E.$$

Since

$$\bar{z}(x_0, Y)-u^{(1)}(x_0, Y)\leqslant \varepsilon_1 \quad \text{for } (x_0, Y)\in \bar{K},$$

we obtain from comparison theorems for partial differential inequalities ([5], Theorem 9.5.3, see also [8], Theorem 37.1)

$$\bar{z}(x, Y)-u^{(1)}(x, Y)\leqslant \tilde{h}_1(x), \tag{54}$$

where $\tilde{h}_1$ is a solution of the Cauchy problem

$$\frac{dz}{dx} = Nz+A(2N+M)\, e^{N(x-x_0)}, \quad z(x_0) = \varepsilon_1.$$

Because

$$\tilde{h}_1(x) = e^{N(x-x_0)}[\varepsilon_1+A(2N+M)(x-x_0)],$$

we obtain from (54) the estimation (52) for $m = 1$.

Suppose that (52) is true for a certain fixed m. From the definition of $u^{(m+1)}(x, Y)$ it follows that

$$\begin{aligned}\bar{z}_x(x, Y)-u_x^{(m+1)}(x, Y) &\leqslant \big[f\big(x, Y, \bar{z}(x, Y), \bar{z}(\cdot)\big)-f\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big)\big]+ \\ &+\sum_{i=1}^{n} g^{(i)}(x, Y)[\bar{z}_{y_i}(x, Y)-u_{y_i}^{(m+1)}(x, Y)]+ \\ &+f_z\big(x, Y, u^{(m)}(x, Y), u^{(m)}(\cdot)\big)\big[\big(\bar{z}(x, Y)-u^{(m+1)}(x, Y)\big)+ \\ &+\big(\bar{z}(x, Y)-u^{(m)}(x, Y)\big)\big].\end{aligned}$$

Thus we see that the function $\bar{z}(x, Y)-u^{(m+1)}(x, Y)$ satisfies the following differential inequality

$$\bar{z}_x(x, Y)-u_x^{(m+1)}(x, Y)\leqslant N[\bar{z}(x, Y)-u^{(m+1)}(x, Y)]+ \\ +(2N+M)h_m(x)+\sum_{i=1}^{n} g^{(i)}(x, Y)[\bar{z}_{y_i}(x, Y)-u_{y_i}^{(m+1)}(x, Y)], \quad (x, Y)\in E, \tag{55}$$

and the initial inequality

$$\bar{z}(x_0, Y)-u^{(m+1)}(x_0, Y)\leqslant \varepsilon_{m+1}, \quad (x_0, Y)\in \bar{K}. \tag{56}$$

Relations (55), (56) together with the comparison theorem for partial differential inequalities, imply

$$(57)\qquad \bar{z}(x, Y) - u^{(m+1)}(x, Y) \leqslant \tilde{h}_{m+1}(x), \qquad (x, Y) \in E,$$

where $\tilde{h}_{m+1}(x)$ is a solution of

$$\frac{dz}{dx} = Nz + (2N+M)h_m(x), \qquad z(x_0) = \varepsilon_{m+1}.$$

Because $\tilde{h}_{m+1}(x) = h_{m+1}(x)$, we obtain from (57) the estimation (52) for $m+1$.

Thus, by induction, relation (52) is true for all m.

Since

$$\sum_{i=0}^{\infty} \frac{(2N+M)^i (x-x_0)^i}{i!} < +\infty \qquad \text{for } x \geqslant x_0$$

and $\lim_{m\to\infty} \varepsilon_m = 0$, then

$$\lim_{m\to\infty} h_m(x) = 0$$

uniformly with respect to $x \in [x_0, x_0+a)$.

In a similar way we can prove (53) and that

$$\lim_{m\to\infty} v^{(m)}(x, Y) = \bar{z}(x, Y)$$

uniformly with respect to $(x, Y) \in E$.

Remark 7. It is easy to prove a theorem analogous to Theorem 5.2 in the case when f_z is non-increasing with respect to z.

Remark 8. If $v^{(0)}(x, Y) - u^{(0)}(x, Y) \leqslant K$ for $(x, Y) \in E$, then estimations (47), (48) are satisfied for $A = \bar{A} = K$.

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